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Bands of η-simple semigroups

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Abstract. In this paper on an arbitrary semigroup we define a few different types of relations and its congruence extensions. Also, we describe the structure of semigroups in which these relations are band congruences. The components of such obtained band decompositions are in some sense simple semigroups.

1. Introduction and preliminaries

Band decompositions of semigroups play a very important role in semigroup theory. The existence of the greatest band decomposition of semigroups was established by Tamura and Kimura [17], 1955. After that, several authors have worked on this important topic. Very interesting decompositions are band decompositions in which components are power joined, periodic and both power joined and periodic semigroups. These decompositions was studied by Tamura [16], Nordahl [14], Iseki [10] and Bogdanović [2], [3].

A semigroup S is *Archimedean* if there exists $n \in \mathbb{Z}^+$ such that $a^n \in SbS$, for all $a, b \in S$. A semigroup S is called *power joined* if for each pair of elements $a, b \in S$ there exist $m, n \in \mathbb{Z}^+$ such that $a^m = b^n$. These semigroups were first considered by Abellanas [1], 1965, for cancellative semigroups only, and Mc Alister [13], 1968, who called then *rational* semigroups. Every power joined semigroup is Archimedean. An element a of a semigroup S is *periodic* if there exist $m, n \in \mathbb{Z}^+$ such that $a^m = a^{m+n}$. A semigroup S is *periodic* if every its element is periodic.

Tamura [16] considered commutative Archimedean semigroups which have a finite number of power joined components. Bands of power joined semigroups are studied by Nordahl [14], in medial case, and by Bogdanović [3], in the general case. Iseki [10] considered periodic semigroup which is the disjoint union of semigroup, each containing only one idempotent. In [2] Bogdanović considered bands of periodic power joined semigroups.

By $\mathbb{Z}^+$ we denote the set of all positive integers. In this paper, on a semigroup S, for $k \in \mathbb{Z}^+$, we define some new equivalence relations η, η_k and τ. If these equivalences are band congruences then they makes band decompositions of η-simple (power joined) semigroups, and band decompositions of two types of periodic power joined semigroups (η_k-simple and τ-simple semigroups). The obtained results generalize the results of above mentioned authors. Also, on a semigroup S, for $k, m, n \in \mathbb{Z}^+$, we define the following relations $\overline{\eta}_{(m,n)}$, $\overline{\eta}_{(k;m,n)}$ and $\overline{\tau}_{(m,n)}$. These relations are congruences and they are generalizations of system of congruences defined by Kopamu in [11]. Some characterizations of semigroups, by congruences which are

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more general then ones introduced by Kopamu in [11], are considered by Bogdanović et al. in [6]. Here we give some equivalent statements in the case when these relations are band congruences.

By S^1 we denote a semigroup S with identity 1. By $E(S)$ we denote the set of all idempotents of a semigroup S. A semigroup which all its elements are idempotents is a *band*.

Let ϱ be an arbitrary relation on a semigroup S. Then the *radical* $R(\varrho)$ of ϱ is a relation on S defined by:

$$(a,b) \in R(\varrho) \Leftrightarrow (\exists p,q \in \mathbb{Z}^+)\ (a^p, b^q) \in \varrho.$$

The radical $R(\varrho)$ was introduced by Shevrin in [15].

An equivalence relation ξ is a *left* (*right*) *congruence* if for all $a, b \in S$, $a\,\xi\,b$ implies $ca\,\xi\,cb$ ($ac\,\xi\,bc$). An equivalence ξ is a congruence if it is both left and right congruence. A congruence relation ξ is a band congruence on S if S/ξ is a band, i.e. if $a\,\xi\,a^2$, for all $a \in S$.

Let ξ be an equivalence on a semigroup S. By $\xi^\flat$ we define the largest congruence relation on S contained in ξ. It is well-known that

$$\xi^\flat = \{(a,b) \in S \times S \,|\, (\forall x, y \in S^1)\ (xay, xby) \in \xi\}.$$

For undefined notions and notations we refer [4], [5], [8] and [9].

2. Preliminary results

First we prove the following lemma which is the helpful result for the further work.

Lemma 2.1. *Let ξ be a congruence relation on a semigroup S. Then $R(\xi) = \xi$ if and only if ξ is a band congruence on S.*

Proof. Let $R(\xi) = \xi$. Since ξ is reflexive, then for every $a \in S$ we have that

$$a^2\,\xi\,a^2 \Leftrightarrow (a^1)^2\,\xi\,(a^2)^1 \Leftrightarrow a\,R(\xi)\,a^2 \Leftrightarrow a\,\xi\,a^2.$$

Thus, ξ is a band congruence on S.

Conversely, let ξ be a band congruence on a semigroup S. Since the inclusion $\xi \subseteq R(\xi)$ always holds, then it remains to prove the opposite inclusion. Also, since ξ is a band congruence on S, then we have that

$$(\forall a \in S)(\forall k \in \mathbb{Z}^+)\ \ a\,\xi\,a^k.$$

Now assume $a, b \in S$ such that $a\,R(\xi)\,b$. Then $a^i\,\xi\,b^j$, for some $i, j \in \mathbb{Z}^+$, and by previous we have that $a\,\xi\,a^i\,\xi\,b^j\,\xi\,b$. Thus $a\,\xi\,b$. Therefore, $R(\xi) \subseteq \xi$, i.e. $R(\xi) = \xi$. □

3. The η relations

On a semigroup S we define the following relations:

$$(a,b) \in \eta \quad \Leftrightarrow \quad (\exists i, j \in \mathbb{Z}^+)\ a^i = b^j,$$
$$(a,b) \in \eta^\flat \quad \Leftrightarrow \quad (\forall x, y \in S^1)\ (xay, xby) \in \eta.$$

It is easy to verify that η is an equivalence relation on a semigroup S.

A semigroup S is *η-simple* if $(\forall a, b \in S)\ (a,b) \in \eta$. These semigroups are well-known in the literature as power joined semigroups.

The important result is the following lemma.

Lemma 3.1. *If ξ is a band congruence on a semigroup S, then $\xi \subseteq \eta$ if and only if every ξ-class of S is an η-simple semigroup.*

Proof. Let A be a ξ-class of S. Then A is a subsemigroup of S, since $a\,\xi\,a^2$, for all $a \in S$. Let $a, b \in A$, then $a\,\xi\,b$, whence $a\,\eta\,b$ in A.

Conversely, let $(a, b) \in \xi$, then $a^i = b^j$, for some $i, j \in \mathbb{Z}^+$, since a and b are in the some ξ-class A of S. Thus $(a, b) \in \eta$. Therefore, $\xi \subseteq \eta$. □

By the following theorem we describe the structure of semigroups in which the relation η is a congruence relation. These semigroups in a different way have been treated by Bogdanović in [3].

Theorem 3.2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of η-simple semigroups;*

(ii) *η is a (band) congruence on S;*

(iii) *$\eta^\flat$ is a band congruence on S;*

(iv) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\eta\,xa^2y$*;*

(v) $R(\eta^\flat) = \eta^\flat$.

Proof. (i)⇒(ii) Let S be a band B of η-simple semigroups S_α, $\alpha \in B$. Assume $a, b, c \in S$ such that $a\,\eta\,b$. Then $a, b \in S_\alpha$ and $c \in S_\beta$, for some $\alpha, \beta \in B$. Also, $ac, bc \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $\alpha, \beta \in B$ and since $S_{\alpha\beta}$, $\alpha, \beta \in B$, is η simple, then $ac\,\eta\,bc$. Similarly we prove that $ca\,\eta\,cb$. Thus η is a congruence relation on S. Further, since $a, a^2 \in S_\alpha$, $\alpha \in B$ and S_α, $\alpha \in B$, is η-simple, then $a\,\eta\,a^2$, i.e. η is a band congruence on S.

(ii)⇒(i) Let (ii) hold. Then S is a band of η-classes. Since $\eta \subseteq \eta$, then by Lemma 3.1 we have that every η-class is an η-simple semigroup. Thus S is a band of η-simple semigroups.

(i)⇒(iv) Let S be a band B of η-simple semigroups S_α, $\alpha \in B$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is η-simple, then $xay\,\eta\,xa^2y$. Thus, (iv) holds.

(iv)⇒(iii) Let (iv) hold. Then by definition for $\eta^\flat$ it is evident that $a\,\eta^\flat\,a^2$, for every $a \in S$. Thus $\eta^\flat$ is a band congruence.

(iii)⇒(i) Let $\eta^\flat$ be a band congruence on S, then S is a band of $\eta^\flat$-classes. Since $\eta^\flat$ is the greatest congruence on S contained in η, then by Lemma 3.1 we have that every $\eta^\flat$-class is an η-simple semigroup. Thus S is a band of η-simple semigroups.

(iii)⇔(v) This equivalence immediately follows by Lemma 2.1. □

Let $m, n \in \mathbb{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(m,n)}$ by

$$(a, b) \in \overline{\eta}_{(m,n)} \iff (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta.$$

If instead of η we assume the equality relation, then we obtain the relation which discussed by S. J. L. Kopamu in [11] and [12]. The main characteristic of previous defined relation gives the following theorem.

Theorem 3.3. *Let S be a semigroup and let $m, n \in \mathbb{Z}^+$. Then $\overline{\eta}_{(m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\eta}_{(m,n)}$ is reflexive and symmetric. Assume that $a\,\overline{\eta}_{(m,n)}\,b$ and $b\,\overline{\eta}_{(m,n)}\,c$. Then for every $x \in S^m$ and $y \in S^n$ there exist $k, l, s, t \in \mathbb{Z}^+$ such that $(xay)^k = (xby)^l$ and $(xby)^s = (xcy)^t$, whence

$$(xay)^{ks} = (xby)^{ls} = (xcy)^{tl},$$

i.e. $xay\,\eta\,xcy$. Thus $\overline{\eta}_{(m,n)}$ is transitive and therefore it is a congruence on S. □

Remark 3.4. Let μ be an equivalence relation on a semigroup S and let $m, n \in \mathbb{Z}^+$. Then a relation $\overline{\mu}_{(m,n)}$ defined on S by

$$(a, b) \in \overline{\mu}_{(m,n)} \iff (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \mu$$

is a congruence relation on S. But, there exists a relation μ which is not equivalence, for example $\mu = \longrightarrow$, for which the relation $\overline{\mu}_{(m,n)}$ is a congruence on S.

The complete description of $\overline{\mu}_{(m,n)}$ congruence, for $\mu = \longrightarrow$, was given by Bogdanović et al. in [7].

Theorem 3.5. *Let $m, n \in \mathbb{Z}^+$. The following conditions on a semigroup S are equivalent:*

(i) $\overline{\eta}_{(m,n)}$ *is a band congruence on S;*

(ii) $(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta\,xa^2y$;

(iii) $\eta \subseteq \overline{\eta}_{(m,n)}$;

(iv) $R(\overline{\eta}_{(m,n)}) = \overline{\eta}_{(m,n)}$.

Proof. (i)⇒(ii) This implication follows immediately.

(ii)⇒(iii) Assume that $a\,\eta\,b$. Then $a^i = b^j$, for some $i, j \in \mathbb{Z}^+$. Then for every $x \in S^m$, $y \in S^n$ and $i, j \in \mathbb{Z}^+$ we have that $xay\,\eta\,xa^2y\,\eta\,xa^iy = xb^jy\,\eta\,xby$. Since η is transitive, we have that $a\,\overline{\eta}_{(m,n)}\,b$. Thus $\eta \subseteq \overline{\eta}_{(m,n)}$.

(iii)⇒(i) Since $a\,\eta\,a^2$, for every $a \in S$, then we have that $a\,\overline{\eta}_{(m,n)}\,a^2$, for every $a \in S$, i.e. $\overline{\eta}_{(m,n)}$ is a band congruence.

(i)⇔(iv) This equivalence immediately follows by Lemma 2.1. □

Proposition 3.6. *Let $m, n \in \mathbb{Z}^+$. If $\overline{\eta}_{(m,n)}$ is a band congruence on a semigroup S, then S is a band of $\overline{\eta}_{(m,n)}$-simple semigroups.*

Proof. Let A be an $\overline{\eta}_{(m,n)}$-class of a semigroup S. Assume $a, b \in A$, then $a\,\overline{\eta}_{(m,n)}\,b$ in S, i.e. $xay\,\eta\,xby$, for every $x \in S^m$ and every $y \in S^n$, whence we have that for every $x \in A^m$ and every $y \in A^n$ is $xay\,\eta\,xby$, i.e. $a\,\overline{\eta}_{(m,n)}\,b$ in A. Thus A is $\overline{\eta}_{(m,n)}$-simple. □

4. The η_k relations

Let $k \in \mathbb{Z}^+$ be a fixed integer. On a semigroup S we define the following relations by

$$(a,b) \in \eta_k \quad\Leftrightarrow\quad a^k = b^k;$$
$$(a,b) \in \eta_k^{\flat} \quad\Leftrightarrow\quad (\forall x, y \in S^1)\ (xay, xby) \in \eta_k.$$

It is easy to verify that η_k is an equivalence relation on a semigroup S.

A semigroup S is *η_k-simple* if $(\forall a, b \in S)\ (a,b) \in \eta_k$. These semigroups are periodic.

Lemma 4.1. *Let $k \in \mathbb{Z}^+$. If ξ is a band congruence on a semigroup S, then $\xi \subseteq \eta_k$ if and only if every ξ-class of S is an η_k-simple semigroup.*

Proof. Let A be a ξ-class of S. Then A is a subsemigroup of S, since $a\,\xi\,a^2$, for all $a \in S$. Let $a, b \in A$, then $a\,\xi\,b$, whence $a\,\eta_k\,b$ in A.

Conversely, let $(a,b) \in \xi$. Since a and b are in the some ξ-class A of S and since A is η_k-simple,then $(a,b) \in \eta_k$. Therefore, $\xi \subseteq \eta_k$. □

By the following theorem we give structural characterization of bands of η_k-simple semigroups.

Theorem 4.2. *Let $k \in \mathbb{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *S is a band of η_k-simple semigroups;*

(ii) $(\forall a, b \in S)\ (\,(ab)^k = (a^kb^k)^k\ \wedge\ a^k = a^{2k}\,)$;

(iii) *η_k is a band congruence on S;*

(iv) *$\eta_k^{\flat}$ is a band congruence on S;*

(v) $(\forall a \in S)(\forall x, y \in S^1)\ xay\,\eta_k\,xa^2y$;

(vi) $R(\eta_k) = \eta_k$ *and* η_k *is a congruence on* S;

(vii) $R(\eta_k^\flat) = \eta_k^\flat$.

Proof. (i)⇒(ii) Let S be a band Y of η_k-simple semigroups S_α, $\alpha \in Y$. For every $a, b \in S$ we have that $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, whence $ab, a^k b^k \in S_{\alpha\beta}$ and so $(ab)^k = (a^k b^k)^k$. Clearly, $a^k = a^{2k}$.

(ii)⇒(iii) It is clear that η_k is an equivalence. Let $a\eta_k b$ and $x \in S$, then $a^k = b^k$ and by hypothesis we have that $(ax)^k = (a^k x^k)^k = (b^k x^k)^k = (bx)^k$, i.e. $ax\,\eta_k\,bx$. Similarly, $xa\,\eta_k, xb$. Thus η_k is a congruence relation on S, and since $a^k = a^{2k}$ we have that η_k is a band congruence on S.

(iii)⇒(i) Let η_k be a band congruence and A be an η_k-class of S. Assume $a, b \in A$, then $a\,\eta_k\,b$ in A and thus A is an η_k-simple semigroup. Therefore, S is a band of η_k-simple semigroups.

(i)⇒(v) Let S be a band Y of η_k-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x, y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is an η_k-simple semigroup then $xay\,\eta_k\,xa^2y$. Thus, (v) holds.

(v)⇒(iv) Let (v) hold. Then by definition for $\eta_k^\flat$ it is evident that $a\,\eta_k^\flat\,a^2$, for every $a \in S$. Thus $\eta_k^\flat$ is a band congruence.

(iv)⇒(i) Let $\eta_k^\flat$ be a band congruence on S, then S is a band of $\eta_k^\flat$-classes. Since $\eta_k^\flat$ is the largest congruence on S contained in η_k, then by Lemma 4.1 we have that every $\eta_k^\flat$-class is η_k-simple semigroup. Thus S is a band of η_k-simple semigroups.

(iii)⇔(vi) and (iv)⇔(vii) These equivalences follow immediately by Lemma 2.1. □

Let $k, m, n \in \mathbb{Z}^+$. On a semigroup S we define a relation $\overline{\eta}_{(k;m,n)}$ by

$$(a, b) \in \overline{\eta}_{(k;m,n)} \iff (\forall x \in S^m)(\forall y \in S^n)\ (xay, xby) \in \eta_k.$$

The following lemma holds.

Lemma 4.3. *Let S be a semigroup and let $k, m, n \in \mathbb{Z}^+$, then $\overline{\eta}_{(k;m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\eta}_{(k;m,n)}$ is reflexive and symmetric. Assume $a, b, c \in S$ such that $a\,\overline{\eta}_{(k;m,n)}\,b$ and $b\,\overline{\eta}_{(k;m,n)}\,c$. Then for every $x \in S^m$ and every $y \in S^n$ we obtain that $(xay)^k = (xby)^k$ and $(xby)^k = (xcy)^k$, whence

$$(xay)^k = (xcy)^k,$$

i.e. $xay\,\eta_{(k;m,n)}\,xcy$. Thus $\overline{\eta}_{(k;m,n)}$ is transitive and therefore it is a congruence on S. □

Theorem 4.4. *Let $k, m, n \in \mathbb{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) $\overline{\eta}_{(k;m,n)}$ *is a band congruence on* S;

(ii) $(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\ xay\,\eta_k\,xa^2y$;

(iii) $R(\overline{\eta}_{(k;m,n)}) = \overline{\eta}_{(k;m,n)}$.

Proof. (i)⇔(ii) This equivalence is evident.

(i)⇔(iii) This equivalence immediately follows by Lemma 2.1. □

Proposition 4.5. *Let $k, m, n \in \mathbb{Z}^+$. If $\overline{\eta}_{(k;m,n)}$ is a band congruence on a semigroup S, then $\eta_k \subseteq \overline{\eta}_{(k;m,n)}$.*

Proof. Since $\overline{\eta}_{(k;m,n)}$ is a band congruence on S, then $xay\,\eta_k\,xa^iy$, for every $i \in \mathbb{Z}^+$ and for all $x \in S^m$, $y \in S^n$, $a \in S$. Assume $a, b \in S$ such that $a\,\eta_k\,b$. Then $a^k = b^k$. Thus for every $x \in S^m$ and $y \in S^n$ we have that

$$xay\,\eta_k\,xa^ky = xb^ky\,\eta_k\,xby.$$

Since η_k is transitive, we obtain that $a\,\overline{\eta}_{(k;m,n)}\,b$. Thus $\eta_k \subseteq \overline{\eta}_{(k;m,n)}$. □

5. The τ relations

Further, by previous defined relations on a semigroup S we define the following relations:

$$(a,b) \in \tau \quad \Leftrightarrow \quad (\exists k \in \mathbb{Z}^+)\ (a,b) \in \eta_k;$$
$$(a,b) \in \tau^\flat \quad \Leftrightarrow \quad (\forall x,y \in S^1)\ (xay,xby) \in \tau.$$

It is easy to verify that the relation τ is an equivalence on a semigroup S.

A semigroup S is *τ-simple* if $(\forall a,b \in S)\ (a,b) \in \tau$.

By the following theorem we describe the structure of bands of τ-simple semigroups. Bogdanović in [2] gave some other characterizations of these semigroups.

Theorem 5.1. *The following conditions on a semigroup S are equivalent:*

(i) *S is a band of τ-simple semigroups;*

(ii) *τ is a band congruence on S;*

(iii) *$\tau^\flat$ is a band congruence on S;*

(iv) $(\forall a \in S)(\forall x,y \in S^1)\ xay\,\tau\,xa^2y$;

(v) *$R(\tau) = \tau$ and τ is a congruence on S;*

(vi) $R(\tau^\flat) = \tau^\flat$.

Proof. (i)⇒(ii) Let S be a band Y of τ-simple semigroups S_α, $\alpha \in Y$. Assume $a,b,c \in S$ such that $a\,\tau\,b$. Then $a^k = b^k$, for some $k \in \mathbb{Z}^+$. So, then $a,b \in S_\alpha$ and $c \in S_\beta$, for some $\alpha,\beta \in Y$. Thus $ac, bc \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $\alpha,\beta \in Y$ and since $S_{\alpha\beta}$, $\alpha,\beta \in Y$, is τ-simple, then $ac\,\tau\,bc$. Similarly, $ca\,\tau\,cb$. Hence, τ is a congruence relation on S. Further, since $a,a^2 \in S_\alpha$, $\alpha \in Y$ and S_α, $\alpha \in Y$, is τ-simple, then $a\,\tau\,a^2$, i.e. τ is a band congruence on S.

(ii)⇒(i) Let (ii) hold. Then S is a band of τ-classes. Let A be a τ-class of S. Then A is a subsemigroup of S. Assume $a,b \in A$, then $a\,\tau\,b$ in A and A is a τ-simple. Therefore, S is a band of τ-simple semigroups.

(i)⇒(iv) Let S be a band Y of τ-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x,y \in S^1$. Then $xay, xa^2y \in S_\alpha$, for some $\alpha \in Y$. Since S_α, $\alpha \in Y$ is τ-simple then $xay\,\tau\,xa^2y$. Thus, (iv) holds.

(iv)⇒(iii) This implication follows immediately.

(iii)⇒(i) Let (iii) hold. Then S is a band of $\tau^\flat$-classes. Let A be an arbitrary $\tau^\flat$-class of S. Then A is a subsemigroup of S. Assume $a,b \in A$, then $a\,\tau^\flat\,b$ in A and since $\tau^\flat \subseteq \tau$, then $a\,\tau\,b$ in A. Thus A is a τ-simple. Therefore, S is a band of τ-simple semigroups.

(ii)⇔(v) and (iii)⇔(vi) These equivalences follows by Lemma 2.1. □

Let $m,n \in \mathbb{Z}^+$. On a semigroup S we define a relation $\overline{\tau}_{(m,n)}$ by

$$(a,b) \in \overline{\tau}_{(m,n)} \ \Leftrightarrow\ (\forall x \in S^m)(\forall y \in S^n)\ (xay,xby) \in \tau.$$

The following theorem holds.

Theorem 5.2. *Let S be a semigroup and let $m,n \in \mathbb{Z}^+$. Then $\overline{\tau}_{(m,n)}$ is a congruence relation on S.*

Proof. It is clear that $\overline{\tau}_{(m,n)}$ is reflexive and symmetric. Assume $a,b,c \in S$ such that $a\,\overline{\tau}_{(m,n)}\,b$ and $b\,\overline{\tau}_{(m,n)}\,c$. Then for every $x \in S^m$ and $y \in S^n$ there exist $k,l \in \mathbb{Z}^+$ such that $(xay)^k = (xby)^k$ and $(xby)^l = (xcy)^l$, whence

$$(xay)^{kl} = (xby)^{kl} = (xby)^{lk} = (xcy)^{lk}.$$

So, we have that $xay\,\eta_{lk}\,xcy$, i.e. $xay\,\tau\,xcy$. Thus $\overline{\tau}_{(m,n)}$ is transitive and therefore it is a congruence on S. □

Theorem 5.3. *Let $m,n \in \mathbb{Z}^+$. Then the following conditions on a semigroup S are equivalent:*

(i) *$\overline{\tau}_{(m,n)}$ is a band congruence on S;*

(ii) $(\forall x \in S^m)(\forall y \in S^n)(\forall a \in S)\; xay\,\tau\, xa^2y$;

(iii) $R(\overline{\tau}_{(m,n)}) = \overline{\tau}_{(m,n)}$.

Proof. (i)⇔(ii) This equivalence follows immediately.
(i)⇔(iii) This equivalence immediately follows by Lemma 2.1. □

Proposition 5.4. *Let $m, n \in \mathbb{Z}^+$. If $\overline{\tau}_{(m,n)}$ is a band congruence on a semigroup S, then $\tau \subseteq \overline{\tau}_{(m,n)}$.*

Proof. Since $\overline{\tau}_{(m,n)}$ is a band congruence on S, then $xay\,\tau\, xa^iy$, for every $i \in \mathbb{Z}^+$ and for all $x \in S^m$, $y \in S^n$, $a \in S$. Assume $a, b \in S$ such that $a\,\tau\, b$. Then $a^k = b^k$, for some $k \in \mathbb{Z}^+$. Thus for every $x \in S^m$, $y \in S^n$ and $k \in \mathbb{Z}^+$ we have that $xay\,\tau\, xa^ky = xb^ky\,\tau\, xby$. Since τ is transitive, then $a\,\overline{\tau}_{(m,n)}\, b$. Therefore $\tau \subseteq \overline{\tau}_{(m,n)}$. □

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