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A commutator approach to Buzano's inequality

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Abstract. Using a 2×2 matrix trick, an inequality involving commutators of certain Hilbert space operators as an operator version of Buzano's inequality, which is in turn a generalization of the Cauchy–Schwarz inequality, is presented. Also a version of the inequality in the framework of Hilbert C^*-modules is stated and a special case in the context of C^*-algebras is presented.

1. Introduction and preliminaries

In [4], Buzano obtained the following extension of the celebrated Cauchy–Schwarz inequality in a real or complex inner product space $\mathscr{H}$:

$$|\langle a, x\rangle\langle x, b\rangle| \leq \frac{1}{2}\left(|\langle a, b\rangle| + \|a\|\|b\|\right)\|x\|^2 \qquad (a, b, x \in \mathscr{H})$$

When $a = b$ this inequality becomes the Cauchy–Schwarz inequality

$$|\langle a, x\rangle|^2 \leq \|a\|^2\|x\|^2.$$

For a real inner product space, Richard [18] independently obtained the following stronger inequality:

$$\left|\langle a, x\rangle\langle x, b\rangle - \frac{1}{2}\langle a, b\rangle\|x\|^2\right| \leq \frac{1}{2}\|a\|\|b\|\|x\|^2 \qquad (a, b, x \in \mathscr{H}).$$

Dragomir [5] showed that this inequality (for real or complex case) is valid with coefficients $\frac{1}{|\alpha|}$ instead of $\frac{1}{2}$, where a non-zero number α satisfies the equality $|1 - \alpha| = 1$. As an application of this inequality, Fujii and Kubo [9] found a bound for roots of algebraic equations. During developing the operator theory and its applications, the authors of [6] have recently extended some numerical inequalities to operator inequalities.

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Some mathematicians have also investigated the operator versions of the Cauchy–Schwarz inequality or its reverse; see [7, 8, 12, 16, 19].

In the next section an operator version of Buzano's inequality is introduced as a commutator inequality and in the last section we state a suitable version of it for Hilbert C^*-modules including C^*-algebras.

In this paper, $\mathbb{B}(\mathscr{H})$ stands for the C^*-algebra of all bounded operators on a complex separable Hilbert space $\mathscr{H}$ equipped with the usual operator norm $\|\cdot\|$. If $\mathscr{H}$ is finite-dimensional with $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. If $x, y \in \mathscr{H}$, the rank-one operator $x \otimes y$ is defined by

$$(x \otimes y)z = \langle z, y\rangle x \qquad (z \in \mathscr{H})$$

For a compact operator $T \in \mathbb{B}(\mathscr{H})$, the singular values of T are defined to be the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$, enumerated as $s_1(T) \geq s_2(T) \geq \cdots$ with their multiplicities counted. If $S \in \mathbb{B}(\mathscr{H})$ and $T \in \mathbb{B}(\mathscr{K})$, we use the direct sum notation $S \oplus T$ for the block-diagonal operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ defined on $\mathscr{H} \oplus \mathscr{K}$. It can be easily seen that the set of singular values of $S \oplus T$ is the union of those of S and T. In particular, the operator norm of $S \oplus T$ is the maximum of the norm of S and T. For $A, B, X \in \mathbb{B}(\mathscr{H})$, the operator $AX - XA$ is called a commutator and the operator $AX - XB$ is said to be a generalized commutator. There are several results related to the singular values and unitarily invariant norms of (generalized) commutators, see [3, 11, 13, 14] and references therein. Recall that a norm $|||\cdot|||$ on $\mathcal{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and all unitary matrices $U, V \in M_n(\mathbb{C})$.

The notion of Hilbert C^*-module is a generalization of that of Hilbert space. Let $\mathscr{A}$ be a C^*-algebra, and let $\mathscr{X}$ be a complex linear space, which is a right $\mathscr{A}$-module satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in \mathscr{X}, a \in \mathscr{A}, \lambda \in \mathbb{C}$. The space $\mathscr{X}$ is called a *(right) pre-Hilbert C^*-module over $\mathscr{A}$* if there exists an $\mathscr{A}$-inner product $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ satisfying

(i) $\langle x, x\rangle \geq 0$ (i.e. $\langle x, x\rangle$ is a positive element of $\mathscr{A}$) and $\langle x, x\rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$;

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$;

for all $x, y, z \in \mathscr{X}$, $\lambda \in \mathbb{C}$, $a \in \mathscr{A}$.

We can define a norm on $\mathscr{X}$ by $\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}}$, where the latter norm denotes that in the C^*-algebra $\mathscr{A}$. A pre-Hilbert $\mathscr{A}$-module is called a *(right) Hilbert C^*-module over $\mathscr{A}$* (or a *(right) Hilbert $\mathscr{A}$-module*) if it is complete with respect to its norm. Any inner product space can be regarded as a pre-Hilbert $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a, b\rangle = a^*b$ $(a, b \in \mathscr{A})$. For more information about C^*-algebras and Hilbert C^*-modules see [17] and [15], respectively.

2. The Hilbert space case

To establish singular value inequalities for Hilbert space operators, we need the following lemma, which is an immediate consequence of the Maximin principle (see, e.g., [2, p. 75] or [10, p. 27]).

Lemma 2.1. *Suppose that $X, Y, Z \in \mathbb{B}(\mathscr{H})$. If Y is compact, then*

$$s_j(XYZ) \leq \|X\|\,\|Z\|\,s_j(Y)$$

for all $j = 1, 2, \ldots$.

Now we state our main result.

Theorem 2.2. *Let $A, B, X \in \mathbb{B}(\mathscr{H})$ be such that A is invertible and it commutes with X. Suppose that, for some Hilbert space $\mathscr{K}$, $\widetilde{A} = A \oplus A' \in \mathbb{B}(\mathscr{H} \oplus \mathscr{K})$, $\widetilde{B} = B \oplus 0 \in \mathbb{B}(\mathscr{H} \oplus \mathscr{K})$ and $\widetilde{X} \in \mathbb{B}(\mathscr{H} \oplus \mathscr{K})$ is any compact extension of X. Then, for $j = 1, 2, \ldots$,*

$$s_j(\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B}) \leq \max\{1, \|1 - A^{-1}B\|\}\|\widetilde{A}\|\,s_j(\widetilde{X}).$$

If $\mathscr{K} = \{0\}$, *then*

$$s_j(AX - XB) \le \|A - B\|\, s_j(X) \le \|1 - A^{-1}B\|\, \|A\|\, s_j(X).$$

Proof. Since $\widetilde{X}$ leaves $\mathscr{H}$ invariant, we can write

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}, \widetilde{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \widetilde{X} = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.$$

It follows from Lemma 2.1 and

$$\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B} = \begin{bmatrix} AX - XB & AY \\ 0 & A'Z \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} I - A^{-1}B & 0 \\ 0 & I \end{bmatrix}$$

that

$$s_j(\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B}) \le \max\{1, \|1 - A^{-1}B\|\}\, \|\widetilde{A}\|\, s_j(\widetilde{X}).$$

If $\mathscr{K} = \{0\}$, then $s_j(AX - XB) = s_j(X(A - B)) \le \|A - B\|\, s_j(X)$ by Lemma 2.1. □

Corollary 2.3. *Let A, B, X be $n \times n$ matrices such that A is invertible and it commutes with X. Suppose that $\widetilde{A} = A \oplus A'$, $\widetilde{B} = B \oplus 0$ and $\widetilde{X}$ is any extension of X to $\mathbb{C}^n \oplus \mathbb{C}^m$ for some m. Then*

(i) $|||\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B}||| \le \max\{1, \|1 - A^{-1}B\|\}\|\widetilde{A}\|\, |||\widetilde{X}|||$ *for every unitarily invariant norm* $|||\cdot|||$ *on* $\mathbb{C}^{n+m}$.

(ii) $|\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B}| \le \max\{1, \|1 - A^{-1}B\|\}\|\widetilde{A}\|\, U|\widetilde{X}|U^*$ *for some unitary matrix* $U \in M_{n+m}(\mathbb{C})$.

Proof. It follows from Theorem 2.2 that we have, for each $k = 1, 2, \ldots, n + m$,

$$\sum_{j=1}^{k} s_j(\widetilde{A}\widetilde{X} - \widetilde{X}\widetilde{B}) \le \sum_{j=1}^{k} \max\{1, \|1 - A^{-1}B\|\}\, \|\widetilde{A}\|\, s_j(\widetilde{X}).$$

The Ky Fan dominance theorem (see, e.g., [2, p. 93]) then completes the proof of (i).

The assertion (ii) follows from the fact that for positive matrices S, T the inequalities $s_j(S) \le s_j(T)$ $(1 \le j \le n + m)$ are equivalent to $S \le UTU^*$ for some unitary matrix U. □

The next result may be considered as a slight generalization of [13, Lemma 3] in the case when $X \in \mathbb{B}(\mathscr{H})$ is a compact operator leaving invariant the range of a projection $P \in \mathbb{B}(\mathscr{H})$.

Corollary 2.4. *Let $P \in \mathbb{B}(\mathscr{H})$ be a non-zero projection on a subspace $\mathscr{K}$ of $\mathscr{H}$, and let $X \in \mathbb{B}(\mathscr{H})$ be a compact operator which leaves $\mathscr{K}$ invariant. Suppose that $C \in \mathbb{B}(\mathscr{H})$ is a contraction satisfying $PC = CP = 0$. Then, for $\alpha \in \mathbb{C}$ and $j = 1, 2, \ldots$,*

$$s_j((P + C)X - \alpha XP) \le \max\{1, |1 - \alpha|\}\, s_j(X). \tag{1}$$

Proof. Since the restriction of the operator $P + C$ to the subspace $\mathscr{K}$ is the identity operator, Theorem 2.2 can be applied for the operators $\widetilde{A} = P + C$, $\widetilde{B} = \alpha P$ and $\widetilde{X} = X$. □

Suppose that $x, a, b \in \mathscr{H}$ and $\|x\| = 1$. Set $P = x \otimes x$, $C = 0$ and $X = x \otimes b$ in inequality (1). Then

$$\|PXa - \alpha XPa\| \le \max\{1, |1 - \alpha|\}\|X\|\|a\|.$$

Since

$$\begin{aligned} \|PXa - \alpha XPa\| &= \|(x \otimes x)(x \otimes b)a - \alpha(x \otimes b)(x \otimes x)a\| \\ &= \|\langle a, b\rangle\langle x, x\rangle x - \alpha\langle a, x\rangle\langle x, b\rangle x\| \\ &= |\langle a, b\rangle - \alpha\langle a, x\rangle\langle x, b\rangle| \end{aligned}$$

and $\|X\| = \|b\|$, we obtain that

$$|\langle a, b\rangle - \alpha\langle a, x\rangle\langle x, b\rangle| \le \max\{1, |1 - \alpha|\}\|b\|\|a\|.$$

If x is an arbitrary non-zero vector in an inner product space $\mathscr{H}$, by completing the space we can assume that $\mathscr{H}$ is a Hilbert space. Then an application of the last inequality for the unit vector $\frac{x}{\|x\|}$ proves the following version of Buzano's inequality. It allows us to regard inequality (1) as an operator version of Buzano's inequality.

Corollary 2.5. *Let x, a, b be vectors in an inner product space $\mathscr{H}$ and $\alpha \in \mathbb{C}$. Then*

$$\left|\langle a, b\rangle \|x\|^2 - \alpha\langle a, x\rangle\langle x, b\rangle\right| \leq \max\{1, |1-\alpha|\}\|b\|\|a\|\|x\|^2. \tag{2}$$

Remark 2.6. An easy inspection of the proof of Dragomir's result [5, Theorem 3.3] shows that he in fact proved inequality (2).

The following result is a slight generalization of [5, Theorem 3.7] (and of Corollary 2.5).

Theorem 2.7. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal family in a Hilbert space $\mathscr{H}$, and let $\{\lambda_i\}_{i=1}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\left|\sum_{i=1}^{\infty} \lambda_i \langle a, e_i\rangle\langle e_i, b\rangle - \langle a, b\rangle\right| \leq \max\{1, \sup_{i\geq 1} |1-\lambda_i|\} \|b\| \|a\|$$

for all $a, b \in \mathscr{H}$. If $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$, then

$$\left|\sum_{i=1}^{\infty} \lambda_i \langle a, e_i\rangle\langle e_i, b\rangle - \langle a, b\rangle\right| \leq \sup_{i\geq 1} |1-\lambda_i| \|b\| \|a\|.$$

Proof. The series $\sum_{i=1}^{\infty} \lambda_i\langle a, e_i\rangle\langle e_i, b\rangle$ converges absolutely, since

$$\sum_{i=1}^{\infty} |\lambda_i\langle a, e_i\rangle\langle e_i, b\rangle| \leq \sup_{i\geq 1} |\lambda_i| \cdot \left(\sum_{i=1}^{\infty} |\langle a, e_i\rangle|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |\langle e_i, b\rangle|^2\right)^{1/2} \leq \sup_{i\geq 1} |\lambda_i| \|a\| \|b\|$$

by the Cauchy–Schwarz inequality in the sequence space l^2 and by Bessel's inequality. Therefore, it is enough to show that, for each positive integer n, we have

$$\left|\sum_{i=1}^{n} \lambda_i \langle a, e_i\rangle\langle e_i, b\rangle - \langle a, b\rangle\right| \leq \max\{1, |1-\lambda_1|, \ldots, |1-\lambda_n|\} \|b\| \|a\|.$$

Set $A := \sum_{i=1}^{n} e_i \otimes e_i$, $B := \sum_{i=1}^{n} \lambda_i e_i \otimes e_i$ and $X := \sum_{i=1}^{n} e_i \otimes b$. Consider the closed subspace $\mathscr{K}$ spanned by vectors $e_1, \ldots, e_n$. Note that A is the identity operator on $\mathscr{K}$, and B leaves $\mathscr{K}$ invariant and it is zero on the orthogonal complement of $\mathscr{K}$. Also, X is an operator from $\mathscr{H}$ to $\mathscr{K}$, so its restriction to $\mathscr{K}$ commutes with A. By Theorem 2.2, we have

$$\|AX - XB\| \leq \max\{1, \|1 - (A|_{\mathscr{K}})^{-1}B|_{\mathscr{K}}\|\}\|A\|\|X\|,$$

and so, for each $a \in \mathscr{H}$,

$$\|(AX - XB)a\| \leq \max\{1, |1-\lambda_1|, \ldots, |1-\lambda_n|\} \left\|\sum_{i=1}^{n} e_i\right\| \|b\|\|a\|.$$

Since

$$(AX - XB)a = \left(\sum_{i=1}^{n} e_i\right)\left(\langle a, b\rangle - \sum_{i=1}^{n} \lambda_i\langle a, e_i\rangle\langle e_i, b\rangle\right),$$

we obtain the desired inequality. When $\{e_i\}_{i=1}^{\infty}$ is a basis of $\mathscr{H}$, we can omit the number 1 in the maximum by the last assertion of Theorem 2.2. □

3. The Hilbert C^*-module case

The following theorem is Buzano's inequality in the context of Hilbert C^*-modules.

Theorem 3.1. *Let $\mathscr{X}$ be a Hilbert C^*-module. If $x, y, z \in \mathscr{X}$ such that $\langle x, z\rangle$ commutes with $\langle z, z\rangle$, then*

$$|2\langle x, z\rangle\langle z, y\rangle - \langle z, z\rangle\langle x, y\rangle| \leq \|x\|\|z\|^2|y|. \tag{3}$$

Proof. For $x, y, z \in \mathscr{X}$, we have

$$\begin{aligned} |2\langle x, z\rangle\langle z, y\rangle - \langle z, z\rangle\langle x, y\rangle| &= |\langle 2z\langle z, x\rangle, y\rangle - \langle x\langle z, z\rangle, y\rangle| \\ &= |\langle 2z\langle z, x\rangle - x\langle z, z\rangle, y\rangle| \\ &\leq \|2z\langle z, x\rangle - x\langle z, z\rangle\|\,|y| \end{aligned} \tag{4}$$

and

$$\begin{aligned} \|2z\langle z, x\rangle - x\langle z, z\rangle\|^2 &= \|\langle 2z\langle z, x\rangle - x\langle z, z\rangle, 2z\langle z, x\rangle - x\langle z, z\rangle\rangle\| \\ &= \|4\langle x, z\rangle\langle z, z\rangle\langle z, x\rangle - 2\langle x, z\rangle\langle z, x\rangle\langle z, z\rangle \\ &\quad -2\langle z, z\rangle\langle x, z\rangle\langle z, x\rangle + \langle z, z\rangle\langle x, x\rangle\langle z, z\rangle\| \\ &\leq \|z\|^4\|x\|^2. \end{aligned} \tag{5}$$

Now (3) follows from (4) and (5). □

Using Theorem 3.1 and the fact that, in a C^*-algebra, the relation $|c| \leq M$ is equivalent to the condition that $|cd| \leq M|d|$ for all d, we get

Corollary 3.2. *If $a, b \in \mathscr{A}$ are elements of a C^*-algebra such that a^*b commutes with b^*b, then*

$$|2a^*bb^* - b^*ba^*| \leq \|a\|\|b\|^2.$$

The following provides a non-trivial example.

Example 3.3. Let $\mathscr{H}$ be a separable complex Hilbert space and let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Define the operator $u : \mathscr{H} \to \mathscr{H}$ by

$$u(e_i) = \begin{cases} e_{i+1} & , \quad i \leq n \\ 0 & , \quad i > n \end{cases}.$$

Then the adjoint operator u^* is defined by $u^*(e_i) = \begin{cases} e_{i-1} & , \quad 2 \leq i \leq n+1 \\ 0 & , \quad i > n+1 \text{ or } i = 1 \end{cases}$. If $\mathscr{K}_1, \mathscr{K}_2$ are the subspaces generated with $\{e_1, \cdots, e_n\}$ and $\{e_2, \cdots, e_{n+1}\}$, respectively, then u^*u is the projection onto $\mathscr{K}_1$ and uu^* is the projection onto $\mathscr{K}_2$. For all $v \in \mathbb{B}(\mathscr{H})$, we clearly have $vu = 0$ on $\mathscr{K}_1^{\perp}$. Therefore, if $v(\mathscr{K}_2) \subseteq \mathscr{K}_1$, then vu commutes with u^*u, so that we have

$$\|2vuu^* - u^*uv\| \leq \|v\|.$$

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