

## An iterative solution to coupled quaternion matrix equations

Caiqin Song<sup>a</sup>, Guoliang Chen<sup>a</sup>, Xiangyun Zhang<sup>a</sup>

<sup>a</sup>Department of Mathematics in East China Normal University, Shanghai, P.R. China

**Abstract.** This note studies the iterative solution to the coupled quaternion matrix equations  $[\sum_{i=1}^p \mathbb{T}_{1i}(X_i), \sum_{i=1}^p \mathbb{T}_{2i}(X_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(X_i)] = [M_1, M_2, \dots, M_p]$ , where  $\mathbb{T}_{si}, s = 1, 2, \dots, p$ ; is a linear operator from  $Q^{m_i \times n_i}$  onto  $Q^{p_s \times q_s}$ ,  $M_s \in Q^{p_s \times q_s}, s = 1, 2, \dots, p. i = 1, 2, \dots, p.$  by making use of a generalization of the classical complex conjugate gradient iterative algorithm. Based on the proposed iterative algorithm, the existence conditions of solution to the above coupled quaternion matrix equations can be determined. When the considered coupled quaternion matrix equations is consistent, it is proven by using a real inner product in quaternion space as a tool that a solution can be obtained within finite iterative steps for any initial quaternion matrices  $[X_1(0), \dots, X_p(0)]$  in the absence of round-off errors and the least Frobenius norm solution can be derived by choosing a special kind of initial quaternion matrices. Furthermore, the optimal approximation solution to a given quaternion matrix can be derived. Finally, a numerical example is given to show the efficiency of the presented iterative method.

### 1. Introduction

Throughout this paper, we need the following notations. We denote the real number field by  $R$ , the complex number field by  $C$  and the quaternion field by  $Q = R \oplus Ri \oplus Rj \oplus Rk$ ,  $i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$ , respectively. For a matrix  $A \in Q^{m \times n}$ , we denote its transpose, conjugate transpose, trace and Frobenius norm by  $A^T, A^H, \text{tr}(A)$  and  $\|A\|$ , respectively. The symbols  $R(A)$  and  $\text{vec}(\cdot)$  stand for null space and the vec operator, i.e., for  $A = (a_1, a_2, \dots, a_n) \in C^{m \times n}$ , where  $a_i (i = 1, 2, \dots, n)$  denotes the  $i$ th column of  $A$ ,  $\text{vec}(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$ .

Let  $LQ^{m \times n, p \times q}$  denote the set of quaternion linear operators from  $Q^{m \times n}$  onto  $Q^{p \times q}$ . Particularly, when  $p = m$  and  $q = n$ ,  $LQ^{m \times n, p \times q}$  is denoted by  $LQ^{m \times n}$ . Let  $\mathcal{I}$  stand for the identity operator on  $Q^{m \times n}$ . In the vector space  $Q^{m \times n}$ , we define real inner product as:  $\langle A, B \rangle = \text{Re}[\text{tr}(A^H B)]$ , for all  $A, B \in Q^{m \times n}$ . Also we have  $\langle A, B \rangle = \langle \bar{A}, \bar{B} \rangle$  and  $\langle A, BC \rangle = \langle B^H A, C \rangle = \langle AC^H, B \rangle$ . The adjoint of  $\mathbb{T} \in Q^{m \times n}$  is denoted by  $\mathbb{T}^*$ . So for all  $X, Y \in Q^{m \times n}$ ,  $\langle \mathbb{T}(X), Y \rangle = \langle X, \mathbb{T}^*(Y) \rangle$ .  $\mathbb{T}$  is called self-adjoint if  $\mathbb{T}^* = \mathbb{T}$ . Two quaternion matrices  $X$  and  $Y$  are said to be orthogonal if  $\langle X, Y \rangle = 0$ .

In the 20th century, quaternion, discovered by Hamilton in 1843, made further appearance in associative algebra, analysis, topology, and physics. Moreover, quaternion matrices play an important role in computer science, quantum physics, signal and color image processing, and so on (e.g. [2, 3, 13, 18, 21, 25, 26, 42]). Some

---

2010 *Mathematics Subject Classification.* Primary 15A06; Secondary 15A24, 47S10

*Keywords.* Coupled quaternion matrix equations, quaternion linear operator, real inner product, least Frobenius norm solution, optimal approximation solution

Received: 28 December 2011; Accepted: 22 April 2012

Communicated by Y. Wei

This project is granted financial support from NSFC (No.11071079, No.10901056), Zhejiang Nature Science Foundation (No.Y6110043) and the Fundamental Research Funds for the Central Universities

*Email address:* glchen@math.ecnu.edu.cn (Guoliang Chen)

important conclusions which hold on the complex field or real field have been generalized to quaternion field, such as the Schur’s theorem [30], Cayley-Hamilton’s theorem [6], the Wilandt-Hoffman’s theorem [37] and Gerschgorin’s theorem [38]. Notice that quaternion algorithm has, up to now, only been proposed for QR decomposition [14], Jacobi algorithm [4] and the Singular Value Decomposition (SVD) [3]. So in this paper, we’ll generalize the conjugate gradient iterative algorithm of complex matrix on the quaternion field.

For the quaternion matrix equation, there are many important results. Jiang and Wei [17] investigate the solution of the quaternion matrix equation  $X - A\bar{X}B = C$  by using real representation of a quaternion matrix. By making use of complex representation of a quaternion matrix, Huang et al. [16] give the solution of quaternion matrix equation  $AXB - CXD = E$ . Over the real quaternion algebra, Wang [31, 32] considered bisymmetric and centro-symmetric solution to certain matrix equations and gave the general solution to the system of quaternion matrix equations  $A_1X = C_1, A_2X = C_2, A_3X = C_3$  and  $A_4X = C_4$ . In addition, there are the following matrix equation results over complex field. Iterative methods in [24, 27, 33] were constructed to obtain the (skew)symmetric solutions of the matrix equations  $A_1XB_1 = C_1, A_2XB_2 = C_2, AXB + CYD = E$  and  $AXB + CX^TD = E$ . In [34], the authors presented the necessary and sufficient conditions for the existence of constant solutions with bi(skew)symmetric constraint to the matrix equations  $A_iX - YB_i = C_i, i = 1, 2, \dots, s$  and  $A_iXB_i - C_iYD_i = E_i, i = 1, 2, \dots, s$ .

Coupled matrix equations have wide applications in several areas, such as stability theory, control theory, perturbation analysis, and some others fields of pure and applied mathematics. For example, in stability analysis of linear jump systems with Markovian transitions, the coupled Lyapunov matrix equations

$$A_i^T P_i + P_i A_i + Q_i + \sum_{j=1}^N p_{ij} P_j = 0, \quad Q_i > 0, \quad i \in I[1, N] \tag{1}$$

and

$$P_i = A_i^T \left( \sum_{j=1}^N p_{ij} P_j \right) + Q_i, \quad Q_i > 0, \quad i \in I[1, N] \tag{2}$$

are often encountered [5, 20], where  $P_i, i \in I[1, N]$  are the matrices to be determined. Due to their wide applications, coupled matrix equations have attracted considerable attention from many researchers. It was pointed out that in [7] that the existence of a positive definite solution to the discrete-time Markovian jump Lyapunov (DMJL) equation (2) is related to the spectral radius of an augmented matrix being less than one.

In addition, the following general coupled Sylvester matrix equations have been investigated

$$A_{i1} X_1 B_{i1} + A_{i2} X_2 B_{i2} + \dots + A_{ip} X_p B_{ip} = E_i, \quad i \in [1, N], \tag{3}$$

where  $A_{ij} \in R^{r_i \times n_j}, B_{ij} \in R^{m_i \times c_j}, E_i \in R^{r_i \times c_i}, i \in [1, N], j \in [1, p]$  are known matrices, and  $X_j \in R^{n_j \times m_j}, j \in [1, p]$  are the matrices to be determined. This kind of matrix equations include all the aforementioned matrix equations as special cases. When  $p = N$ , by using the hierarchical identification principle, iterative algorithm were proposed in [8] for obtaining the unique least-square solution by introducing the block-matrix inner product. Recently, from an optimization point of view gradient based iterations were constructed in [9] to solve the general coupled matrix equations (3). The gradient based iterative algorithm [10–12, 41] and least squares based iterative algorithm [41] for solving (coupled) matrix equations are a novel and computationally efficient numerical algorithms and were presented based on the hierarchical identification principle which regards the unknown matrix as the system parameter matrix to be identified. Meanwhile, Wu et al. [39] have considered the following so-called coupled Sylvester-conjugate matrix by means of the hierarchical identification principle

$$\sum_{\eta=1}^p (A_{i\eta} X_{\eta} B_{i\eta} + C_{i\eta} \bar{X}_{\eta} D_{i\eta}) = F_i, \quad i \in I[1, N], \tag{4}$$

where  $A_{i\eta}, C_{i\eta} \in C^{m_i \times r_\eta}, B_{i\eta}, D_{i\eta} \in C^{s_\eta \times n_i}, F_i \in C^{m_i \times n_i}, i \in I[1, N], \eta \in I[1, p]$  are the given known matrices, and  $X_\eta \in C^{r_\eta \times s_\eta}, \eta \in I[1, p]$  are the matrices to be determined. At the same time, Wu et al. [40] proposed a finite iterative method for the so-called Sylvester-conjugate matrix equation (4). In [19, 41], the following linear equations

$$\sum_{i=1}^r A_i X B_i + \sum_{j=1}^s C_j X^T D_j = E, \tag{5}$$

where  $A_i, B_i, C_j, D_j, i = 1, \dots, r; j = 1, \dots, s$  and  $E$  are some known constant matrices of appropriate dimensions and  $X$  is a matrix to be determined, was considered. In [35], the special case of equation (5)  $AXB + CX^T D = E$  was considered by the iterative algorithm. A more special case of (5), namely, the matrix equation  $AX + X^T B = C$ , was investigated by Piao et al. [22]. The Moore-Penrose generalized inverse was used in [22] to find explicit solutions to this matrix equation.

However, it is not considered how to computer the iterative solution of the above matrix equation over quaternion field. So in this paper, we extend the conjugate graduate iterative algorithm from complex field to the quaternion field and give the iterative solution of the following quaternion matrix equations based on the complex representation of a quaternion matrix. It should be remarked that all the aforementioned matrix equations can all be rewritten as the following quaternion matrix equations system:

$$\left[ \sum_{i=1}^p \mathbb{T}_{1i}(X_i), \sum_{i=1}^p \mathbb{T}_{2i}(X_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(X_i) \right] = [M_1, M_2, \dots, M_p],$$

where  $\mathbb{T}_{si} \in LQ^{m_i \times n_i, p_s \times q_s}$  and  $M_s \in Q^{p_s \times q_s}, i = 1, 2, \dots, p; s = 1, 2, \dots, p$ . In the present paper, we mainly consider the following two problems:

**Problem I.** For given  $\mathbb{T}_{si} \in LQ^{m_i \times n_i, p_s \times q_s}, M_s \in Q^{p_s \times q_s}$ , find  $X_i, i = 1, 2, \dots, p$  such that

$$\left[ \sum_{i=1}^p \mathbb{T}_{1i}(X_i), \sum_{i=1}^p \mathbb{T}_{2i}(X_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(X_i) \right] = [M_1, M_2, \dots, M_p]. \tag{6}$$

**Problem II.** Let  $S$  denote the solution set of Problem I, for given  $\widehat{X}_i \in Q^{m_i \times n_i}$ , find  $\overline{X}_i \in S, i = 1, 2, \dots, p$ , such that

$$\sum_{i=1}^p \|\overline{X}_i - \widehat{X}_i\|^2 = \min_{(X_1, \dots, X_p) \in S} \left\{ \sum_{i=1}^p \|X_i - \widehat{X}_i\|^2 \right\}. \tag{7}$$

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we propose an iterative algorithm to obtain a solution or the least Frobenius norm solution of Problem I and present some basic properties of the algorithm. Some numerical examples are given in Section 4 to show the efficiency of the proposed iterative method. Finally, we put some conclusions in Section 5.

## 2. Preliminary

In this section, some concepts on quaternion matrix are given. A new real inner product is firstly defined for quaternion matrix space over the real field  $R$ . This inner product will play a very important role in this paper.

### 2.1. Complex representation of a quaternion matrix

Some well-known equalities for complex and real matrices hold for quaternion matrices, whereas some others are no more valid. Here is a short list of relations that hold for quaternion matrices  $A \in Q^{N \times M}$  and  $B \in Q^{M \times P}$ :

- (1)  $(AB)^H = B^H A^H$ ;
- (2)  $\overline{AB} \neq \overline{A} \overline{B}$  in general;
- (3)  $(A^H)^{-1} = (A^{-1})^H$ ;

(4)  $(\overline{A})^{-1} \neq \overline{A^{-1}}$  in general.

See [42] for a more complete list. The most common way to study quaternion matrix is to use its complex representation of a quaternion matrix introduced in [42]. For any quaternion matrix  $A = B_1 + B_2j \in Q^{m \times n}$ , the author [42] defined a complex representation

$$A_\sigma = \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix} \in C^{2m \times 2n}. \tag{8}$$

The complex matrix  $A_\sigma$  was called complex representation of a quaternion matrix  $A$ . The complex representation of a quaternion matrix has the following results, which can be found in [1].

- Proposition 2.1.** (1) If  $A, B \in Q^{m \times n}, a \in R$ , then  $(A + B)_\sigma = A_\sigma + B_\sigma, (aA)_\sigma = aA_\sigma$ ;  
 (2) If  $A \in Q^{m \times n}, B \in Q^{n \times s}$ , then  $(AB)_\sigma = A_\sigma B_\sigma, (A^H)_\sigma = (A_\sigma)^H$ ;  
 (3) If  $A \in Q^{m \times m}$ , then  $A$  is nonsingular if and only if  $A_\sigma$  is nonsingular, and  $(A_\sigma)^{-1} = (A^{-1})_\sigma$ ;  
 (4) If  $A \in Q^{m \times m}$ , then  $A$  is unitary if and only if  $A_\sigma$  is unitary;  
 (5)  $\overline{A}_\sigma = Q_m^{-1} A_\sigma Q_n$ , in which  $Q_t = \begin{bmatrix} 0 & I_t \\ -I_t & 0 \end{bmatrix}$  is a unitary matrix,  $I_t$  is a  $t \times t$  identity matrix.

From Proposition 2.1, we know that  $\sigma : Q^{m \times n} \rightarrow \sigma(Q^{m \times n})$  is an isomorphism of vector space, and  $\sigma : Q^{m \times n} \rightarrow \sigma(Q^{m \times n})$  is an isomorphism of algebra.

2.2. Quaternion matrix norm

**Definition 2.2.** A function  $v : A \in Q^{m \times n} \rightarrow R$  is a quaternion matrix norm on  $Q^{m \times n}$  if it satisfies the following conditions:

- (1) definiteness,  $A \neq 0 \implies v(A) > 0$ ;
- (2) homogeneity,  $v(aA) = |a|v(A)$ ;
- (3) the triangle inequality,  $v(A + B) \leq v(A) + v(B)$ ,

where  $A, B \in Q^{m \times n}$  are arbitrary quaternion matrices, and  $a$  is an arbitrary quaternion.

For  $A = A_1 + A_2i + A_3j + A_4k, A_t \in R^{m \times n}, t = 1, \dots, 4$ , we define

$$v_1(A) = \sqrt{\|A_1\|_F^2 + \|A_2\|_F^2 + \|A_3\|_F^2 + \|A_4\|_F^2} = \frac{1}{\sqrt{2}} \|A_\sigma\|_F.$$

Obviously,  $v_1$  satisfies (1) and (3).

Let  $\alpha = \alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k \in Q, \alpha_l \in R, (l = 1, 2, 3, 4)$ . Then by carefully computation, we obtain that

$$\begin{aligned} \alpha A &= (\alpha_1A_1 - \alpha_2A_2 - \alpha_3A_3 - \alpha_4A_4) + (\alpha_1A_2 + \alpha_2A_1 + \alpha_3A_4 - \alpha_4A_3)i + (\alpha_1A_3 - \alpha_2A_4 + \alpha_3A_1 + \alpha_4A_2)j \\ &\quad + (\alpha_1A_4 + \alpha_2A_3 - \alpha_3A_2 + \alpha_4A_1)k, \end{aligned}$$

and  $v_1^2(\alpha A) = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)(\|A_1\|_F^2 + \|A_2\|_F^2 + \|A_3\|_F^2 + \|A_4\|_F^2) = |\alpha|^2 v_1^2(A)$ .

Similarly,  $v_1(\alpha A) = |\alpha|v_1(A)$ , that is,  $v_1$  also satisfy (2), so  $v_1$  is a quaternion matrix norm, denoted by  $\|\cdot\|_{(F)}$ , which has the following properties:

- (1)  $\|A\|_{(F)} = \frac{1}{\sqrt{2}} \|A_\sigma\|_F$ ;
- (2)  $\|AB\|_{(F)} \leq \|A\|_{(F)} \|B\|_{(F)}$ ;
- (3) Unitary invariant norm;
- (4)  $\|A\|_{(F)} = \sqrt{\sum \sigma_i^2(A)}$ , where  $\sigma_i(A)$ s are singular value of  $A$ ;
- (5)  $\|A\|_{(F)} = \sqrt{\text{trace}(A^H A)}$ ;
- (6)  $\|A\|_{(F)} = \sqrt{\sum |a_{ij}|^2}$ .

Therefore, it is a natural generality of Frobenius norm of complex matrix.

2.3. Real inner product of Hilbert quaternion matrix space

**Definition 2.3.** A real inner product space is a vector space  $V$  over the real field  $R$  together with an inner product defined by a real-valued function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow R$ , satisfying the following three axioms for all vectors  $x, y \in V$  and all scalars  $a \in R$ :

- (1) Symmetry:  $\langle x, y \rangle = \langle y, x \rangle$ ;
- (2) Linearity in the first argument:  $\langle ax, y \rangle = a\langle x, y \rangle, \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ ;
- (3) Positive-definiteness:  $\langle x, x \rangle > 0$  for all  $x \neq 0$ .

Two vectors  $u, v \in V$  are said to be orthogonal if  $\langle u, v \rangle = 0$ .

The following theorem defines a real inner product on quaternion space  $Q^{m \times n}$  over the real field  $R$ .

**Theorem 2.4.** In the quaternion Hilbert space  $Q^{m \times n}$  over the real field  $R$ , an inner product can be defined as

$$\langle A, B \rangle = \text{Re}[\text{tr}(A^H B)], \tag{9}$$

for  $A, B \in Q^{m \times n}$ . This inner product space is denoted as  $(Q^{m \times n}, R, \langle \cdot, \cdot \rangle)$ .

*Proof.* (1) For  $A, B \in Q^{m \times n}$ , according to the properties of the trace, we have

$$\langle A, B \rangle = \text{Re}[\text{tr}(A^H B)] = \frac{1}{2} \text{Re}[\text{tr}((A_\sigma)^H B_\sigma)] = \frac{1}{2} \text{Re}[\text{tr}((B_\sigma)^H A_\sigma)] = \text{Re}[\text{tr}(B^H A)] = \langle B, A \rangle.$$

(2) For  $a \in R$ , we have

$$\langle aA, B \rangle = \text{Re}[\text{tr}((aA)^H B)] = \text{Re}[\text{tr}(aA^H B)] = a \text{Re}[\text{tr}(A^H B)] = a\langle A, B \rangle,$$

$$\langle A + B, C \rangle = \text{Re}[\text{tr}((A + B)^H C)] = \text{Re}[\text{tr}(A^H C)] + \text{Re}[\text{tr}(B^H C)] = \langle A, C \rangle + \langle B, C \rangle.$$

(3) It is well known that  $\text{tr}(A^H A) > 0$  for all  $A \neq 0$ . Thus,  $\langle A, A \rangle = \text{Re}(\text{tr}(A^H A)) > 0$  for all  $A \neq 0$ .

According to Definition 2.3, all the above arguments reveal that the space  $Q^{m \times n}$  over real field  $R$  with the inner product defined in (9) is an inner product space. This completes the proof.  $\square$

2.4. Relationship between quaternion matrix equations and its complex representation matrix equations

The complex representation matrix equations of quaternion matrix equations (6) can be stated as the following matrix equations

$$\left[ \sum_{i=1}^p \mathbb{B}_{1i}(X_i)_\sigma, \sum_{i=1}^p \mathbb{B}_{2i}(X_i)_\sigma, \dots, \sum_{i=1}^p \mathbb{B}_{pi}(X_i)_\sigma \right] = [(M_1)_\sigma, (M_2)_\sigma, \dots, (M_p)_\sigma], \tag{10}$$

where  $\mathbb{B}_{si} \in LC^{2m_i \times 2n_i, 2p_s \times 2q_s}$  and  $(M_s)_\sigma \in C^{2p_s \times 2q_s}, i = 1, 2, \dots, p; s = 1, 2, \dots, p$ .

Meanwhile, we define the following matrix equations

$$\left[ \sum_{i=1}^p \mathbb{B}_{1i}(Y_i), \sum_{i=1}^p \mathbb{B}_{2i}(Y_i), \dots, \sum_{i=1}^p \mathbb{B}_{pi}(Y_i) \right] = [(M_1)_\sigma, (M_2)_\sigma, \dots, (M_p)_\sigma], \tag{11}$$

where  $\mathbb{B}_{si} \in LC^{2m_i \times 2n_i, 2p_s \times 2q_s}$  and  $(M_s)_\sigma \in C^{2p_s \times 2q_s}, s = 1, 2, \dots, p$ .

So we have the following relationship between the solutions  $Y_i$  to the matrix equations (11) and the solutions  $X_i$  to the matrix equations (6). It can be stated as the following Theorem 2.5.

**Theorem 2.5.** The quaternion matrix equations (6) has a solution  $X_i \in Q^{m_i \times n_i}$  if and only if the complex representation matrix equations (11) has a solution  $Y_i \in C^{2m_i \times 2n_i}$ , in which case, if  $Y_i$  is a solution to (11), then the following matrices are a solution to the quaternion matrix equations (6):

$$X_i = \frac{1}{4} \begin{bmatrix} I_n & -jI_n \end{bmatrix} (Y_i + Q_n \bar{Y}_i Q_p^{-1}) \begin{bmatrix} I_p \\ jI_p \end{bmatrix}, i = 1, 2, \dots, p. \tag{12}$$

*Proof.* We only prove that if matrix

$$Y_i = \begin{bmatrix} Y_i^{11} & Y_i^{12} \\ Y_i^{21} & Y_i^{22} \end{bmatrix}, Y_i \in C^{2m_i \times 2n_i}, i = 1, 2, \dots, p. \tag{13}$$

is a solution to the complex representation matrix equations (11), then  $X_i$  in (12) is a solution to the quaternion matrix equations (6).

By (5) of Proposition 2.1, we have  $A_\sigma = Q_m \overline{A_\sigma} Q_n^{-1}$ , i.e. if  $Y_i$  is a solution to (11), then  $Q_n \overline{Y_i} Q_p^{-1}$  is also a solution to (11). Thus the following matrix

$$\hat{Y}_i = \frac{1}{2}(Y_i + Q_n \overline{Y_i} Q_p^{-1}), \tag{14}$$

is also a solution to (11).

It is easy to get by direct calculation

$$\hat{Y}_i = \begin{pmatrix} \hat{Y}_i^{(1)} & \hat{Y}_i^{(2)} \\ -\hat{Y}_i^{(2)} & \hat{Y}_i^{(1)} \end{pmatrix}, \tag{15}$$

where

$$\hat{Y}_i^{(1)} = \frac{1}{2}(Y_i^{11} + \overline{Y_i}^{22}), \hat{Y}_i^{(2)} = \frac{1}{2}(Y_i^{12} - \overline{Y_i}^{21}). \tag{16}$$

From (15) we construct a quaternion matrix

$$X_i = \hat{Y}_i^{(1)} + \hat{Y}_i^{(2)}j = \frac{1}{2} \begin{bmatrix} I_n & -jI_n \end{bmatrix} \hat{Y}_i \begin{bmatrix} I_p \\ jI_p \end{bmatrix}. \tag{17}$$

Clearly  $(X_i)_\sigma = \hat{Y}_i$ . So  $X_i$  is a solution to (6).  $\square$

### 3. Main result

In this section, we first propose an iterative algorithm to solve Problem I, then present some basic properties of the algorithm. We also consider finding the least Frobenius norm solution of Problem I. In the sequel, the least norm solution always means the least Frobenius norm solution.

#### Algorithm 1.

Step 1. Input  $\mathbb{T}_{si} \in LQ^{m_i \times n_i, p_s \times q_s}, M_s \in Q^{p_s \times q_s}, i, s = 1, 2, \dots, p$ , and arbitrary  $[X_1(0), \dots, X_p(0)] \in \mathbb{S}, i = 1, 2, \dots, p; s = 1, 2, \dots, p$ ;

Step 2. Compute

$$R(1) = \text{diag}((M_1 - \sum_{i=1}^p \mathbb{T}_{1i}(X_i(1))), M_2 - \sum_{i=1}^p \mathbb{T}_{2i}(X_i(1)), \dots, M_p - \sum_{i=1}^p \mathbb{T}_{pi}(X_i(1))) = \text{diag}(R_1(1), R_2(1), \dots, R_p(1)),$$

$$P_i(1) = \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(1)), i = 1, 2, \dots, p; k := 0;$$

Step 3. If  $R(k) = 0$  then stop and  $(X_1(k), X_2(k), \dots, X_p(k))$  is the constraint solution group;

else if  $R(k) \neq 0$  but  $P_m(k) = 0$  for all  $m = 1, 2, \dots, p$ , then stop and the coupled quaternion matrix equations (6) are not consistent;

else  $k := k + 1; \sum_{i=1}^p \|R_i(k)\|_{(F)}^2 \neq 0$  but  $P_k = 0$ , then stop; else,  $k := k + 1$ ;

Step 4. Compute

$$\begin{aligned}
 X_i(k) &= X_i(k-1) + \frac{\|R(k-1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(k-1)\|_{(F)}^2} P_i(k-1); \\
 R(k) &= \text{diag} \left( M_1 - \sum_{i=1}^p \mathbb{T}_{1i}(X_i(k)), M_2 - \sum_{i=1}^p \mathbb{T}_{2i}(X_i(k)), \dots, M_p - \sum_{i=1}^p \mathbb{T}_{pi}(X_i(k)) \right) \\
 &= R(k-1) - \frac{\|R(k-1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(k-1)\|_{(F)}^2} \text{diag} \left( \sum_{i=1}^p \mathbb{T}_{1i}(P_i(k-1)), \sum_{i=1}^p \mathbb{T}_{2i}(P_i(k-1)), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(P_i(k-1)) \right) \\
 &= \text{diag} (R_1(k), R_2(k), \dots, R_p(k)), \\
 Z_i(k) &= \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(k)), \\
 P_i(k) &= Z_i(k) + \frac{\|R(k)\|_{(F)}^2}{\|R(k-1)\|_{(F)}^2} P_i(k-1) = \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(k)) + \frac{\|R(k)\|_{(F)}^2}{\|R(k-1)\|_{(F)}^2} P_i(k-1), i = 1, 2, \dots, p;
 \end{aligned}$$

Step 5. Go to Step 3.

Some basic properties of Algorithm 1 are listed in the following lemmas.

**Lemma 3.1.** For the sequences  $\{R_i(k)\}$ ,  $\{P_i(k)\}$  and  $\{Z_i(k)\}$ ,  $i = 1, 2, \dots, p$  generated by Algorithm 1, they follow that

$$\sum_{i=1}^p \langle R_i(m+1), R_i(n) \rangle = \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), Z_i(n) \rangle, \quad m, n = 1, \dots, k; m \neq n. \quad (18)$$

*Proof.* By Algorithm 1, we have

$$\begin{aligned}
 \sum_{i=1}^p \langle R_i(m+1), R_i(n) \rangle &= \sum_{i=1}^p \left\langle R_i(m) - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{t=1}^p \mathbb{T}_{it}(P_t(m)), R_i(n) \right\rangle \\
 &= \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \sum_{t=1}^p \langle \mathbb{T}_{it}(P_t(m)), R_i(n) \rangle \\
 &= \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \sum_{t=1}^p \langle P_t(m), \mathbb{T}_{it}^*(R_i(n)) \rangle \\
 &= \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{t=1}^p \langle P_t(m), Z_t(n) \rangle \\
 &= \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), Z_i(n) \rangle.
 \end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.2.** Assume that Problem I is consistent. If there exists a positive integer  $m$  such that  $\|R_i(m)\|_{(F)}^2 \neq 0$  for all  $m = 1, 2, \dots, k$ , then the sequences  $\{X_i(m)\}$ ,  $\{R_i(m)\}$  and  $\{P_i(m)\}$  generated by Algorithm 1 satisfy

$$\sum_{i=1}^p \langle R_i(m), R_i(n) \rangle = 0 \quad \text{and} \quad \sum_{i=1}^p \langle P_i(m), P_i(n) \rangle = 0, \quad (m, n = 1, 2, \dots, k, m \neq n). \tag{19}$$

*Proof.* Note that  $\langle A, B \rangle = \langle B, A \rangle$  holds for arbitrary quaternion matrices  $A$  and  $B$ . We only need to prove the conclusion holds for all  $0 \leq n < m \leq k$ . For  $m = 1$ , by Lemma 3.1, we have

$$\begin{aligned} \sum_{i=1}^p \langle R_i(2), R_i(1) \rangle &= \sum_{i=1}^p \langle R_i(1), R_i(1) \rangle - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(1), Z_i(1) \rangle \\ &= \sum_{i=1}^p \|R_i(1)\|_{(F)}^2 - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(1), P_i(1) \rangle = 0, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^p \langle P_i(2), P_i(1) \rangle &= \sum_{i=1}^p \left\langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(2)) + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} P_i(1), P_i(1) \right\rangle \\ &= \sum_{i=1}^p \left\langle R_s(2), \sum_{s=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle + \sum_{i=1}^p \left\langle \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} P_i(1), P_i(1) \right\rangle \\ &= \sum_{s=1}^p \left\langle R_s(1) - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)), \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 \\ &= \sum_{s=1}^p \left\langle R_s(1), \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{s=1}^p \left\langle \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)), \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 \\ &= \sum_{s=1}^p \left\langle R_s(1), \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{s=1}^p \left\langle \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)), \sum_{i=1}^p \mathbb{T}_{si}(P_i(1)) \right\rangle + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 \\ &= \sum_{s=1}^p \left\langle R_s(1), (R_i(1) - R_i(2)) \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \right\rangle \\ &\quad - \frac{\|R(1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2} \sum_{s=1}^p \left\langle \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} (R_i(1) - R_i(2)), \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} (R_i(1) - R_i(2)) \right\rangle + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 \\ &= \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \langle R_i(1), R_i(1) - R_i(2) \rangle - \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \langle R_i(1) - R_i(2), R_i(1) - R_i(2) \rangle + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 \\ &= \sum_{t=1}^p \|P_t(1)\|_{(F)}^2 - \frac{\sum_{t=1}^p \|P_t(1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} (\|R(1)\|_{(F)}^2 + \|R(2)\|_{(F)}^2) + \frac{\|R(2)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \|P_i(1)\|_{(F)}^2 = 0. \end{aligned}$$

Assume that  $\sum_{i=1}^p \langle R_i(m), R_i(n) \rangle = 0$  and  $\sum_{i=1}^p \langle P_i(m), P_i(n) \rangle = 0$  for all  $0 \leq n < m, 0 < m \leq k$ , we shall show that  $\sum_{i=1}^p \langle R_i(m+1), R_i(n) \rangle = 0$  and  $\sum_{i=1}^p \langle P_i(m+1), P_i(n) \rangle = 0$  hold for all  $0 \leq n < m+1, 0 < m+1 \leq k$ .

By the hypothesis and Lemma 3.1, for the case where  $0 \leq n < m$ , we have

$$\begin{aligned} \sum_{i=1}^p \langle R_i(m+1), R_i(n) \rangle &= \sum_{i=1}^p \langle R_i(m), R_i(n) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \langle P_i(m), Z_i(n) \rangle \\ &= -\frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \left\langle P_i(m), P_i(n) - \frac{\|R(n)\|_{(F)}^2}{\|R(n-1)\|_{(F)}^2} P_i(n-1) \right\rangle \\ &= -\frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(n) \rangle + \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \frac{\|R(n)\|_{(F)}^2}{\|R(n-1)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(n-1) \rangle = 0, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^p \langle P_i(m+1), P_i(n) \rangle &= \sum_{i=1}^p \langle Z_i(m+1) + \frac{\|R(m+1)\|_{(F)}^2}{\|R(m)\|_{(F)}^2} P_i(m), P_i(n) \rangle \\ &= \sum_{i=1}^p \langle Z_i(m+1), P_i(n) \rangle + \frac{\|R(m+1)\|_{(F)}^2}{\|R(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(n) \rangle \\ &= \sum_{i=1}^p \langle Z_i(m+1), P_i(n) \rangle \\ &= \frac{\sum_{i=1}^p \langle R_i(m+1), R_i(n) \rangle - \sum_{i=1}^p \langle R_i(m+1), R_i(n+1) \rangle}{\frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2}} = 0. \end{aligned}$$

For the case  $n = m$ , we have

$$\begin{aligned} \sum_{i=1}^p \langle R_i(m+1), R_i(m) \rangle &= \sum_{i=1}^p \langle R_i(m), R_i(m) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), Z_i(m) \rangle \\ &= \sum_{i=1}^p \|R_i(m)\|_{(F)}^2 - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \left\langle P_i(m), P_i(m) - \frac{\|R(m)\|_{(F)}^2}{\|R(m-1)\|_{(F)}^2} P_i(m-1) \right\rangle \\ &= \|R(m)\|_{(F)}^2 - \|R(m)\|_{(F)}^2 + \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \frac{\|R(m)\|_{(F)}^2}{\|R(m-1)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(m-1) \rangle = 0, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^p \langle P_i(m+1), P_i(m) \rangle &= \sum_{i=1}^p \langle Z_i(m+1) + \frac{\|R(m+1)\|_{(F)}^2}{\|R(m)\|_{(F)}^2} P_i(m), P_i(m) \rangle \\ &= \sum_{i=1}^p \langle Z_i(m+1), P_i(m) \rangle + \frac{\|R(m+1)\|_{(F)}^2}{\|R(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(m) \rangle \\ &= \frac{\sum_{i=1}^p \langle R_i(m+1), R_i(m) \rangle - \sum_{i=1}^p \langle R_i(m+1), R_i(m+1) \rangle}{\|R(m)\|_{(F)}^2} + \frac{\|R(m+1)\|_{(F)}^2}{\|R(m)\|_{(F)}^2} \sum_{i=1}^p \|P_i(m)\|_{(F)}^2 = 0. \end{aligned}$$

Then by the principle of induction, we complete the proof.  $\square$

**Lemma 3.3.** Assume that  $X^* = [X_1^*, X_2^*, \dots, X_p^*]$  is a solution of Problem I. Let  $X(k) = [X_1(k), X_2(k), \dots, X_p(k)]$ , then the sequences  $\{X_i(k)\}, \{R(k)\}$  and  $\{P(k)\}$  generated by Algorithm 1 satisfy the following equality:

$$\sum_{i=1}^p \langle P_i(n), X_i^* - X_i(m) \rangle = \|R(n)\|_{(F)}^2, \quad n \geq m, \tag{20}$$

$$\sum_{i=1}^p \langle P_i(n), X_i^* - X_i(m) \rangle = 0, \quad n < m. \tag{21}$$

*Proof.* By induction we first demonstrate the conclusion

$$\sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m) \rangle = \|R(m)\|_{(F)}^2; \quad m = 1, 2, \dots.$$

For  $m = 1$ , by Algorithm 1 we have

$$\begin{aligned} \sum_{i=1}^p \langle P_i(1), X_i^* - X_i(1) \rangle &= \sum_{i=1}^p \langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(1)), X_i^* - X_i(1) \rangle \\ &= \sum_{i=1}^p \langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_i(1)), X_i^* - X_i(1) \rangle \\ &= \sum_{i=1}^p R_i(1), \sum_{s=1}^p \mathbb{T}_{si}(X_i^* - X_i(1)) \\ &= \sum_{s=1}^p \langle R_s(1), M_s - \sum_{i=1}^p \mathbb{T}_{si}(X_1) \rangle \\ &= \sum_{s=1}^p \langle R_s(1), R_s(1) \rangle \\ &= \sum_{s=1}^p \|R_s(1)\|_{(F)}^2 \\ &= \|R(1)\|_{(F)}^2. \end{aligned}$$

Assume that the conclusion holds for  $1 \leq m \leq u$ . For  $m = u + 1$ , we have

$$\begin{aligned} \sum_{i=1}^p \langle P_i(u+1), X_i^* - X_i(u+1) \rangle &= \sum_{i=1}^p \left\langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(u+1)) + \frac{\|R(u+1)\|_{(F)}^2}{\|R(u)\|_{(F)}^2} P_i(u), X_i^* - X_i(u+1) \right\rangle \\ &= \sum_{i=1}^p \left\langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(u+1)), X_i^* - X_i(u+1) \right\rangle + \frac{\|R(u+1)\|_{(F)}^2}{\|R(u)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(u), X_i^* - X_i(u+1) \rangle \\ &= \sum_{s=1}^p \langle R_s(u+1), \sum_{i=1}^p \mathbb{T}_{si}(X_i^* - X_i(u+1)) \rangle + \frac{\|R(u+1)\|_{(F)}^2}{\|R(u)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(u), X_i^* - X_i(u+1) \rangle \\ &= \sum_{s=1}^p \langle R_s(u+1), \sum_{i=1}^p \mathbb{T}_{si}(X_i^* - X_i(u+1)) \rangle + \frac{\|R(u+1)\|_{(F)}^2}{\|R(1)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(u), X_i^* - X_i(u) \rangle - \frac{\|R(u+1)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(u)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(u), P_i(u) \rangle \\ &= \|R(u+1)\|_{(F)}^2 + \frac{\|R(u+1)\|_{(F)}^2}{\|R(u)\|_{(F)}^2} \|R(u)\|_{(F)}^2 - \|R(u+1)\|_{(F)}^2 \\ &= \|R(u+1)\|_{(F)}^2. \end{aligned}$$

By the principle of introduction, the conclusion holds for  $m = 1, 2, \dots$ . Now we assume that

$$\sum_{i=1}^p \langle P_i(m+r), X_i^* - X_i(m) \rangle = \|R(m+r)\|_{(F)}^2,$$

For  $r = 0, 1, 2, \dots, k$  using the previous results, gives us

$$\begin{aligned} \sum_{i=1}^p \langle P_i(m+r+1), X_i^* - X_i(m) \rangle &= \sum_{i=1}^p \left\langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(m+r+1)) + \frac{\|R(m+r+1)\|_{(F)}^2}{\|R(m+r)\|_{(F)}^2} P_i(m+r), X_i^* - X_i(m+r+1) \right\rangle \\ &= \sum_{i=1}^p \left\langle \sum_{s=1}^p \mathbb{T}_{si}^*(R_s(m+r+1)), X_i^* - X_i(m) \right\rangle - \frac{\|R(m+r+1)\|_{(F)}^2}{\|R(m+r)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m+r), X_i^* - X_i(m) \rangle \\ &= \sum_{s=1}^p \langle R_s(m+r+1), \sum_{i=1}^p \mathbb{T}_{si}(X_i^* - X_i(m)) \rangle + \frac{\|R(m+r+1)\|_{(F)}^2}{\|R(m+r)\|_{(F)}^2} \|R(m+r)\|_{(F)}^2 \\ &= \sum_{s=1}^p \langle R_s(m+r+1), M_s - \sum_{i=1}^p \mathbb{T}_{si}(X_i(m)) \rangle + \|R(m+r+1)\|_{(F)}^2 \\ &= \sum_{s=1}^p \langle R_s(m+r+1), R_s(m) \rangle + \|R(m+r+1)\|_{(F)}^2 \\ &= \|R(m+r+1)\|_{(F)}^2. \end{aligned}$$

By the principle of induction, the conclusion (20) holds. It follows from Algorithm 1 that

$$\begin{aligned} \sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m+1) \rangle &= \sum_{i=1}^p \left\langle P_i(m), X_i^* - X_i(m) - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} P_i(m) \right\rangle \\ &= \sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m) \rangle - \frac{\|R(m)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m)\|_{(F)}^2} \sum_{i=1}^p \langle P_i(m), P_i(m) \rangle \\ &= \|R(m)\|_{(F)}^2 - \|R(m)\|_{(F)}^2 = 0. \end{aligned}$$

Hence suppose that  $\sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m+r) \rangle = 0$  for  $r = 1, 2, \dots$ . By using (19), we have

$$\begin{aligned} \sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m+r+1) \rangle &= \sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m+r) - \frac{\|R(m+r)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m+r)\|_{(F)}^2} P_i(m+r) \rangle \\ &= \sum_{i=1}^p \langle P_i(m), X_i^* - X_i(m+r) \rangle - \frac{\|R(m+r)\|_{(F)}^2}{\sum_{t=1}^p \|P_t(m+r)\|_{(F)}^2} \sum_{t=1}^p \langle P_i(m), P_t(m+r) \rangle = 0. \end{aligned}$$

Hence the conclusion (21) holds by the principle of induction. The proof is completed.  $\square$

**Remark 3.4.** Lemma 3.3 implies that if Problem I is consistent, then  $\|R(k)\|_{(F)}^2 \neq 0$  implies that  $\sum_{i=1}^p P_i(k) \neq 0$ .

Else if there exists a positive number  $m$  such that  $\|R(m)\|_{(F)}^2 \neq 0$  but  $\sum_{i=1}^p P_i(m) = 0$ , then Problem I must be inconsistent. Hence the solvability of Problem I can be determined by Algorithm 1 in the absence of roundoff errors.

**Theorem 3.5.** Suppose that Problem I is consistent, then for any arbitrary initial quaternion matrix pair  $[X_1(1), X_2(1), \dots, X_p(1)]$ , a quaternion matrix solution pair of Problem I can be obtained within finite number of iterations in the absence of roundoff errors.

*Proof.* Firstly, in the space  $Q^{p_1 \times q_1} \times Q^{p_2 \times q_2} \times \dots \times Q^{p_p \times q_p}$ , we define an inner product as

$$\langle (A_1, A_2, \dots, A_p), (B_1, B_2, \dots, B_p) \rangle = \operatorname{Re} \left[ \sum_{i=1}^p \operatorname{tr}(B_i^H A_i) \right] \tag{22}$$

for  $(A_1, A_2, \dots, A_p), (B_1, B_2, \dots, B_p) \in Q^{p_1 \times q_1} \times Q^{p_2 \times q_2} \times \dots \times Q^{p_p \times q_p}$ . Denote  $d = \sum_{i=1}^p m_i n_i$ . From above it is known that the space  $Q^{p_1 \times q_1} \times Q^{p_2 \times q_2} \times \dots \times Q^{p_p \times q_p}$  with the inner product defined in (22) is a  $4d$ -dimensional. According to Lemma 3.2,  $\sum_{i=1}^p \langle R_i(m), R_i(n) \rangle = 0$ , for  $m, n = 0, 1, \dots, 4d - 1$ , and  $m \neq n$ . Thus  $(R_1(k), \dots, R_p(k)), k = 0, 1, 2, \dots, 4d - 1$  is a group orthogonal basis of the previously defined inner product space. In addition, it follows from Lemma 3.2 that  $\sum_{i=1}^p \langle R_i(4d), R_i(m) \rangle = 0$ , for  $k = 0, 1, \dots, 4d - 1$ . Consequently, it is derived from the property of an inner product space that  $(R_1(4d), \dots, R_p(4d)) = 0$ . So  $(X_1(4d), \dots, X_p(4d))$  are a group of exact solution to the Problem I.  $\square$

Next we consider finding the least norm solution of Problem I. The following lemmas are needed for our derivation.

**Lemma 3.6.** ([31]) Suppose that the consistent system of linear equation  $Ax = b$  has a solution  $x^* \in R(A^H)$ , then  $x^*$  is the unique least norm solution of linear equation.

**Lemma 3.7.** ([27]) For  $A \in LC^{m \times n, p \times q}$ , there exists a unique matrix  $M \in C^{pq \times mn}$ , such that  $\operatorname{vec}(A(X)) = M \operatorname{vec}(X)$  for all  $X \in C^{m \times n}$ .

According to Lemma 3.7 and the definition of self-adjoint operator, one can easily obtain the following corollary.

**Corollary 3.8.** Let  $A$  and  $M$  be the same as those in Lemma 4, and  $A^*$  be the self-adjoint operator of  $A$ , then  $\operatorname{vec}(A^*(Y)) = M^H \operatorname{vec}(Y)$  for all  $Y \in C^{p \times q}$ .

**Theorem 3.9.** Assume that Problem I is consistent. If we choose the initial quaternion matrix  $X_i(1) = \sum_{s=1}^p \mathbb{T}_{si}^*(H_i)$ , for  $i = 1, 2, \dots, p$ , where  $H_i$  is an arbitrary matrix, or more especially, let  $X_i(1) = 0, i = 1, 2, \dots, p$ , then the solution  $[X_1^*, X_2^*, \dots, X_p^*]$  obtained by Algorithm 1 is the least norm solution.

*Proof.* We only need to prove that the solution  $[(X_1^*)_\sigma, (X_2^*)_\sigma, \dots, (X_p^*)_\sigma] = [Y_1^*, Y_2^*, \dots, Y_p^*]$  is the least norm solution to its complex representation matrix equations (11). By (12) and (6) has a solution if and only if its complex representation matrix equations (11) has a solution. So the Problem I is equivalent to the following Problem I'.

Problem I'. For given  $\mathbb{B}_{si} \in LC^{2m_i \times 2n_i, 2p_s \times 2q_s}$  and  $M_s \in C^{2p_s \times 2q_s}, i = 1, 2, \dots, p; s = 1, 2, \dots, p$ , find  $Y_i, i = 1, 2, \dots, p$  such that

$$\left[ \sum_{i=1}^p \mathbb{B}_{1i}(X_i), \sum_{i=1}^p \mathbb{B}_{2i}(Y_i), \dots, \sum_{i=1}^p \mathbb{B}_{pi}(Y_i) \right] = [(M_1)_\sigma, (M_2)_\sigma, \dots, (M_p)_\sigma]. \tag{23}$$

If Problem I' has a solution  $[Y_1, Y_2, \dots, Y_p]$ , then  $[Y_1, Y_2, \dots, Y_p]$  must be a solution of the system (11). Conversely, if the system (11) has a solution  $[Y_1, Y_2, \dots, Y_p]$ , then it is easy to verify that  $[X_1, X_2, \dots, X_p]$  is a solution of Problem I. Therefore, the solvability of Problem I' is equivalent to that of the system (11).

If we choose the initial matrix  $Y_i(1) = (X_i(1))_\sigma = \sum_{s=1}^p \mathbb{B}_{si}^*((H_i)_\sigma), i = 1, 2, \dots, p; s = 1, 2, \dots, p$ , where  $H_i$  is an arbitrary quaternion matrix in  $Q^{m_i \times n_i}$ , by Algorithm 1 and Theorem 3.5, we can obtain the solution  $[Y_1, Y_2, \dots, Y_p]$  of Problem I' within finite iteration steps, which can be represented as  $Y_i(1)^* = (X_i^*)_\sigma = \sum_{s=1}^p \mathbb{B}_{si}^*(Y_i), i, s = 1, 2, \dots, p$ . Since the solution set of Problem I' is a subset of that of the system (11), next we shall prove that  $[Y_1, Y_2, \dots, Y_p]$  is the least norm solution of the system (11), which implies that  $[Y_1, Y_2, \dots, Y_p]$  is the least norm solution of Problem I'.

Let  $E_s$  be the matrices such that  $\text{vec}(\sum_{i=1}^p \mathbb{B}_{si}^*((H_s)_\sigma)) = E_i \begin{pmatrix} \text{vec}((H_1)_\sigma) \\ \vdots \\ \text{vec}((H_p)_\sigma) \end{pmatrix}$  for all  $E_i \in C^{\sum_{s=1}^p 4p_s q_s \times \sum_{i=1}^p 4m_i n_i}, s = 1, 2, \dots, p; (H_i)_\sigma \in C^{2m_i \times 2n_i}, i = 1, \dots, p$ , are arbitrary matrices. Then the system (11) is equivalent to the following system of linear equations:

$$\begin{pmatrix} \text{vec}(\sum_{i=1}^p \mathbb{B}_{1i}^*((H_i)_\sigma)) \\ \text{vec}(\sum_{i=1}^p \mathbb{B}_{2i}^*((H_i)_\sigma)) \\ \vdots \\ \text{vec}(\sum_{i=1}^p \mathbb{B}_{pi}^*((H_i)_\sigma)) \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_p \end{pmatrix} \begin{pmatrix} \text{vec}((H_1)_\sigma) \\ \vdots \\ \text{vec}((H_p)_\sigma) \end{pmatrix} \\ = \left( \begin{pmatrix} E_1^H & E_2^H & \dots & E_p^H \end{pmatrix} \right)^H \begin{pmatrix} \text{vec}((H_1)_\sigma) \\ \vdots \\ \text{vec}((H_p)_\sigma) \end{pmatrix} \in R \left( \begin{pmatrix} E_1^H & E_2^H & \dots & E_p^H \end{pmatrix} \right)^H.$$

Obviously, if we consider  $Y_i(1) = \sum_{s=1}^p \mathbb{B}_{si}^*((H_i)_\sigma)$ , for  $i = 1, 2, \dots, p$ , where  $H_i$  is an arbitrary matrix, or more especially, let  $Y_i(1) = 0, i = 1, 2, \dots, p$ , then all  $Y_i(k), i = 1, 2, \dots, p$ , generated by Algorithm 1 satisfy  $\begin{pmatrix} \text{vec}(Y_1(k)) \\ \vdots \\ \text{vec}(Y_p(k)) \end{pmatrix} \in R \left( \begin{pmatrix} E_1^H & E_2^H & \dots & E_p^H \end{pmatrix} \right)$ . Then it follows from Lemma 3.6 that  $[Y_1, \dots, Y_p]$  is the least norm solution of the system (11). It's showed that  $[X_1, \dots, X_p]$  is the least norm solution of Problem I.

Now we assume that the Problem I is consistent, and its solution set  $S$  is not non-empty, then we have

$$\left[ \sum_{i=1}^p \mathbb{T}_{1i}(X_i), \sum_{i=1}^p \mathbb{T}_{2i}(X_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(X_i) \right] = [F_1, F_2, \dots, F_p]$$

$$\iff \left[ \sum_{i=1}^p \mathbb{T}_{1i}(X_i - \tilde{X}_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(X_i - \tilde{X}_i) \right] = [F_1 - \sum_{i=1}^p \mathbb{T}_{1i}(\tilde{X}_i), F_2 - \sum_{i=1}^p \mathbb{T}_{2i}(\tilde{X}_i), \dots, F_p - \sum_{i=1}^p \mathbb{T}_{pi}(\tilde{X}_i)].$$

Define

$$\widehat{X}_i = X_i - \tilde{X}_i, \quad \widehat{F}_j = F_j - \sum_{i=1}^p \mathbb{T}_{ji}(\tilde{X}_i) \quad i = 1, 2, \dots, p; \quad j = 1, 2, \dots, p.$$

for all  $\tilde{X}_i \in S, i = 1, 2, \dots, p$ .

So the Problem II is equivalent to finding the least norm solution of the following quaternion matrix equations.

$$\left[ \sum_{i=1}^p \mathbb{T}_{1i}(\widehat{X}_i), \sum_{i=1}^p \mathbb{T}_{2i}(\widehat{X}_i), \dots, \sum_{i=1}^p \mathbb{T}_{pi}(\widehat{X}_i) \right] = [\widehat{F}_1, \widehat{F}_2, \dots, \widehat{F}_p].$$

Therefore, we can apply the Algorithm 1 to find the least norm solution of the quaternion matrix equation. And the least norm solution of the Problem II is  $X_i = \widehat{X}_i + \tilde{X}_i, i = 1, 2, \dots, p. \quad \square$

#### 4. Applications

Denote  $q^i = -iqi = q_a + iq_b - jq_c - kq_d, q^j = -jqj = q_a - iq_b + jq_c - kq_d, q^k = -kqk = q_a - iq_b - jq_c + kq_d$ . Their conjugate vectors are  $\bar{q}^i = q_a - iq_b + jq_c + kq_d, \bar{q}^j = q_a + iq_b - jq_c + kq_d, \bar{q}^k = q_a + iq_b + jq_c - kq_d$  and  $q_a = \frac{1}{2}(q + \bar{q}), q_b = \frac{1}{2i}(q - \bar{q}^i), q_c = \frac{1}{2j}(q - \bar{q}^j), q_d = \frac{1}{2k}(q - \bar{q}^k)$ . Then  $\bar{q} = \frac{1}{2}(q^i + q^j + q^k - q)$ .

In the second-order statistics of quaternion random signals [28], it always occurs to the model  $q^a = Aq^r$ , for all  $q^a = [q^T, (q^i)^T, (q^j)^T, (q^k)^T]^T, q^r \in R^{4n \times 1}$ , i.e.

$$\begin{bmatrix} q \\ q^i \\ q^j \\ q^k \end{bmatrix} = \begin{bmatrix} I & iI & jI & kI \\ I & iI & -jI & -kI \\ I & -iI & jI & -kI \\ I & -iI & -jI & kI \end{bmatrix} \begin{bmatrix} q_a \\ q_b \\ q_c \\ q_d \end{bmatrix}$$

for all  $I \in R^{n \times n}$  is the identity matrix,  $q = [q_1, q_2, \dots, q_n]^T \in Q^{n \times 1}; q^i, q^j, q^k \in Q^{n \times 1}, q_a, q_b, q_c, q_d \in R^{n \times 1}$ . In linear mean-squared error (MSE) estimation, it needs to compute complete second-order estimation. Its model in complex field [23] is  $\widehat{y} = h^T x$ , and its model in quaternion field [29] is

$$\widehat{y} = E[y|x, x^i, x^j, x^k] + iE[y^i|x, x^i, x^j, x^k] + jE[y^j|x, x^i, x^j, x^k] + kE[y^k|x, x^i, x^j, x^k].$$

It is to find the the least norm solution of the following quaternion matrix equation [29]:

$$y = g^H x + h^H x^i + u^H x^j + v^H x^k.$$

Consider the following general linear vector-sensor signal model [3]

$$S_i(t) = \sum_{j=1}^p h_{ij} g_j(t - (i-1)\tau_p) + b_i(t), \quad i = 1, 2, \dots, n,$$

where  $h_{ij} \in R, \tau_p$ , are known scalars,  $g_j(t) \in Q^N, b_i(t), S_i(t)$  are the determined quaternion vector. We need to solve the coupled quaternion vector equations

$$Y_i = \sum_{j=1}^p \alpha_j X_j + Z_i + C_i$$

where  $Y_i, i = 1, 2, \dots, n, X_j, j = 1, 2, \dots, p, Z_i$  are the determined quaternion vectors,  $Z_i, i = 1, 2, \dots, n$ , are the known quaternion vector and  $\alpha_j, j = 1, 2, \dots, p$ , are the known scalars.

### 5. Numerical examples

In this section, we shall give a numerical example to illustrate the efficiency of Algorithm 1.

**Example 5.1.** Consider the matrix equation

$$\begin{cases} A_{11}X_1B_{11} + A_{12}X_2B_{12} = C_1 \\ A_{21}X_1B_{21} + A_{22}X_2B_{22} = C_2 \end{cases} \tag{25}$$

where

$$\begin{aligned} A_{11} &= \begin{pmatrix} -1 & 2j & k & 1+i \\ 2+k & 2i & 3j & k \\ 2+i & 1-k & 2+j & 2 \\ 2+j+k & 2k & 5j & k \end{pmatrix}, B_{11} = \begin{pmatrix} 1+j & 2+k & i+j & 3+k \\ 5j & 2k & 4i & 1 \\ 1 & j-k & 1 & 4k \\ 2 & 6j & 2 & 1+j \end{pmatrix}, \\ A_{12} &= \begin{pmatrix} 1+j & 4k & 2+i & 6+j \\ 2+k & 2+j & 6+i & 1+j+k \\ 3j+2k & 2-4k & 1-j & 6+2k \\ 2+4j & 3-4k & 5j & 2k \end{pmatrix}, B_{12} = \begin{pmatrix} 2+j & 1+k & 3j & k \\ 4j+k & 3i & 2 & 6i \\ 4j & 2k & 3 & 1+k \\ 1 & 2j & 3i & 0 \end{pmatrix}, \\ A_{21} &= \begin{pmatrix} j & i & k & 1 \\ 2i & 2j & 2k & 4 \\ 1+i & 5+k & 3 & 1 \\ k & 1+j & k & 0 \end{pmatrix}, B_{21} = \begin{pmatrix} 4+j & 3+k & 2+i & 4 \\ 4j & 1+i & 2j & k \\ 2 & 3+i & 3+k & 1+2i \\ 4+k & 2+j & 2+k & 0 \end{pmatrix}, \\ A_{22} &= \begin{pmatrix} 0 & 2j & 0 & 1+k \\ 0 & 11+j & 21i & k \\ 4 & 0 & 2+j & 1+i \\ 0 & -j & -k & 1 \end{pmatrix}, B_{22} = \begin{pmatrix} -3j & 12k & 11+i & 2k \\ 21+k & 23j & k & 2j \\ 1+j & 2j & 4k & 1+i \\ 2i & 11 & 3i & 2 \end{pmatrix}, \\ C_1 &= \begin{pmatrix} -58+i+88j+33k & -22+24i-29j+71k & 5-27i+71j-48k & 52+67i+5j-5k \\ -68+41i+53j+26k & -52+68i+21j-18k & 4-4i+30j+3k & -74+63i-22j-108k \\ -75+19i+98j+83k & -42+38i+41j+90k & 30+42i+23j+25k & -87+51i+7j-7k \\ -121+73i-25j-2k & 79+74i-17j-49k & -42-11i+9j+26k & -142+15i-81j-110k \end{pmatrix}, \\ C_2 &= \begin{pmatrix} -72+21i-3j+28k & -41+6i-43j-38k & -24-21i-9j-22k & -12+2i-5j-6k \\ -86+497i+582j+437k & -402-233i-259j+812k & -352-56i+84j-131k & -31-3i-27j+191k \\ 127+121i+91j+90k & -133+55i+199j+295k & -29+113i+36j+73k & -19+21i+71j+42k \\ 36+15i+24j-18k & -11+18i+77j+54k & -21+15i+15j+27k & -9i+10j \end{pmatrix}. \end{aligned}$$

It can be verified that the generalized coupled matrix equation (25) are consistent and have a unique solution pair  $(X_1^*, X_2^*)$  as follows:

$$X_1^* = \begin{pmatrix} -1 & 2j & 2k & 1+j \\ 2 & 1+2k & 2i & 2k \\ 2j & 2i & 3k & 2i \\ k & 2j & 2 & 4i \end{pmatrix}, X_2^* = \begin{pmatrix} 1+k & 1+i & j & 0 \\ 1+j & 2j & k & 1+i \\ 1+i & 1+j & 0 & 2j \\ k & 1 & 2 & k \end{pmatrix}.$$

If we let the initial matrix  $(X_1, X_2) = 10^{-6}I_4$ , applying Algorithm 1, we obtain the solutions, that are

$$X_1(799) = \begin{pmatrix} -0.9999 & 2.0000j & 1.9999k & 1.0000 + 0.9999j \\ 2.0000 & 1.0000 + 2.0000k & 2.0000i & 2.0000k \\ 2.0000j & 2.0000i & 3.0000k & 1.9999i \\ 1.0000k & 2.0000j & 1.9999 & 4.0000i \end{pmatrix},$$

$$X_2(799) = \begin{pmatrix} 0.9999 + 1.0000k & 1.0000 + 0.9999i & 1.0000j & 0.0000 \\ 1.0000 + 0.9999j & 2.0000j & 1.0000k & 0.9999 + 1.0000i \\ 1.0000 + 1.0000i & 1.0000 + 0.9999j & 0.0000 & 2.0000j \\ 0.9999k & 1.0000 & 2.0000 & 1.0000k \end{pmatrix},$$

with corresponding residual

$$\|R(799)\|_{(F)} = \|\text{diag}(C_1 - A_{11}X_1(799)B_{11} - A_{12}X_2(799)B_{12}, C_2 - A_{21}X_1(799)B_{21} - A_{22}X_2(799)B_{22})\|_{(F)} = 6.2826 \times 10^{-11}.$$

The obtained results are presented in Figure 1, where

$$e_k = \frac{\|(X_1(k), X_2(k)) - (X_1^*, X_2^*)\|_{(F)}}{\|(X_1^*, X_2^*)\|_{(F)}}, \quad r_k = \|R(k)\|_{(F)}.$$

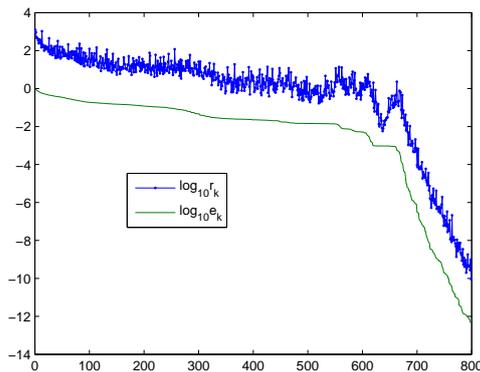


Figure 1: The relative error of solution and the residual

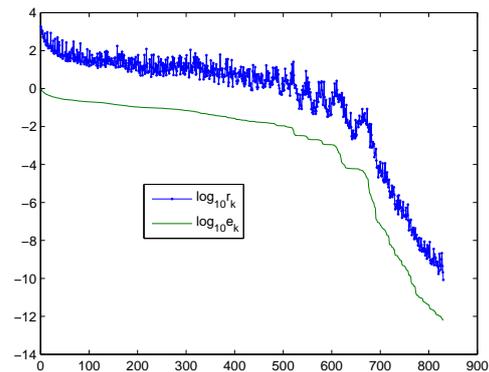


Figure 2: The relative error of solution and the residual

Now let

$$\widehat{X}_1 = \begin{pmatrix} -i & j & k & 0 \\ 1+j & 1+k & 1 & 1 \\ 1 & 1-j & 2k & 1 \\ 0 & 1+2j & 1 & 2j \end{pmatrix}, \widehat{X}_2 = \begin{pmatrix} 2j & 2i & 2k & 0 \\ 2 & 1-4k & 2k & 2i \\ 2 & 2j & 1-6k & 0 \\ 1-i & 2j & 0 & 2k \end{pmatrix}.$$

By Algorithm 1 for the generalized coupled Sylvester matrix equation (25), with the initial matrix pair  $(\overline{X}_1, \overline{Y}_1) = 0$ , we have the least Frobenius norm solution of the generalized coupled Sylvester matrix equation (25) by the following form:

$$\overline{X}_1^* = \overline{X}_1(830) = \begin{pmatrix} -1.0000 + 1.0000i & 0.9999j & 0.9999k & 0.9999 + 1.0000j \\ 0.9999 - 1.0000j & 0.9999k & -1.0000 + 1.9999i & -1.0000 + 1.9999k \\ -1.0000 + 2.0000j & -0.9999 + 1.9999i + 1.0000j & 0.9999k & -1.0000 + 2.0000i \\ 1.0000k & -0.9999 & 0.9999 & 3.9999i - 1.9999j \end{pmatrix},$$

$$\bar{X}_2^* = \bar{X}_2(830) = \begin{pmatrix} 0.9999 - 2j + k & 1 - 0.9999i & 0.9999 - 1.9999k & 0.0000 \\ -0.9999 + 0.9999j & -1 + 2j + 3.9999k & -0.9999k & 0.9999 - 0.9999i \\ -0.9999 + 0.9999i & 0.9999 - 0.9999j & -0.9999 + 6k & 1.9999j \\ 0.9999 + 1.0000i + 0.9999k & 0.9999 - 2.0000j & 1.9999 & -k \end{pmatrix}$$

with corresponding residual

$$\|R(830)\|_{(F)} = \|\text{diag}(\bar{C}_1 - A_{11}\bar{X}_1(830)B_{11} - A_{12}\bar{X}_2(830)B_{12}, \bar{C}_2 - A_{21}\bar{X}_1(830)B_{21} - A_{22}\bar{X}_2(830)B_{22})\|_{(F)} = 5.9374 \times 10^{-11}.$$

The obtained results are presented in Figure 2. Therefore, the solution of the matrix equation nearness problem is

$$\begin{aligned} \widetilde{X}_1 &= \bar{X}_1^* + \widehat{X}_1 = \begin{pmatrix} -1.0000 & 1.9999j & 1.9999k & 0.9999 + 1.0000j \\ 1.9999 & 1 + 1.9999k & 1.9999i & 1.9999k \\ 2.0000j & 0.0001 + 1.9999i & 2.9999k & 2i \\ 1.0000k & 0.0001 + 2j & 1.9999 & 3.9999i + 0.0001j \end{pmatrix}, \\ \widetilde{X}_2 &= \bar{X}_2^* + \widehat{X}_2 = \begin{pmatrix} 0.9999 + 1.0000k & 1.0000 + 0.0001i & 0.9999j + 0.0001k & 0.0000 \\ 1.0001 + 0.9999j & 2.0000j - 0.0001k & 1.0000k & 0.9999 + 1.0001i \\ 1.0001 + 0.9999i & 0.9999 + 1.0001j & 0.0001 & 1.9999j \\ 0.0001 + 0.9999k & 0.9999 & 1.9999 & 1.0000k \end{pmatrix} \end{aligned}$$

From the Figure 1, we can see that the residual and the relative error of the solution are declining gradually with the increase of the iterative steps. So we can say that the Algorithm 1 can converge to exact solution of quaternion matrix equations (25). From the Figure 2, we can see that the Algorithm 1 can solve the optimal solution of the matrix equation (25). But for any quaternion matrix  $A = A_1 + A_2i + A_3j + A_4k$ ,  $i^2 = j^2 = k^2 = -1$ , complex matrix can be stated as the special case of quaternion matrix. So when we solve the quaternion matrix equations problem by means of the iterative algorithm, generally its convergence rate is not fast, such as an operable iterative method called LSQR-Q algorithm [36] for finding the minimum-norm solution of the QLS problem by applying real representation of a quaternion matrix, which is more appropriate to large scale system.

### References

- [1] A. Ben-Israel, T.N.E. Greville, Generalized Inverse: Theory and Applications, Springer, New York, 2003.
- [2] N.L. Bihan, S.J. Sangwine, Color image decomposition using quaternion singular value decomposition, in: IEEE International Conference on Visual Information Engineering of quaternion (VIE), Guildford, UK, 2003.
- [3] N.L. Bihan, J. Mars, Singular value decomposition of matrices: a new tool for vector-sensor signal processing, Signal Processing 84 (2004) 1177-1199.
- [4] N.L. Bihan, S.J. Sangwine, Jacobi method for quaternion matrix singular value decomposition, App. Math. Comput. 187 (2007) 1265-1271.
- [5] I. Borno, Parallel computation of the solutions of coupled algebraic Lyapunov equations, Automatica 31(9) (1995) 1345-1347.
- [6] L.X. Chen, Generalization of Cayley-Hamilton theorem over quaternion field, Sci. Bull. 17 (1991) 1291-1293.
- [7] O.L.V. Costa, M.D. Fragoso, Stability results for discrete-time linear systems with Markovian jumping parameters, J. Math. Anal. Appl. 179(1) (1993) 154-178.
- [8] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Transactions Automatic Control 50(8) (2005) 1216-1221.
- [9] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM Journal on Control and Optimization 44 (2006) 2269-2284.
- [10] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, System and Control Letters 54 (2005) 95-107.
- [11] F. Ding, T. Chen, Hierarchical gradient-based identification of multivariable discrete-time systems, Automatica 41(2) (2005) 315-325.
- [12] F. Ding, T. Chen, Hierarchical least squares identification methods for multivariable systems, IEEE Transactions on Automatic Control 50(3) (2005) 397-402.
- [13] D.R. Farenick, B.A.F. Pidkowich, The spectral theorem in quaternions, Linear Algebra Appl. 371 (2003) 75-102.
- [14] A.B. Gersther, R. Byers, V. Mehrmann, A quaternion QR algorithm, Numer. Math. 55 (1989) 83-95.
- [15] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.
- [16] L.P. Huang, The Matrix Equation  $AXB - CXD = E$  over the Quaternion Field, Linear Algebra and its Applications. 234 (1996) 197-208.

- [17] T.S. Jiang, M.S. Wei, On solution of the Quaternion Matrix Equation  $X - \tilde{A}XB = C$  and Its Application, *Acta Mathematica Sinica*. 21 (2005) 483-490.
- [18] S.D. Leo, G. Scolarici, Right eigenvalue equation in quaternionic quantum mechanics, *J. Phys. A* 33 (2000) 2971-2995.
- [19] Z.Y. Li, Y. Wang, B. Zhou, G.R. Duan, Least squares solution with the minimum-norm to general matrix equations via iteration, *Appl. Math. Comput.* 215 (2010) 3547-3562.
- [20] M. Mariton, *Jump Linear System in Automatic Control*, Marcel Dekker, New York, Basel, 1990.
- [21] C.E. Moxey, S.J. Sangwine, T.A. Ell, Hypercomplex correlation techniques for vector imagines, *IEEE Trans. Signal Process.* 51(7) (2003) 1177-1199.
- [22] F.X. Piao, Q.L. Zhang, Z.F. Wang, The solution to matrix equation  $AX + X^T C = B$ , *J. Fran. Inst.* 344 (2007) 1056-1062.
- [23] B. Picinbono, P. Chevalier, Widely linear estimation with complex data, *IEEE Trans. Signal Process.* 43(8) (1995) 2030-2033.
- [24] Y.X. Peng, X.Y. Hu, L. Zhang, An iterative method for symmetric solutions and optimal approximation solution of the system of matrix equations  $A_1XB_1 = C_1, A_2XB_2 = C_2$ , *Appl. Math. Comput.* 183 (2006) 1127-1137.
- [25] S.J. Sangwine, T.A. Ell, C.E. Moxey, Vector phase correlation, *Electron. Lett.* 37(25) (2001) 1513-1515.
- [26] S.J. Sangwine, N.L. Bihan, Quaternion Singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion householder transformations, *Appl. Math. Comput.* 182(1) (2006) 727-738.
- [27] X.P. Shen, G.L. Chen, An iterative method for the symmetric and skew symmetric solutions of a linear matrix equation  $AXB + CYD = E$ , *Journal of Computational and Applied Mathematics* 233 (2010) 3030-3040.
- [28] C.C. Took, D.P. Mandic, Augmented second-order statistics of quaternion random signals, *Signal Process.* 91 (2011) 214–224.
- [29] C.C. Took, D.P. Mandic, New developments in statistical signal processing of quaternion random variables with applications in wind forecasting, *Proceeding of the International Symposium on Evolving Intelligent Systems at the AISB 2010 convention*, 2010, pp. 58-62.
- [30] B.X. Tu, Generalization of Schur theorem over quaternion field, *Math. Ann.* A 2 (1998) 131-138(in Chinese).
- [31] Q.W. Wang, Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations, *Comput. Math. Appl.* 49 (2005) 641-650.
- [32] Q.W. Wang, The general solution to a system of real quaternion matrix equations, *Comput. Math. Appl.* 49 (2005) 665-675.
- [33] M.H. Wang, X.H. Cheng, M.S. Wei, Iterative algorithms for solving the matrix equation  $AXB + CX^T D = E$ , *Appl. Math. Comput.* 187 (2007) 622-629.
- [34] Q.W. Wang, J.H. Sun, S.Z. Li, Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra, *Linear Algebra and its Applications* 353 (2002) 169-182.
- [35] M.H. Wang, X.H. Cheng, M.S. Wei, Iterative algorithms for solving the matrix equation  $AXB + CX^T D = E$ , *Appl. Math. Comput.* 187 (2007) 622-629.
- [36] M.H. Wang, M.S. Wei, Y. Feng, An iterative algorithm for least squares problem in quaternionic quantum theory, *Comput. Phys. Commun.* 179 (2008) 203-207.
- [37] J.L. Wu, The generalization of Wieland-Hoffman theorems on quaternion field, *Math. Res. Exp.* 15 (1995) 121-124.
- [38] J.L. Wu, Distribution and estimation for eigenvalues of real quaternion matrices, *Comput. Math. Appl.* 55 (2008) 1998-2004.
- [39] A.G. Wu, G. Feng, G.R. Duan, W.J. Wu, Iterative solutions to coupled Sylvester-conjugate matrix equations, *Computers and Mathematics with Applications* 60(1) (2010) 54-66.
- [40] A.G. Wu, B. Li, Y. Zhang, G.R. Duan, Finite iterative solutions to coupled Sylvester-conjugate matrix equations, *Applied Mathematics Modelling* 35(3) (2011) 1065-1080.
- [41] L. Xie, J. Ding, F. Ding, Gradient based iterative solution for general linear matrix equations, *Computers and Mathematics with Applications* 58 (2009) 1441-1448.
- [42] F.Z. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* 251 (1997) 21-57.