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An iterative algorithm to compute the Bott-Duffin inverse and generalized Bott-Duffin inverse

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Abstract. Let L be a subspace of C^n and P_L be the orthogonal projector of C^n onto L. For $A \in C^{n\times n}$, the generalized Bott-Duffin (B-D) inverse $A^{(\dagger)}_{(L)}$ is given by $A^{(\dagger)}_{(L)} = P_L(AP_L + P_{L^\perp})^\dagger$. In this paper, by defined a non-standard inner product, a finite formulae is presented to compute Bott-Duffin inverse $A^{(-1)}_{(L)} = P_L(AP_L+P_{L^\perp})^{-1}$ and generalized Bott-Duffin inverse $A^{(\dagger)}_{(L)} = P_L(AP_L+P_{L^\perp})^\dagger$ under the condition A is $L-zero$ (i.e., $AL\cap L^\perp = \{0\}$). By this iterative method, when taken the initial matrix $X_0 = P_L A^* P_L$, the Bott-duffin inverse $A^{(-1)}_{(L)}$ and generalized Bott-duffin inverse $A^{(\dagger)}_{(L)}$ can be obtained within a finite number of iterations in absence of roundoff errors. Finally a given numerical example illustrates that the iterative algorithm dose converge.

1. Introduction

The Bott-Duffin (B-D) inverse was first introduced by Bott and Duffin in their famous paper [2]. Many properties and applications of the B-D inverse have been developed in [1, 11]. Later, Chen in his paper [5] defined the generalized B-D inverse of a square matrix and gave some properties and applications. Wang and Wei in [10] and Wei and Xu in [12] discussed the perturbation theory for the B-D inverse and showed the B-D condition number $\mathcal{K}_{BD}(A) =\| A \| \cdot \| A^{(-1)}_{(L)} \|$ to be minimum in the inequality of error analysis and the perturbation bound of the solution of the constrained system. Recently, in [6], Liu et al. use the projection methods, which is an applications of the generalization of the Bott-Duffin inverse, for solving sparse linear systems. Chen et al. in [3, 4], Xue and Chen in [13], and Zhang et al. in [14], established the perturbation theory of the generalized B-D inverse $A^{\dagger}_{(L)}$ under $L-$ zero matrices, presented the expression of $A^{\dagger}_{(L)}$ and point the $A^{\dagger}_{(L)}$ under $L-$ zero matrices popularize that in [5].

The authors also did some works on the computation of generalized inverses. In [8], the authors gave a full-rank representation and the minor of the generalized inverse $A^{(2)}_{T,S}$. In [9], they obtain a representation of $A^{(2)}_{T,S}$ based on Gaussian elimination. Until now, we could not see using finite iterative algorithm to compute the B-D inverse and generalized B-D inverse. In this paper, we will first introduce a non-standard inner product and then develop a finite iterative formulae for the the Bott-duffin inverse $A^{(-1)}_{(L)}$ and generalized

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Bott-duffin inverse $A^{(\dagger)}_{(L)}$. In the end of the paper, a numerical example demonstrate that the iterative method is quite efficient.

2. Notations and preliminaries

Throughout the paper, let $C^{n\times n}$ (resp. $C^{m\times n}$) denote the set of all $n \times n$ (resp. $m \times n$) matrices over C. L is a subspace of C^n and P_L is the orthogonal projector onto L. For any $A \in C^{n\times n}$, we write $R(A)$ for its range, $N(A)$ for its nullspace. A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. Recall that the Bott-Duffin inverse of $A \in C^{n\times n}$ is the matrix by $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1} = (P_LAP_L)^\dagger$ when $AP_L + P_{L^\perp}$ is nonsingular. The generalized Bott-Duffin inverse of A is $A^{(\dagger)}_{(L)} = P_L(AP_L + P_{L^\perp})^\dagger$. When A is L-zero $A^{(\dagger)}_{(L)} = (P_LAP_L)^\dagger$.

Let L be a subspace of C^n, The restricted conjugate transpose on L of a complex matrix A is defined as $A^*_L = P_LA^*P_L$. In the same way, in the space $C^{n\times n}$, a restricted inner product on the subspace L is defined as $< A, B >_L =< P_LAP_L, B >= tr(P_LA^*P_LB)$ for all $A, B \in C^{n\times n}$, which is called non-standard inner product. Then the restricted norm on L of a matrix A generated by this inner product is the Frobenius norm of the matrix P_LAP_L denoted by $\| A \|_L$.

For a complex matrix $A \in C^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ is defined to be unique solution of the following four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

A matrix X is called $\{i, j, \ldots, k\}$ inverse of A if it satisfies $(i), (j), \ldots, (k)$ from among the equations $(1) - (4)$.

The $\{2\}$ inverse of a matrix $A \in C^{m\times n}$ with range T and nullspace S is defined as following:

Let $A \in C^{m\times n}$ be of rank r, T be a subspace of C^n of dimension $s \le r$ and S be a subspace of C^m of dimension $m - s$. If X satisfies $XAX = X$, $R(X) = T$ and $N(X) = S$, then X is called the generalized inverse $A^{(2)}_{T,S}$ of A. When $s = r$, $A^{(2)}_{T,S} = A^{(1,2)}_{T,S}$.

In this paper the following Lemmas are needed in what follows:

Lemma 2.1. ([1]) *Let $A \in C^{m\times n}$ be of rank r, any two of the following three statements imply the third:*

$$X \in A\{1\}$$
$$X \in A\{2\}$$
$$rankA = rankX.$$

Lemma 2.2. ([1]) *Let $A \in C^{n\times n}$, L be a subspace of C^n. If $AP_L + P_{L^\perp}$ is nonsingular, then*
(1) $A^{(-1)}_{(L)} = (AP_L)^{(1,2)}_{L,L^\perp} = (P_LA)^{(1,2)}_{L,L^\perp} = (P_LAP_L)^{(1,2)}_{L,L^\perp}$,
(2) $(A^{(-1)}_{(L)})^{(-1)}_{(L)} = P_LAP_L$.

Lemma 2.3. ([4]) *Let L be a subspace of C^n with $dimL = k \le r(A)$ and let the columns of $n \times k$ matrix U form an orthogonal basis for L. The following statements are equivalent:*
(1) $AL \cap L^\perp = \{0\}$, i.e., A is L-zero;
(2) $N(A) \cap L = N(AL)$, i.e., $N(P_LAP_L) = N(AP_L)$;
(3) $A^{(\dagger)}_{(L)} = (P_LAP_L)^\dagger = (P_LAP_L)^{(1,2)}_{R(P_LA^*P_L),N(P_LA^*P_L)}$;
(4) $r(AU) = r(U^*AU)$.

Lemma 2.4. ([1]) *Let L and M be complementary subspaces of C^n, the projector $P_{L,M}$ has the following properies*
(1) $P_{L,M}A = A$ if and only if $R(A) \subset L$,
(2) $AP_{L,M} = A$ if and only if $N(A) \supset M$.

Throughout the paper, we assume that $AP_L + P_{L^\perp}$ is nonsingular or $AL \cap L^\perp = \{0\}$ (i.e., A is L-zero).
About the restricted inner product on subspace L, we have the following property.

Lemma 2.5. *Let L be the subspace of $\mathbb{C}^n$, $A, B \in \mathbb{C}^{n\times n}$, then we have:*

$$< A, B >_L = < A, P_L B P_L > = < P_L A P_L, B P_L > = \overline{< B, A >_L} = < B^*, P_L A^* P_L > .$$

According to the definition and the properties of inner product, the above equalities are right.

3. Iterative method for computing $A^{(\dagger)}_{(L)}$ and $A^{(-1)}_{(L)}$

In this section we first introduce an iterative method to obtain a solution of the matrix equation $P_L AXAP_L = P_L AP_L$, where $A \in C^{n\times n}$. We then show that if $AP_L + P_{L^\perp}$ is nonsingular or $AP_L + P_{L^\perp}$ is singular but A is L-zero, then for any initial matrix X_0 with $R(X_0) \subset P_L A^*$, the matrix sequence $\{X_k\}$ generated by the iterative method converges to its a solution within at most n^2 iteration steps in absence of the roundoff errors. We also show that if let the initial matrix $X_0 = P_L A^* P_L$, then the solution X^* obtained by the iterative method is the generalized Bott-Duffin inverse $A^{(\dagger)}_{(L)}$.

First we present the iteration method for solving the matrix equation $P_L AXAP_L = P_L AP_L$, the iteration method as follows:

Algorithm 3.1:

1. Input matrices $A \in C^{n\times n}$, $P_L \in C^{n\times n}$ and $X_0 \in C^{n\times n}$ with $R(X_0) \subset R(P_L A^*)$;
2. Calculate

$$R_0 = A - AX_0A; \qquad P_0 = A(R_0)^*_L A; \qquad k := 0.$$

3. If $P_L R_k = 0$, then stop; otherwise, $k := k + 1$;
4. Calculate

$$\begin{aligned}
X_k &= X_{k-1} + \frac{\| R_{k-1} \|_L^2}{\| P_{k-1} \|_L^2}(P_{k-1})^*_L; \\
R_k &= A - AX_kA = R_{k-1} - \frac{\| R_{k-1} \|_L^2}{\| P_{k-1} \|_L^2} A(P_{k-1})^*_L A; \\
P_k &= A(R_k)^*_L A + \frac{\| R_k \|_L^2}{\| R_{k-1} \|_L^2} P_{k-1};
\end{aligned}$$

5. Goto step 3.

About Algorithm 3.1, we have the following basic properties.

Theorem 3.2. *In Algorithm 3.1, if we take the initial matrix $X_0 = A^*_L$, then the sequences $\{X_k\}$ and $\{P_k\}$ generalized by it such that*

(1) $R(X_k) \subset R(P_L A^* P_L)$, $N(X_k) \supset N(P_L A^* P_L)$ and $R(P_k) \subset R(AP_L)$, $N(P_k) \supset N(P_L A)$;

(2) if $P_L R_k P_L = 0$, $AP_L + P_{L^\perp}$ is singular and A is L-zero, then $X_k = A^{(\dagger)}_{(L)}$;

(3) if $P_L R_k P_L = 0$ and $AP_L + P_{L^\perp}$ is nonsingular, then $X_k = A^{(-1)}_{(L)}$.

Proof. (1) To prove the conclusion, we use the induction.

When $s = 0$, we have $X_0 = A^*_L = P_L A^* P_L$ and $P_0 = A(R_0)^*_L A = AP_L R^*_0 P_L A$. This implies the conclusion is right.

When $s = 1$, we have

$$X_1 = X_0 + \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} P_L A^* P_L R_0 P_L A^* P_L = P_L A^* P_L \left(P_L + \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} P_L R_0 P_L A^* P_L \right) = \left(P_L + \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} P_L A^* P_L R_0 P_L \right) P_L A^* P_L$$

and

$$P_1 = AP_LR_1^*P_LA + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2}P_0 = AP_L\left(P_LR_1^*P_LA + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2}P_LR_0^*P_LA\right) = \left(AP_LR_1^*P_L + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2}AP_LR_0^*P_L\right)P_LA.$$

Assume that conclusion holds for all s $(0 < s < k)$. Then there exist matrices U, V, W, and Y such that $X_s = P_LA^*P_LU = VP_LA^*P_L$ and $P_s = AP_LW = YP_LA$.

Further, we have that

$$X_{s+1} = X_s + \frac{\| R_s \|_L^2}{\| P_s \|_L^2}P_LP_s^*P_L = P_LA^*P_L\left(U + \frac{\| R_s \|_L^2}{\| P_s \|_L^2}Y^*P_L\right) = \left(V + \frac{\| R_s \|_L^2}{\| P_s \|_L^2}P_LW^*\right)P_LA^*P_L$$

and

$$P_{s+1} = AP_LR_{s+1}^*P_LA + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2}P_s = AP_L\left(P_LR_{s+1}^*P_LA + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2}W\right) = \left(AP_LR_{s+1}^*P_L + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2}Y\right)P_LA.$$

This implies that $R(X_{s+1}) \subset R(P_LA^*P_L)$ and $N(X_{s+1}) \supset N(P_LA^*P_L)$, and $R(P_{s+1}) \subset R(AP_L)$ and $N(P_{s+1}) \supset N(P_LA)$.

By the principle of induction, the conclusion holds for all $k = 0, 1, \cdots$

(2) According to Algorithm 3.1 and the results in (1) , we know that, if $P_LR_kP_L = 0$, then we have $X_k \in (P_LAP_L)\{1\}$. This implies $r(X_k) \geq r(P_LAP_L)$, then by the conclusion of (1), we can easy get $r(X_k) = r(P_LAP_L)$. From Lemma 2.1 we know $X_k \in (P_LAP_L)\{1, 2\}$ with range $R(P_LA^*P_L)$ and null space $N(P_LA^*P_L)$. If $AP_L + P_{L^\perp}$ is singular and A is L-zero, by Lemma 2.3 we know $X_k = A_{(L)}^{(\dagger)}$.

(3) If $AP_L + P_{L^\perp}$ is nonsingular, then $r(P_LAP_L) = r(AP_L) = dimL$. It is not difficult to deduce $R(P_LA) = R(P_L) = L$ and $N(A^*P_L) = L^\perp$. This means $X_k \in (P_LAP_L)\{1, 2\}$ with range L and null space $L^\perp$. By Lemma 2.2, $X_k = A_{(L)}^{(-1)}$. □

Theorem 3.3. *Let $\widetilde{X}$ be an solution of matrix equation $P_LAXAP_L = P_LAP_L$ with $R(\widetilde{X}) \subset L$ and $N(\widetilde{X}) \subset L^\perp$, then for any initial matrix X_0 with $R(X_0) \subset L$ and $N(X_0) \subset L^\perp$, the sequences $\{X_i\}$, $\{R_i\}$ and $\{P_i\}$ generalized by Algorithm 3.1 satisfy $< P_i, P_L(\widetilde{X} - X_i)^*P_L >_L = \| R_i \|_L^2, (i = 0, 1, 2, \cdots)$.*

Proof. First by Lemma 2.4 and the properties of $\widetilde{X}$, we have $P_L\widetilde{X}P_L = \widetilde{X}$.

Next we prove the conclusion by induction. By Algorithm 3.1 and Lemma 2.4, when $i = 0$, we have

$$\begin{aligned}
< P_0, P_L(\widetilde{X} - X_0)^*P_L >_L &= < P_LP_0P_L, P_L(\widetilde{X} - X_0)^*P_L > \\
&= < P_0, P_L(\widetilde{X} - X_0)^*P_L > \\
&= < AP_LR_0^*P_LA, (\widetilde{X} - X_0)^* > \\
&= < P_LR_0^*P_L, A^*(\widetilde{X} - X_0)^*A^* > \\
&= < R_0^*, P_LA^*(\widetilde{X} - X_0)^*A^*P_L > \\
&= < R_0^*, P_LR_0^*P_L > = \| R_0 \|_L^2 .
\end{aligned}$$

And when $i = 1$, we have

$$\begin{aligned}
< P_1, P_L(\widetilde{X} - X_1)^*P_L >_L &= < P_LP_1P_L, P_L(\widetilde{X} - X_1)^*P_L > \\
&= < P_1, P_L(\widetilde{X} - X_1)^*P_L > \\
&= < P_1, (\widetilde{X} - X_1)^* >
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle AP_LR_1^*P_LA + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2}P_0, (\widetilde{X} - X_1)^* \right\rangle \\
&= < AP_LR_1^*P_LA, (\widetilde{X} - X_1)^* > + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} < P_0, (\widetilde{X} - X_1)^* > \\
&= < P_LR_1^*P_L, R_1^* > + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} < P_0, (\widetilde{X} - X_0)^* > - \frac{\| R_1 \|_L^2}{\| P_0 \|_L^2} < P_0, (P_LP_0^*P_L)^* > \\
&= \| R_1 \|_L^2 .
\end{aligned}
$$

Assume that the conclusion holds for $i = s(s > 0)$, that $< P_s, P_L(\widetilde{X} - X_s)^*P_L >_L = \| R_s \|_L^2$, then $i = s + 1$, we have

$$
\begin{aligned}
< P_{s+1}, P_L(\widetilde{X} - X_{s+1})^*P_L >_L &= < P_LP_{s+1}P_L, P_L(\widetilde{X} - X_{s+1})^*P_L > \\
&= < P_{s+1}, P_L(\widetilde{X} - X_{s+1})^*P_L > \\
&= < P_{s+1}, (\widetilde{X} - X_{s+1})^* > \\
&= \left\langle AP_LR_{s+1}^*P_LA + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2}P_s, (\widetilde{X} - X_{s+1})^* \right\rangle \\
&= < AP_LR_{s+1}^*P_LA, (\widetilde{X} - X_{s+1})^* > + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} < P_s, (\widetilde{X} - X_{s+1})^* > \\
&= < P_LR_{s+1}^*P_L, R_{s+1}^* > + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} < P_s, (\widetilde{X} - X_s)^* > - \frac{\| R_{s+1} \|_L^2}{\| P_s \|_L^2} < P_s, P_LP_sP_L > \\
&= \| R_{s+1} \|_L^2 .
\end{aligned}
$$

By the principle of induction, the conclusion $< P_i, P_L(\widetilde{X} - X_i)^*P_L >_L = \| R_i \|_L^2$ holds for all $i = 0, 1, 2, \cdots$ □

Remark 1. From Theorem 2.3 we know that if $P_LR_iP_L \neq 0$, then $P_LP_iP_L \neq 0$. This result shows that if $P_LR_iP_L \neq 0$, then Algorithm 3.1 can not be terminated.

Theorem 3.4. *For the sequences $\{R_i\}$ and $\{P_i\}$ generated by Algorithm 3.1 with the $X_0 = P_LA^*P_L$, if there exists a positive number k such that $R_i \neq 0$ for all $i = 0, 1, 2, \cdots k$, then we have*

$$< R_i, R_j >_L = 0, \quad < P_i, P_j >_L = 0, \quad (i \neq j, i, j = 0, 1, \cdots, k).$$

Proof. According to Lemma 2.5, we know that $< A, B >_L = \overline{< B, A >_L}$ holds for all matrices A and B in $C^{n\times n}$, so we only need prove the conclusion hold for all $0 \leq i < j \leq k$. Using induction and two steps are required.

Step1. Show that $< R_i, R_{i+1} >_L = 0$ and $< P_i, P_{i+1} >_L = 0$ for all $i = 0, 1, 2, \cdots, k$. To prove this conclusion, we also use induction. According to Lemma 2.5 and Algorithm 3.1, when $i = 0$, we have

$$
\begin{aligned}
< R_0, R_1 >_L = < P_LR_0P_L, R_1 > &= \left\langle P_LR_0P_L, R_0 - \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2}AP_LP_0^*P_LA \right\rangle \\
&= < P_LR_0P_L, R_0 > - \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} < P_LR_0P_L, AP_LP_0^*P_LA > \\
&= \| R_0 \|_L^2 - \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} < A^*P_LR_0P_LA^*, P_LP_0^*P_L > \\
&= \| R_0 \|_L^2 - \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} < P_0^*, P_LP_0^*P_L > \\
&= \| R_0 \|_L^2 - \frac{\| R_0 \|_L^2}{\| P_0 \|_L^2} \| P_0 \|_L^2 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
< P_0, P_1 >_L = < P_L P_0 P_L, P_1 > &= \left\langle P_L P_0 P_L, A P_L R_1^* P_L A + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} P_0 \right\rangle \\
&= < P_L P_0 P_L, A P_L R_1^* P_L A > + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} < P_L P_0 P_L, P_0 > \\
&= < A^* P_L P_0 P_L A^*, P_L R_1^* P_L > + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} \| P_0 \|_L^2 \\
&= \frac{\| P_0 \|_L^2}{\| R_0 \|_L^2} < (R_0 - R_1)^*, P_L R_1^* P_L > + \frac{\| R_1 \|_L^2}{\| R_0 \|_L^2} \| P_0 \|_L^2 = 0.
\end{aligned}
$$

Assume that conclusion holds for all $i \leq s (0 < s < k)$. Then

$$
\begin{aligned}
< R_s, R_{s+1} >_L &= < P_L R_s P_L, R_{s+1} > \\
&= \left\langle P_L R_s P_L, R_s - \frac{\| R_s \|_L^2}{\| P_s \|_L^2} A P_L P_s^* P_L A \right\rangle \\
&= < P_L R_s P_L, R_s > - \frac{\| R_s \|_L^2}{\| P_s \|_L^2} < P_L R_s P_L, A P_L P_s^* P_L A > \\
&= \| R_s \|_L^2 - \frac{\| R_s \|_L^2}{\| P_s \|_L^2} < A^* P_L R_s P_L A^*, P_L P_s^* P_L > \\
&= \| R_s \|_L^2 - \frac{\| R_s \|_L^2}{\| P_s \|_L^2} \left\langle (P_s - \frac{\| R_s \|_L^2}{\| R_{s-1} \|_L^2} P_{s-1})^*, P_L P_s^* P_L \right\rangle \\
&= \| R_s \|_L^2 - \frac{\| R_s \|_L^2}{\| P_s \|_L^2} \| P_s \|_L^2 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
< P_s, P_{s+1} >_L = < P_L P_s P_L, P_{s+1} > &= \left\langle P_L P_s P_L, A P_L R_{s+1}^* P_L A + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} P_s \right\rangle \\
&= < A^* P_L P_s P_L A^*, P_L R_{s+1}^* P_L > + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} < P_L P_s P_L, P_s > \\
&= \frac{\| P_s \|_L^2}{\| R_s \|_L^2} < (R_s - R_{s+1})^*, P_L (R_{s+1}^* P_L > + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} \| P_s \|_L^2 \\
&= -\frac{\| P_s \|_L^2}{\| R_s \|_L^2} \| R_{s+1} \|_L^2 + \frac{\| R_{s+1} \|_L^2}{\| R_s \|_L^2} \| P_s \|_L^2 = 0.
\end{aligned}
$$

By the principle of induction, $< R_i, R_{i+1} >_L = 0$, and $< P_i, P_{i+1} >_L = 0$, hold for all $i = 0, 1, \cdots, k$.

Step2. Assume that $< R_i, R_{i+l} >_L = 0$, and $< P_i, P_{i+l} >_L = 0$, hold for all $0 \leq i \leq k$ and $1 < l < k$, show that $< R_i, R_{i+l+1} >_L = 0$, and $< P_i, P_{i+l+1} >_L = 0$.

$$
\begin{aligned}
< R_i, R_{i+l+1} >_L = < P_L R_i P_L, R_{i+l+1} > &= \left\langle P_L R_i P_L, R_{i+l} - \frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2} A P_L P_{i+l}^* P_L A \right\rangle \\
&= -\frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2} < P_L R_i P_L, A P_L P_{i+l}^* P_L A > \\
&= -\frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2} < A^* P_L R_i P_L A^*, P_L P_{i+l}^* P_L > .
\end{aligned}
$$

If $i = 0$, we have $A^*P_LR_0P_LA^* = P_0^*$. Then the above equation becomes

$$-\frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2} < A^*P_LR_iP_LA^*, P_LP_{i+l}^*P_L >= -\frac{\| R_l \|_L^2}{\| P_l \|_L^2} < P_0*, P_LP_l^*P_L >= 0.$$

If $i \geq 1$, we have

$$-\frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2} < A^*P_LR_iP_LA^*, P_LP_{i+l}^*P_L >= -\frac{\| R_{i+l} \|_L^2}{\| P_{i+l} \|_L^2}\left\langle P_i - \frac{\| R_i \|_L^2}{\| R_{i-1} \|_L^2}P_{i-1}, P_LP_{i+l}P_L\right\rangle = 0$$

and

$$\begin{aligned} < P_i, P_{i+l+1} >_L =< P_LP_iP_L, P_{i+l+1} > &= \left\langle P_LP_iP_L, AP_LR_{i+l+1}^*P_LA + \frac{\| R_{i+l+1} \|_L^2}{\| R_{i+l} \|_L^2}P_{i+l}\right\rangle \\ &= < P_LP_iP_L, AP_LR_{i+l+1}^*P_LA > + \frac{\| R_{i+l+1} \|_L^2}{\| R_{i+l} \|_L^2} < P_LP_iP_L, P_{i+l} > \\ &= < A^*P_LP_iP_LA^*, P_LR_{i+l+1}^*P_L > \\ &= \frac{\| P_i \|_L^2}{\| R_i \|_L^2} < (R_{i+1} - R_i)^*, P_LR_{i+l+1}^*P_L >= 0. \end{aligned}$$

From step 1 and step 2, we have by principle induction that $< R_i, R_j >_L= 0$, and $< P_i, P_j >_L= 0$, hold for all $i, j = 0, 1, \cdots, k, i \neq j$. □

Remark 2. Theorem 3.4 implies that, for an initial matrix $X_0 = P_LA^*P_L$, since the $R_0, R_1, \cdots$ are orthogonal each other, based on restricted inner product on subspace L, in the finite dimension matrix space $C^{n\times n}$, it is certain there exists a positive number $k \leq n^2$ such that $\| R_k \|_L= 0$. Then by Theorem 2.2, the Bott-duffin inverse $A_{(L)}^{(-1)}$ and generalized Bott-duffin inverse $A_{(L)}^{(\dagger)}$ can be obtained within at most n^2 iteration steps.

4. Numerical examples

In this section, we will give some numerical examples to illustrate our results. All the tests are performed by MATLAB6.1 and the initial iterative matrices are chosen as $X_0 = P_LA^*P_L$. Because of the influence of the error of roundoff, we regard the matrix P_LAP_L as zero matrix if $\| A \|_L< 10^{-10}$.

Example 3.1. Given matrices A and L as follows.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ L = span\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right\}$$

If we set

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ 0 & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \end{pmatrix},$$

then $r(AU) = r(U^*AU) = 1$ so that A is L-zero by Lemma 2.3. By computing

$$P_L = UU^* = \begin{pmatrix} \frac{17}{18} & \frac{2}{9} & \frac{1}{18} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{1}{18} & -\frac{2}{9} & \frac{17}{18} \end{pmatrix},\ P_LA^*P_L = \frac{1}{81}\begin{pmatrix} \frac{187}{2} & 22 & \frac{11}{2} \\ 2 & 1 & -2 \\ \frac{7}{2} & -14 & \frac{119}{2} \end{pmatrix}.$$

Using Algorithm 3.1 and iterate 3 steps, we have X_3 as follow:

$$X_3 = \begin{pmatrix} 0.57894736842105 & 0.13622291021672 & 0.03405572755418 \\ 0.05263157894737 & 0.01238390092879 & 0.00309597523220 \\ 0.36842105263158 & 0.08668730650155 & 0.02167182662539 \end{pmatrix}$$

with

$$\| R_3 \|_L^2 = \| A - AX_3A \|_L^2 = 9.830326866758750 \times 10^{-32}$$

On other hand, by computing, we obtain that

$$A_{(L)}^{(\dagger)} = \begin{pmatrix} \frac{11}{19} & \frac{44}{323} & \frac{11}{323} \\ \frac{1}{19} & \frac{4}{323} & \frac{1}{323} \\ \frac{7}{19} & \frac{28}{323} & \frac{7}{322} \end{pmatrix}$$

Then from the above data, we can find that the iterative sequence$\{X_k\}$ converges to $A_{(L)}^{(\dagger)}$.

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