Filomat 26:4 (2012), 839–844
DOI 10.2298/FIL1204839G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Remarks on $S_i$-separation axioms

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**Abstract.** In this paper, a space is a pair $(X, \mathscr{K})$, where $X$ is a set and $K \subset \exp X$. This paper gives some new characterizations for $S_i$-separation axioms in the space $(X, \mathscr{K})$ $(i = 1, 2)$. As some corollaries of these results, some characterizations for $T_i$-separation axioms in the space $(X, \mathscr{K})$ are obtained $(i = 1, 2)$.

## 1. Introduction

In the general theory of topological spaces, separation axioms had played an important role. In a series of papers, the ordinary separation axioms are modified in the way that the role of open sets is given to other classes of sets (see e.g. [2, 3, 5, 7, 8]). Moreover, Arenas et al. [1, 6] studied some weak separation axioms related with Alexandroff topological spaces. In [3], A. Császár discussed some lower separation axioms $T_0$, $T_1$, $T_2$, $S_1$ and $S_2$ in generalized topological spaces, and gave some "nice" characterizations for these separation axioms. Having gained some enlightenment from results on separation axioms obtained by A. Császár in [3], this paper investigates $T_i$-separation axioms and $S_i$-separation axioms $(i = 1, 2)$, and obtain some new characterizations for these separation axioms in the space $(X, \mathscr{K})$.

In this paper, a space is a pair $(X, \mathscr{K})$, where $X$ is a set and $\mathscr{K} \subset \exp X$. Throughout this paper, we use the following notations.

**Notation 1.1.** Let $(X, \mathscr{K})$ be a space and $A \subset X$.
(1) $\kappa A = \{x : x \in K \in \mathscr{K} \textit{ implies } K \bigcap A \neq \emptyset\}$.
(2) $\chi A = \bigcap\{K : A \subset K \in \mathscr{K}\}$.
(3) $\overline{\chi} A = \bigcap\{\kappa K : A \subset K \in \mathscr{K}\}$.

**Remark 1.2.** ([3]) (1) In the sense of Notation 1.1, if no $K \in \mathscr{K}$ satisfies $x \in K$, then $x \in \kappa A$.
(2) In particular, $\chi A = X$ and $\overline{\chi} A = X$ if there do not exist sets $K \subset \mathscr{K}$ satisfying $A \subset K$.

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2010 *Mathematics Subject Classification*. Primary 54A05; Secondary 54D10
*Keywords*. $S_1$-separation axiom, $S_2$-separation axiom, $T_0$-separation axiom, $T_1$-separation axiom, $T_2$-separation axiom
Received: 11 October 2011; Revised: 02 January 2012; Accepted: 13 January 2012
Communicated by Ljubiša D.R. Kočinac
Research supported by the National Natural Science Foundation of China (No. 10971185 and 11061004) and the Fok Ying Tong Education Foundation(No. 114002)
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## 2. Preliminaries

Let us recall $T_i$-separation axioms ($i = 0, 1, 2$) and $S_i$-separation axioms ($i = 1, 2$), which come from [3].

**Definition 2.1.** Let $(X, \mathscr{K})$ be a space.

(1) $T_0$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathscr{K}$ containing precisely one of $x$ and $y$.

(2) $T_1$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathscr{K}$ such that $x \in K$ and $y \notin K$.

(3) $T_2$-separation axiom: $x, y \in X$ and $x \neq y$ imply the existence of $K, K' \in \mathscr{K}$ such that $x \in K$, $y \in K'$ and $K \bigcap K' = \emptyset$.

(4) $S_1$-separation axiom: If $x, y \in X$ and there exists $K \in \mathscr{K}$ such that $x \in K$ and $y \notin K$, then there exists $K' \in \mathscr{K}$ satisfying $y \in K'$ and $x \notin K'$.

(5) $S_2$-separation axiom: If $x, y \in X$ and there exists $K \in \mathscr{K}$ such that $x \in K$ and $y \notin K$, then there exist $K', K'' \in \mathscr{K}$ satisfying $x \in K'$, $y \in K''$ and $K' \bigcap K'' = \emptyset$.

**Remark 2.2.** Some of the consequences of these separation axioms are valid in this generality. In particularly, the following hold.

(1) $T_2$-separation axiom $\Longrightarrow$ $T_1$-separation axiom $\Longrightarrow$ $T_0$-separation axiom.

(2) $S_2$-separation axiom $\Longrightarrow$ $S_1$-separation axiom.

(3) $T_1$-separation axiom $\Longleftrightarrow$ $T_0$- and $S_1$-separation axiom.

(4) $T_2$-separation axiom $\Longleftrightarrow$ $T_0$- and $S_2$-separation axiom.

The following belong to A. Császár [3].

**Lemma 2.3.** ([3]) *Let $(X, \mathscr{K})$ be a space. Given $x, y \in X$, $\kappa\{x\} \neq \kappa\{y\}$ if and only if there exists $K \in \mathscr{K}$ containing precisely one of $x$ and $y$. Thus, $\mathscr{K}$ satisfies $T_0$-separation axiom if and only if $\kappa\{x\} \neq \kappa\{y\}$ for all $x, y \in X$.*

**Lemma 2.4.** ([3]) *Let $(X, \mathscr{K})$ be a space. $\mathscr{K}$ satisfies $S_1$-separation axiom if and only if $x \in K \in \mathscr{K}$ implies $\kappa\{x\} \subset K$.*

**Definition 2.5.** ([3]) Let $X$ be a set. A mapping $\lambda : \exp X \longrightarrow \exp X$ is called an envelope operation (or briefly an envelope) on X if the following hold (We write $\lambda A$ for $\lambda(A)$).

(1) $A \subset \lambda A$ for $A \subset X$.

(2) $\lambda A \subset \lambda B$ for $A \subset B \subset X$.

(3) $\lambda\lambda A = \lambda A$ for $A \subset X$.

**Lemma 2.6.** ([3]) *Let $\kappa : \exp X \longrightarrow \exp X$ and $\chi : \exp X \longrightarrow \exp X$ be defined as Notation 1.1. Then both $\kappa$ and $\chi$ are envelopes on X, and hence the following hold.*

(1) *$x \in \kappa\{x\}$, $x \in \chi\{x\}$ and $\kappa\{x\} \subset \overline{\chi}\{x\}$.*

(2) *If $x \in \kappa\{y\}$, then $\kappa\{x\} \subset \kappa\{y\}$.*

(3) *If $x \in \chi\{y\}$, then $\chi\{x\} \subset \chi\{y\}$.*

## 3. The main results

For a space $(X, \mathscr{K})$ and $x \in X$, we write $\mathscr{K}_x = \{K : x \in K \in \mathscr{K}\}$ for the sake of convenience. Consequently, $x \in K \in \mathscr{K}$ if and only if $K \in \mathscr{K}_x$. Thus, "$K \in \mathscr{K}_x$" denotes "$x \in K \in \mathscr{K}$" in this section.

By the definitions of $\kappa$, $\chi$ and $\overline{\chi}$, the following remark is obvious.

**Remark 3.1.** Let $(X, \mathscr{K})$ be a space and $x \in X$. Then the following hold.

(1) $\kappa\{x\} = \{y : K \in \mathscr{K}_y \text{ implies } x \in K\}$, i.e., $y \in \kappa\{x\}$ if and only if $x \in K$ for each $K \in \mathscr{K}_y$.

(2) $\chi\{x\} = \bigcap\{K : K \in \mathscr{K}_x\}$, i.e., $y \in \chi\{x\}$ if and only if $y \in K$ for each $K \in \mathscr{K}_x$.

(3) $\overline{\chi}\{x\} = \bigcap\{\kappa K : K \in \mathscr{K}_x\}$.

(4) $y \notin \kappa\{x\}$ if and only if there exists $K \in \mathscr{K}_y$ such that $x \notin K$.

(5) $y \notin \chi\{x\}$ if and only if there exists $K \in \mathscr{K}_x$ such that $y \notin K$.

(6) $y \notin \overline{\chi}\{x\}$ if and only if there exists $K \in \mathscr{K}_x$ such that $y \notin \kappa K$.

**Lemma 3.2.** *Let $(X, \mathscr{K})$ be a space and $x \in X$. Then $\kappa\{x\} = X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$.*

*Proof.* Let $y \in \kappa\{x\}$. By Remark 3.1(1), $x \in K$ for each $K \in \mathscr{K}_y$. So, for each $K \in \mathscr{K}$, $y \notin K$ if $K \notin \mathscr{K}_x$. That is, for each $K \in \mathscr{K} - \mathscr{K}_x$, $y \notin K$. It follows that $y \notin \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$, and so $y \in X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$. On the other hand, let $y \in X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$. Then we have $y \in \kappa\{x\}$ by reversing the proof above. This proves that $\kappa\{x\} = X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$. □

**Lemma 3.3.** *Let $(X, \mathscr{K})$ be a space and $x, y \in X$. Then the following are equivalent.*

(1) *$K \bigcap \{x, y\} = \{x, y\}$ for each $K \in \mathscr{K}_x$.*
(2) *$y \in \chi\{x\}$.*
(3) *$x \in \kappa\{y\}$.*
(4) *$\mathscr{K}_x \subset \mathscr{K}_y$.*
(5) *$\chi\{y\} \subset \chi\{x\}$.*
(6) *$\kappa\{x\} \subset \kappa\{y\}$.*

*Proof.* (1) $\Longrightarrow$ (2): Let $K \bigcap \{x, y\} = \{x, y\}$ for each $K \in \mathscr{K}_x$. Then $y \in K$ for each $K \in \mathscr{K}_x$. By Remark 3.1(2), $y \in \chi\{x\}$.

(2) $\Longrightarrow$ (3): It holds from Remark 3.1(1),(2).

(3) $\Longrightarrow$ (4): Let $x \in \kappa\{y\}$. By Lemma 3.2, $x \in X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_y\}$, i.e., $x \notin \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_y\}$. So, if $K \in \mathscr{K} - \mathscr{K}_y$, then $x \notin K$. Consequently, if $K \in \mathscr{K}_x$, then $K \in \mathscr{K}_y$. This proves that $\mathscr{K}_x \subset \mathscr{K}_y$.

(4) $\Longrightarrow$ (5): Let $\mathscr{K}_x \subset \mathscr{K}_y$. Then $\chi\{y\} = \bigcap\{K : K \in \mathscr{K}_y\} \subset \bigcap\{K : K \in \mathscr{K}_x\} = \chi\{x\}$.

(5) $\Longrightarrow$ (1) Let $\chi\{y\} \subset \chi\{x\}$. For each $K \in \mathscr{K}_x$, since $y \in \chi\{y\} \subset \chi\{x\}$, $y \in K$. Note that $x \in K$. It follows that $K \bigcap \{x, y\} = \{x, y\}$.

(4) $\Longrightarrow$ (6) Let $\mathscr{K}_x \subset \mathscr{K}_y$. Then $\mathscr{K} - \mathscr{K}_y \subset \mathscr{K} - \mathscr{K}_x$, and hence $\bigcup\{K : K \in \mathscr{K} - \mathscr{K}_y\} \subset \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\}$. By Lemma 3.2, $\kappa\{x\} = X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_x\} \subset X - \bigcup\{K : K \in \mathscr{K} - \mathscr{K}_y\} = \kappa\{y\}$.

(6) $\Longrightarrow$ (3) Let $\kappa\{x\}) \subset \kappa\{y\}$. Then $x \in \kappa\{x\} \subset \kappa\{y\}$. □

**Lemma 3.4.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*

(1) *$(X, \mathscr{K})$ satisfies $S_1$-separation axiom.*
(2) *For every pair $x, y \in X$, $x \notin \chi\{y\}$ implies $y \notin \chi\{x\}$.*
(3) *For every pair $x, y \in X$, $x \in \chi\{y\}$ implies $y \in \chi\{x\}$.*
(4) *For every pair $x, y \in X$, $x \notin \kappa\{y\}$ implies $y \notin \kappa\{x\}$.*
(5) *For every pair $x, y \in X$, $x \in \kappa\{y\}$ implies $y \in \kappa\{x\}$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $S_1$-separation axiom. Let $x, y \in X$ and $x \notin \chi\{y\}$. Then there exists $K \in \mathscr{K}$ such that $y \in K$ and $x \notin K$. Since $(X, \mathscr{K})$ satisfies $S_1$-separation axiom, there exists $K' \in \mathscr{K}$ such that $x \in K'$ and $y \notin K'$. So $y \notin \chi\{x\}$.

(2) $\Longrightarrow$ (1): Assume that (2) holds. Let $x, y \in X$ and let there exist $K \in \mathscr{K}$ such that $x \in K$, $y \notin K$. Then $y \notin \chi\{x\}$. Since (2) holds, $x \notin \chi\{y\}$. So there exists $K' \in \mathscr{K}$ such that $y \in K'$ and $x \notin K'$. This proves that $(X, \mathscr{K})$ satisfies $S_1$-separation axiom.

(2) $\Longleftrightarrow$ (3): It is clear.

(4) $\Longleftrightarrow$ (5): It is clear.

(3) $\Longrightarrow$ (5): Assume that (3) holds. Let $x, y \in X$ and $x \in \kappa\{y\}$. By Lemma 3.3, $y \in \chi\{x\}$, and so $x \in \chi\{y\}$. By Lemma 3.3 again, $y \in \kappa\{x\}$.

(5) $\Longrightarrow$ (3): The proof is similar to that of (3) $\Longrightarrow$ (5). □

**Theorem 3.4.1.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*

(1) *$(X, \mathscr{K})$ satisfies $S_1$-separation axiom.*
(2) *For each $x \in X$, $\chi\{x\} = \kappa\{x\}$.*
(3) *For each $x \in X$, $\chi\{x\} \subset \kappa\{x\}$.*
(4) *For each $x \in X$, $\kappa\{x\} \subset \chi\{x\}$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $S_1$-separation axiom. Let $x \in X$. If $y \notin \kappa\{x\}$, then there exists $K \in \mathscr{K}_y$ such that $x \notin K$. Since $(X, \mathscr{K})$ satisfies $S_1$-separation axiom, there exists $K' \in \mathscr{K}_x$ such that $y \notin K'$, and so $y \notin \chi\{x\}$. This proves that $\chi\{x\} \subset \kappa\{x\}$. On the other hand, if $y \notin \chi\{x\}$, then there exists $K \in \mathscr{K}_x$ such that $y \notin K$. Since $(X, \mathscr{K})$ satisfies $S_1$-separation axiom, there exists $K' \in \mathscr{K}_y$ such that $x \notin K'$, and so $y \notin \kappa\{x\}$. This proves that $\kappa\{x\} \subset \chi\{x\}$. Consequently, $\chi\{x\} = \kappa\{x\}$

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (4): Assume that (3) holds. Let $x \in X$. If $y \in \kappa\{x\}$, then $x \in \chi\{y\}$ from Lemma 3.3. Since $\chi\{y\} \subset \kappa\{y\}$, $x \in \kappa\{y\}$. By Lemma 3.3 again, $y \in \chi\{x\}$. This proves that $\kappa\{x\} \subset \chi\{x\}$.

(4) $\Longrightarrow$ (1): Assume that (4) holds. Let $x \in K \in \mathscr{K}$, then $\kappa\{x\} \subset \chi\{x\}$. By the definition of $\chi\{x\}$, $\chi\{x\} \subset K$. So $\kappa\{x\} \subset \chi\{x\} \subset K$. By Lemma 2.4, $(X, \mathscr{K})$ satisfies $S_1$-separation axiom. □

Let $(X, \mathscr{K})$ be a space. Recall $\mathscr{K}$ is called a generalized topology in $X$ if $\mathscr{K}' \subset \mathscr{K}$ implies $\bigcup\{K : K \in \mathscr{K}'\} \in \mathscr{K}$; $(X, \mathscr{K})$ is called a generalized topological space if $\mathscr{K}$ is a generalized topology in $X$. We call a family $\{F_x : x \in X\}$ of subsets of a set $X$ constitutes a partition of $X$ if for every pair $x, y \in X$, $F_x = F_y$ or $F_x \bigcap F_y = \emptyset$. In [3], A. Császár obtained the following proposition.

**Proposition 3.5.** ([3]) *Let $(X, \mathscr{K})$ be a generalized topological space. If $(X, \mathscr{K})$ satisfies $S_1$-separation axiom, then $\{\kappa\{x\} : x \in X\}$ constitutes a partition of $X$.*

The following theorem improve Proposition 3.5 by omitting "generalized topological" in Proposition 3.5.

**Theorem 3.5.1.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*
(1) *$(X, \mathscr{K})$ satisfies $S_1$-separation axiom.*
(2) *$\{\kappa\{x\} : x \in X\}$ constitutes a partition of $X$.*
(3) *$\{\chi\{x\} : x \in X\}$ constitutes a partition of $X$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $S_1$-separation axiom. Let $x, y \in X$ and $\kappa\{x\} \bigcap \kappa\{y\} \neq \emptyset$, Then there exists $z \in \kappa\{x\} \bigcap \kappa\{y\}$. By Lemma 3.4, $x \in \kappa\{z\}$ since $z \in \kappa\{x\}$. So $\kappa\{z\} \subset \kappa\{x\}$ and $\kappa\{x\} \subset \kappa\{z\}$ by Lemma 2.6. It follows that $\kappa\{x\} = \kappa\{z\}$. Similarly, $\kappa\{y\} = \kappa\{z\}$. Thus $\kappa\{x\} = \kappa\{y\}$. This proves that $\{\kappa\{x\} : x \in X\}$ constitutes a partition of $X$.

(2) $\Longrightarrow$ (3): Assume that (2) holds. Let $x, y \in X$ and $\chi\{x\} \bigcap \chi\{y\} \neq \emptyset$, Then there exists $z \in \chi\{x\} \bigcap \chi\{y\}$. By Lemma 2.6, $\chi\{z\} \subset \chi\{x\}$ since $z \in \chi\{x\}$. On the other hand, $x \in \kappa\{z\}$ by Lemma 3.3. Thus $x \in \kappa\{x\} \bigcap \kappa\{z\} \neq \emptyset$, so $\kappa\{x\} = \kappa\{z\}$, and hence $z \in \kappa\{z\} = \kappa\{x\}$. So $x \in \chi\{z\}$. It follows that $\chi\{x\} \subset \chi\{z\}$. This proves that $\chi\{x\} = \chi\{z\}$. Similarly, $\chi\{y\} = \chi\{z\}$. Consequently, $\chi\{x\} = \chi\{y\}$. So $\{\chi\{x\} : x \in X\}$ constitutes a partition of $X$.

(3) $\Longrightarrow$ (1): Assume that (3) holds. Let $x, y \in X$ and $y \notin \chi\{x\}$. By Lemma 3.4, it suffices to prove that $x \notin \chi\{y\}$. Since $y \in \chi\{y\}$, so $\chi\{x\} \neq \chi\{y\}$, and hence $\chi\{x\} \bigcap \chi\{y\} = \emptyset$. Since $x \in \chi\{x\}$, so $x \notin \chi\{y\}$. □

**Theorem 3.5.2.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*
(1) *$(X, \mathscr{K})$ satisfies $S_2$-separation axiom.*
(2) *$x \in K \in \mathscr{K}$ implies $\overline{\chi}\{x\} \subset K$.*
(3) *$\overline{\chi}\{x\} = \kappa\{x\}$ for each $x \in X$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $S_2$-separation axiom. Let $x \in K \in \mathscr{K}$ and $y \in \overline{\chi}\{x\}$. It suffices to prove that $y \in K$. In fact, if $y \notin K$, then there exist $K', K'' \in \mathscr{K}$ such that $x \in K'$, $y \in K''$ and $K' \bigcap K'' = \emptyset$. Thus $y \notin \kappa K'$. Note that $K' \in \mathscr{K}_x$. So $y \notin \overline{\chi}\{x\}$. This is a contradiction.

(2) $\Longrightarrow$ (1): Assume that (2) holds. Let $x, y \in X$ and let there exist $K \in \mathscr{K}$ such that $x \in K$, $y \notin K$. Then $y \notin \overline{\chi}\{x\}$. So there exists $K' \in \mathscr{K}_x$ such that $y \notin \kappa K'$. It follows that there exists $K'' \in \mathscr{K}_y$ such that $K' \bigcap K'' = \emptyset$. This proves that $(X, \mathscr{K})$ satisfies $S_2$-separation axiom.

(1) $\Longrightarrow$ (3): Assume that $(X, \mathscr{K})$ satisfies $S_2$-separation axiom. Let $x \in X$. By Lemma 3.3, we only need to prove $\overline{\chi}\{x\} \subset \kappa\{x\}$. Let $y \in \overline{\chi}\{x\}$. It suffices to prove that $y \in \kappa\{x\}$. In fact, if $y \notin \kappa\{x\}$, then there exists $K \in \mathscr{K}_y$ such that $x \notin K$. And so there exist $K', K'' \in \mathscr{K}$ such that $x \in K'$, $y \in K''$ and $K' \bigcap K'' = \emptyset$. Thus $y \notin \kappa K'$. Note that $K' \in \mathscr{K}_x$. So $y \notin \overline{\chi}\{x\}$. This is a contradiction.

(3) $\Longrightarrow$ (1): Assume that (3) holds. Let $x, y \in X$ and let there exist $K \in \mathscr{K}$ such that $x \in K$, $y \notin K$. Then $x \notin \kappa\{y\}$, and hence $x \notin \overline{\chi}(y)$. So there exists $K' \in \mathscr{K}_y$ such that $x \notin \kappa K'$. It follows that there exists $K'' \in \mathscr{K}_x$ such that $K' \bigcap K'' = \emptyset$. This proves that $(X, \mathscr{K})$ satisfies $S_2$-separation axiom. □

Taking Lemma 3.4 and Theorem 3.5.1 into account, the following question is interesting.

**Question 3.6.** *Let $(X, \mathscr{K})$ be a space. Are the following equivalent.*
(1) *$(X, \mathscr{K})$ satisfies $S_2$-separation axiom.*
(2) *For every pair $x, y \in X$, $x \notin \overline{\chi}\{y\}$ implies $y \notin \overline{\chi}\{x\}$.*
(3) *$\{\overline{\chi}\{x\} : x \in X\}$ constitutes a partition of $X$.*

The following answer the above question.

**Proposition 3.7.** *Let $(X, \mathscr{K})$ be a space. Then, for every pair $x, y \in X$, $x \notin \overline{\chi}\{y\}$ implies $y \notin \overline{\chi}\{x\}$.*

*Proof.* Let $x, y \in X$. If $x \notin \overline{\chi}\{y\}$, then there exists $K \in \mathscr{K}_y$ such that $x \notin \kappa K$, and hence there exists $K' \in \mathscr{K}_x$ such that $K' \bigcap K = \emptyset$. Thus, $y \notin \kappa K'$. So $y \notin \overline{\chi}\{x\}$. □

**Proposition 3.8.** *Let $(X, \mathscr{K})$ be a space. If $(X, \mathscr{K})$ satisfies $S_2$-separation axiom, then $\{\overline{\chi}\{x\} : x \in X\}$ constitutes a partition of $X$.*

*Proof.* Assume that $(X, \mathscr{K})$ satisfies $S_2$-separation axiom. By Remark 2.2(2) and Proposition 3.6, $\{\kappa\{x\} : x \in X\}$ constitutes a partition of $X$. Also, by Theorem 3.5.2, $\overline{\chi}\{x\} = \kappa\{x\}$ for each $x \in X$. So $\{\overline{\chi}\{x\} : x \in X\}$ constitutes a partition of $X$. □

**Example 3.9.** There exists a space $(X, \mathscr{K})$ such that $\{\overline{\chi}\{x\} : x \in X\}$ constitutes a partition of $X$, and $(X, \mathscr{K})$ does not satisfy $S_1$-separation axiom.

Put $X = \{a, b, c, d\}$ and $\mathscr{K} = \{\{a\}, \{a, b\}, \{c\}, \{c, d\}\}$. It is not difficult to check that $\kappa K = \{a, b\}$ if $K \in \{\{a\}, \{a, b\}\}$, and $\kappa K = \{c, d\}$ if $K \in \{\{c\}, \{c, d\}\}$. So $\overline{\chi}\{x\} = \{a, b\}$ if $x \in \{a, b\}$, and $\overline{\chi}\{x\} = \{c, d\}$ if $x \in \{c, d\}$. Then $\{\overline{\chi}\{x\} : x \in X\}$ constitutes a partition $\{\{a, b\}, \{c, d\}\}$ of $X$. Since $\kappa\{a\} = \{a, b\}$ and $\chi\{a\} = \{a\}$, $\kappa\{a\} \neq \chi\{a\}$, so $(X, \mathscr{K})$ does not satisfy $S_1$-separation axiom.

As some applications of Theorem 3.4.1, Theorem 3.5.1 and Theorem 3.5.2, we give some characterizations of $T_i$-separation axiom ($i = 1, 2$).

**Theorem 3.9.1.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*
(1) *$(X, \mathscr{K})$ satisfies $T_1$-separation axiom.*
(2) *For every pair $x, y \in X$, $x \neq y$ implies $\kappa\{x\} \bigcap \kappa\{y\} = \emptyset$.*
(3) *For each $x \in X$, $\kappa\{x\} = \{x\}$.*
(4) *For every pair $x, y \in X$, $x \neq y$ implies $\chi\{x\} \bigcap \chi\{y\} = \emptyset$.*
(5) *For each $x \in X$, $\chi\{x\} = \{x\}$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $T_1$-separation axiom. Let $x, y \in X$ and $x \neq y$. By Remark 2.2(3), $(X, \mathscr{K})$ satisfies $T_0$- and $S_1$-separation axiom. So $\kappa\{x\} \neq \kappa\{y\}$ from Lemma 2.3, and hence $\kappa\{x\} \bigcap \kappa\{y\} = \emptyset$ from Theorem 3.5.1.

(2) $\Longrightarrow$ (3): Assume that (2) holds. Let $x \in X$, then $x \in \kappa\{x\}$. For each $y \in X - \{x\}$, since $y \in \kappa\{y\}$ and $\kappa\{x\} \bigcap \kappa\{y\} = \emptyset$, $y \notin \kappa\{x\}$. It follows that $\kappa\{x\} = \{x\}$.

(3) $\Longrightarrow$ (1): Assume that (3) holds. For every pair $x, y \in X$, if $x \neq y$, then $\kappa\{x\} = \{x\} \neq \{y\} = \kappa\{y\}$, So $(X, \mathscr{K})$ satisfies $T_0$-separation axiom from Lemma 2.3. On the other hand, for each $x \in X$, $\kappa\{x\} = \{x\} \subset \chi\{x\}$. By Theorem 3.4.1, $(X, \mathscr{K})$ satisfies $S_1$-separation axiom. Thus, $(X, \mathscr{K})$ satisfies $T_1$-separation axiom from Remark 2.2(3).

(1) $\Longrightarrow$ (4): Assume that $(X, \mathscr{K})$ satisfies $T_1$-separation axiom. Then $(X, \mathscr{K})$ satisfies $S_1$-separation axiom from Remark 2.2(3). Let $x, y \in X$ and $x \neq y$. Then $\chi\{x\} = \kappa\{x\}$ and $\chi\{y\} = \kappa\{y\}$ from Theorem 3.4.1. By the above (1) $\Longrightarrow$ (2), $\kappa\{x\} \bigcap \kappa\{y\} = \emptyset$. It follows that $\chi\{x\} \bigcap \chi\{y\} = \emptyset$.

(4) $\Longrightarrow$ (5) $\Longrightarrow$ (1): The proof can be completed by a similar way as in the proof of (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), so we omit it. □

**Theorem 3.9.2.** *Let $(X, \mathscr{K})$ be a space. Then the following are equivalent.*

(1) *$(X, \mathscr{K})$ satisfies $T_2$-separation axiom.*

(2) *$x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathscr{K}$ such that $x \in K$ and $y \notin \kappa K$.*

(3) *$x, y \in X$ and $x \neq y$ imply the existence of $K \in \mathscr{K}$ such that $x \in K \subset \kappa K \subset X - \{y\}$.*

(4) *For each $x \in X$, $\overline{\chi}\{x\} = \{x\}$.*

(5) *For every pair $x, y \in X$, $x \neq y$ implies $\overline{\chi}\{x\} \bigcap \overline{\chi}\{y\} = \emptyset$.*

*Proof.* (1) $\Longrightarrow$ (2): Assume that $(X, \mathscr{K})$ satisfies $T_2$-separation axiom. Let $x, y \in X$ and $x \neq y$. By Remark 2.2(1),(4), $(X, \mathscr{K})$ satisfies $S_2$- and $T_1$-separation axiom. By Theorem 3.5.2 and Theorem 3.9.1, $\overline{\chi}\{x\} = \kappa\{x\} = \{x\}$, and hence $y \notin \overline{\chi}\{x\}$. Thus, there exists $K \in \mathscr{K}$ such that $x \in K$ and $y \notin \kappa K$.

(2) $\Longrightarrow$ (3): Assume that (2) holds. Let $x, y \in X$ and $x \neq y$. Then there exists $K \in \mathscr{K}$ such that $x \in K$ and $y \notin \kappa K$. Thus $x \in K \subset \kappa K \subset X - \{y\}$.

(3) $\Longrightarrow$ (4): Assume that (3) holds. Let $x \in X$. If $y \in X$ and $x \neq y$, then there exists $K \in \mathscr{K}$ such that $x \in K \subset \kappa K \subset X - \{y\}$. Thus, $K \in \mathscr{K}_x$ and $y \notin \kappa K$, so $y \notin \overline{\chi}\{x\}$. This proves that $\overline{\chi}\{x\} = \{x\}$.

(4) $\Longrightarrow$ (1): Assume that (4) holds. Let $x \in K \in \mathscr{K}$. Then $\overline{\chi}\{x\} = \{x\} \subset K$. So $(X, \mathscr{K})$ satisfies $S_2$-separation axiom from Theorem 3.5.2. In addition, $\{x\} \subset \chi\{x\} \subset \overline{\chi}\{x\} = \{x\}$, so $\chi\{x\} = \{x\}$. By Theorem 3.9.1, $(X, \mathscr{K})$ satisfies $T_1$-separation axiom. It follows that $(X, \mathscr{K})$ satisfies $T_2$-separation axiom from Remark 2.2(1),(4).

(4) $\Longrightarrow$ (5): Assume that (4) holds. Let $x, y \in X$ and $x \neq y$. Then $\overline{\chi}\{x\} \bigcap \overline{\chi}\{y\} = \{x\} \bigcap \{y\} = \emptyset$.

(5) $\Longrightarrow$ (4): Assume that (5) holds. Let $x \in X$, then $x \in \overline{\chi}\{x\}$. For each $y \in X - \{x\}$, since $y \in \overline{\chi}\{y\}$ and $\overline{\chi}\{x\} \bigcap \overline{\chi}\{y\} = \emptyset$, $y \notin \overline{\chi}\{x\}$. It follows that $\overline{\chi}\{x\} = \{x\}$. □

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