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Cyclic contractions and fixed point theorems

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Abstract. In this manuscript, the existence and uniqueness of fixed points of a class of cyclic operators defined on a closed subset of a Banach space is discussed. Fixed point theorems for some contractions from this class are introduced and illustrative examples are given.

1. Introduction and preliminaries

The theory of existence and uniqueness of fixed points has been developing since the work of Banach [2] in 1922 and numerous results have been obtained so far (see e.g. [9]–[6]). Recently, the study of best proximity points of cyclic type contractions has been a subject of considerable interest [11]–[17]. Various types of cyclic contractions acting on complete metric spaces have been defined and studied thoroughly from this point of view [1]-[20].

In 1986 , Nova [15] defined a class $D(a, b)$ of operators acting on a subset of a Banach space and proved the existence and uniqueness of fixed points even if the operator is discontinuous.

Definition 1.1. (See [15]) Let K be a subset of a Banach space X. An operator T defined on K is said to belong to class $D(a, b)$ if

$$\left\|Tx - Ty\right\| \leq a\left\|x - y\right\| + b[\|x - Tx\| + \left\|y - Ty\right\|] \tag{1}$$

for all x and y in K, where $0 \leq a, b \leq 1$.

If an operator T is in class $D(k, 0)$ with $0 < k < 1$, then T is a contraction. with $0 < k < 1$.

2. Main results

In this section we define a class of cyclic operators and investigate the existence and uniqueness of fixed points of these operators. We also give examples of discontinuous cyclic operators with unique fixed points.

Define a class $D(a, b)$ of cyclic operators on a union of closed subsets of a Banach space as follows.

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Definition 2.1. Let K_1 and K_2 be closed subsets of a Banach space X. An operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ is said to belong to class $D(a, b)$ if it satisfies

$$\left\|Tx - Ty\right\| \leq a \left\|x - y\right\| + b[\|x - Tx\| + \left\|y - Ty\right\|] \tag{2}$$

for all $x \in K_1$ and $y \in K_2$, where $0 \leq a, b \leq 1$.

It is clear that if T belongs to the class $D(k, 0)$ with $0 < k < 1$, then T is a cyclic contraction [13].

Example 2.2. Let $K_1 = \left[0, \frac{1}{2}\right]$ and $K_2 = \left[\frac{1}{3}, 1\right]$. Define the operator T as follows:

$$Tx = \begin{cases} \frac{2}{5} & if \quad 0 \leq x \leq \frac{1}{2} \\ \frac{2}{3}(1 - x) & if \quad \frac{1}{2} < x \leq 1 \end{cases}$$

We will show that T is in the class $D\left(\frac{1}{4}, \frac{1}{4}\right)$. Take $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{3}, \frac{1}{2}]$. Then

$$\left\|Tx - Ty\right\| = \left|\frac{2}{5} - \frac{2}{5}\right| = 0.$$

Now let $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$. Then

$$\begin{aligned} \left\|Tx - Ty\right\| &= \left|\tfrac{2}{5} - \tfrac{2}{3} + \tfrac{2}{3}y\right| = \left|\tfrac{2}{3}y - \tfrac{4}{15}\right| \\ &= \left|\tfrac{1}{4}x - \tfrac{1}{4}y + \tfrac{1}{4}x - \tfrac{1}{10} + \tfrac{5}{12}y - \tfrac{1}{6}\right| \\ &\leq \tfrac{1}{4}\left|x - y\right| + \tfrac{1}{4}\left[\left|x - \tfrac{2}{5}\right| + \left|\tfrac{5}{3}y - \tfrac{2}{3}\right|\right] \\ &= \tfrac{1}{4}\left|x - y\right| + \tfrac{1}{4}\left[\left|x - Tx\right| + \left|y - Ty\right|\right] \end{aligned}$$

Observe that T has a unique fixed point $p = \frac{2}{5}$.

In what follows we investigate the existence of fixed points of operators in class $D(a, b)$. The first proposition gives uniqueness conditions of the fixed point of an operator provided that the fixed point exists.

Proposition 2.3. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2) with $0 \leq a < 1$, $0 \leq b \leq 1$. If*

$$F(T) = \{x \in K_1 \cup K_2 : Tx = x\} \neq \emptyset,$$

then $F(T)$ consists of a single point.

Proof. Assume the contrary, that is, let $z, w \in K_1 \cup K_2$ be two distinct fixed points of T. Then

$$\|z - w\| = \|Tz - Tw\| \leq a \|z - w\| + b[\|z - Tz\| + \|w - Tw\|] = a \|z - w\|$$

which is possible only if $z = w$, since $a < 1$. □

Remark 2.4. In the previous proposition, $F(T) \subset K_1 \cap K_2$. Indeed, the case $F(T) = \emptyset$ is trivial. Suppose $F(T) \neq \emptyset$. Take $z \in F(T)$ and without loss of generality take $z \in K_1$. Since $Tz \in K_2$ and $z = Tz$ hence $z \in K_2$.

Next we discuss the problem of existence of fixed points.

Proposition 2.5. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2) with $0 \le a + 2b < 1$. Then $\inf_{x \in K_1 \cup K_2} \|x - Tx\| = 0$.*

Proof. Take an arbitrary point $x_0 \in K_1$ and define the sequence of Picard's iterates $x_{n+1} = Tx_n = T^n x_0$ for all $n = 0, 1, \ldots$. Then we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &= \|Tx_{n+1} - Tx_n\| \\ &\le a\,\|x_{n+1} - x_n\| + b[\|x_{n+1} - Tx_{n+1}\| + \|x_n - Tx_n\|] \\ &= (a+b)\,\|x_{n+1} - x_n\| + b\,\|x_{n+1} - x_{n+2}\|. \end{aligned}$$

This implies $(1-b)\,\|(I-T)x_{n+1}\| \le (a+b)\,\|(I-T)x_n\|$, which results in

$$\|(I-T)x_{n+1}\| \le \left(\frac{a+b}{1-b}\right)^{n+1} \|(I-T)x_0\|.$$

Since $a + b < 1 - b$, then $\dfrac{a+b}{1-b} < 1$ and hence, $\|x_{n+1} - Tx_{n+1}\| \to 0$ as $n \to \infty$, that is, $\inf_{x \in K_1 \cup K_2} \|x - Tx\| = 0$. □

We now define asymptotically regular operators, for details see [15].

Definition 2.6. Let X be a Banach space, T be a mapping of X into itself and x be a point in X. The mapping T is called asymptotically regular in x if $\left\|T^{n+1}x - T^n x\right\| \to 0$ as $n \to \infty$.

The next result gives a condition for a cyclic operator to be asymptotically regular.

Proposition 2.7. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2) with $0 \le a + 2b < 1$. Then T is asymptotically regular at any point $x \in K_1 \cup K_2$.*

Proof. Due to Proposition 2.5 we have

$$\left\|T^n x_0 - T^{n+1} x_0\right\| = \|x_n - x_{n+1}\| = \|(I-T)x_n\| \to 0$$

as $n \to 0$ for an arbitrary $x_0 \in K_1 \cup K_2$ which completes the proof. □

The existence and uniqueness of a fixed point of a cyclic operator of class $D(a, b)$ is discussed in the next theorem which gives the necessary and sufficient conditions for the convergence of a sequence of Picard's iterates to the fixed point of the operator.

Theorem 2.8. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2) with $0 \le a, b < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies*

$$\lim_{n \to \infty} (x_n - Tx_n) = 0$$

if and only if the sequence $\{x_n\}$ converges to the unique fixed point of T.

Proof. We proof the necessity first. Let $x_0 \in K_1$. Define $x_n = Tx_{n-1} = T^n x_0$.Then

$$\|Tx_n - Tx_m\| \le a\,\|x_n - x_m\| + b\,\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}.$$

By the triangle inequality we have

$$\|Tx_n - Tx_m\| \le a\{\|x_n - Tx_n\| + \|Tx_n - Tx_m\| + \|x_m - Tx_m\|\} + b\,\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}.$$

which implies

$$\|Tx_n - Tx_m\| \leq \frac{a+b}{1-a}\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}.$$

Observe that from the hypothesis, the right hand side of the inequality tends to 0, as $n \to \infty$, hence $\{Tx_n\}$ is a Cauchy sequence. Since $K_1 \cup K_2$ is complete, then then it converges to a limit, say $z \in K_1 \cup K_2$, that is,

$$\lim_{n\to\infty} Tx_n = z.$$

Note that the subsequence $\{x_{2n}\} \in K_1$ and the subsequence $\{x_{2n+1}\} \in K_2$. Thus $z \in K_1 \cap K_2 \neq \emptyset$. Then we employ the triangle inequality and the fact that $b < 1$ to get

$$\|z - Tz\| \leq \frac{1+a}{1-b}\|z - x_n\| + \frac{1+b}{1-b}\|x_n - Tx_n\|.$$

It follows from $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty}(Tx_n - x_n) = 0$ that z is the fixed point of T which is unique by the Proposition 2.3.

To prove the sufficiency part, assume that T has a fixed point $z \in K_1 \cup K_2$ such that $\lim_{n\to\infty} x_n = z$ for the sequence $\{x_n\} \in K_1 \cup K_2$. From the triangle inequality it follows that

$$\|Tx_n - x_n\| - \|x_n - z\| \leq \|Tx_n - z\| \leq a\|x_n - z\| + b\|z - x_n - Tx_n\|.$$

This implies

$$(1-b)\|x_n - Tx_n\| \leq (1+a)\|x_n - z\|.$$

Hence, $Tx_n - x_n \to 0$ as $n \to \infty$ which completes the proof. □

Corollary 2.9. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operators $R : K_1 \to K_2$ and $S : K_2 \to K_1$ satisfy*

$$\|Rx - Sy\| \leq a\|x - y\| + b[\|Rx - Sx] + \|Sy - Ry\|, \ x \in K_1, \quad y \in K_2 \tag{3}$$

with $0 \leq a, b < 1$. Then the sequence $\{x_n\}$ in $K_1 \cup K_2$ satisfies

$$\lim_{n\to\infty}(Rx_n - Sx_n) = 0$$

if and only if the sequence $\{x_n\}$ converges to the unique common fixed point of S and R.

Proof. Let

$$Tx = \begin{cases} Rx & \text{if} \quad x \in K_1 \\ Sx & \text{if} \quad x \in K_2 \end{cases}$$

Notice that T is well defined since (3) implies $Rx = Sx$ if $x \in K_1 \cap K_2$. Thus, Theorem 2.8 yields the required result for $T : K_1 \cup K_2 \to K_1 \cup K_2$. □

One can generalize the definition of cyclic operators and state the following corollary which gives conditions for the uniqueness of fixed points of these generalized cyclic operators.

Corollary 2.10. *Let $\{K_i, i = 1, \dots p\}$ be non-empty closed subsets of a Banach space X and let $T : \cup_{i=1}^{p} K_i \to \cup_{i=1}^{p} K_i$ satisfies the following conditions:*

(1) $T(K_i) \subseteq K_{i+1}$ for $1 \leq i \leq p$ and $K_{p+1} = K_1$.

(2) There exists $0 \leq a, b < 1$ *such that*

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|Tx - x\| + \|Ty - y\|\}$$

for all $x \in K_i$, $y \in K_{i+1}$ *and* $1 \leq i \leq p$.

Then T has a unique fixed point.

Proof. It is sufficient to prove that for a given $x \in \cup_{i=1}^{p} K_i$, infinitely many terms of the sequence $T^n x$ lie in each K_i. Thus, $\cap_{i=1}^{p} K_i \neq \emptyset$. Then the operator

$$T : \cap_{i=1}^{p} K_i \to \cap_{i=1}^{p} K_i$$

satisfies the conditions of the Theorem 1 in [15]. □

We generalize the operator defined in Example 1 in the following way:

Example 2.11. Let $K_1 = \left[0, \frac{1}{2}\right]$, $K_2 = \left[\frac{1}{4}, \frac{3}{4}\right]$ and $K_3 = \left[\frac{1}{6}, 1\right]$. Define the operator T as follows:

$$Tx = \begin{cases} \frac{2}{5} & if \quad 0 \leq x \leq \frac{1}{2} \\ \frac{2}{3}(1 - x) & if \quad \frac{1}{2} < x \leq 1 \end{cases}$$

Observe that

$$T(K_1) = \left\{\frac{2}{5}\right\} \subset K_2, T(K_2) = \left[\frac{1}{6}, \frac{2}{5}\right] \subset K_3, T(K_3) = \left[0, \frac{2}{5}\right] \subset K_1$$

We have shown in Example 1 that this operator T is in class $D\left(\frac{1}{4}, \frac{1}{4}\right)$ and has a unique fixed point.

If we impose an additional condition on the operator, more precisely on the constants a and b we get the following theorem.

Theorem 2.12. *Let* K_1 *and* K_2 *be closed subsets of a Banach space X. Suppose that the operator* $T : K_1 \cup K_2 \to K_1 \cup K_2$ *with* $T(K_1) \subset K_2$ *and* $T(K_2) \subset K_1$ *satisfies (2) with* $0 \leq a + 2b < 1$. *Then*

(i) T has a unique fixed point p in $K_1 \cap K_2$.

(ii) $\|Tx - p\| < \|x - p\|$ *for all* $x \in K_1 \cup K_2$ *where p is the fixed point of T.*

Proof. (i) Take a point $x_0 \in K_1$. Define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. Then we have

$$\|x_{n+1} - Tx_{n+1}\| = \|Tx_n - Tx_{n+1}\| \leq a\|x_n - x_{n+1}\| + b\left[\|x_n - Tx_n\| + \|x_{n+1} - Tx_{n+1}\|\right].$$

This inequality implies

$$(1 - b)\|x_{n+1} - Tx_{n+1}\| \leq \left(\frac{a + b}{1 - a}\right)\|x_n - Tx_n\| \leq \ldots \leq \left(\frac{a + b}{1 - a}\right)^n \|x_0 - Tx_0\|.$$

Hence, we obtain $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, implying that $\lim_{n\to\infty} x_n = p$, where p is the fixed point of T. Since the subsequence $\{x_{2n}\} \in K_1$ and the subsequence $\{x_{2n+1}\} \in K_2$, then $p \in K_1 \cap K_2$. The uniqueness follows from Proposition 2.3.

(ii) Let p be the fixed point of T and $x \in K_1 \cup K_2$. Then using (2) and the triangle inequality, we have

$$\begin{aligned} \|Tx - p\| &\leq \|Tx - Tp\| + \|Tp - p\| \\ &\leq a\|x - p\| + b\left[\|x - Tx\| + \|p - Tp\|\right] \\ &\leq a\|x - p\| + b\left[\|x - p\| + \|p - Tx\|\right]. \end{aligned}$$

This inequality implies

$$\left\|Tx - p\right\| \leq \left(\frac{a+b}{1-b}\right)\left\|x - p\right\| < \left\|x - p\right\|$$

as $\frac{a+b}{1-b} < 1$, which completes the proof. □

In our last theorem we give some properties of the set of fixed points of cyclic operators.

Theorem 2.13. *Let K_1 and K_2 be closed subsets of a Banach space X. Suppose that the operator $T : K_1 \cup K_2 \to K_1 \cup K_2$ with $T(K_1) \subset K_2$ and $T(K_2) \subset K_1$ satisfies (2) with $0 \leq a, b < 1$. Let $F(T)$ be the set of fixed points of the operator T. If $F(T) \neq \emptyset$, then $F(T) \subset K_1 \cap K_2$. Moreover, if T satisfies*

$$\left\|Tx - p\right\| \leq \left\|x - p\right\| \tag{4}$$

for every $x \in K_1 \cup K_2$ and $p \in F(T)$, then $F(T)$ is closed.

Proof. The first part of the theorem follows from the fact that T is cyclic. In other words, let $F(T) = \{p \in K_1 \cup K_2 | Tp = p\} \neq \emptyset$ and assume that $F(T)$ is not a subset of $K_1 \cap K_2$. Without loss of generality suppose that there exists $p \in F(T)$ such that $p \in K_1 \backslash K_2$. Since $T(K_1) \subset K_2$, then $Tp \in K_2$, that is $p \in K_2$. However, this contradicts our assumption, thus, $F(T) \subset K_1 \cap K_2$.

Now, let $\{p_n\}$ be a convergent sequence in $F(T)$ which converges to a limit p. Using triangle inequality and (4) we have

$$\left\|Tp - p\right\| \leq \left\|Tp - p_n\right\| + \left\|p_n - p\right\| \leq 2\left\|p_n - p\right\|.$$

Since $\lim_{n\to\infty}\left\|p_n - p\right\| = 0$, then $Tp = p$, that is, $p \in F(T)$. Hence, $F(T)$ is closed. □

Remark 2.14. If the set $K_1 \cup K_2$ in theorem 2.13 is convex, then the set $F(T)$ of fixed points of T is also convex. The proof can be easily done imitating the proof of Theorem 3 in [6].

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