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On the multidimensional Hilbert-type inequalities involving the Hardy operator

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Abstract. This paper deals with the multidimensional Hilbert-type inequalities involving the Hardy operator and homogeneous kernels. The main results are established in the setting with the non-conjugate exponents. After reduction to the conjugate case, the inequalities with the best possible constant factors are obtained in some general cases. As an application, some particular settings are considered in order to obtain the multidimensional extensions of some recent results, known from the literature.

1. Introduction

Let p and q be conjugate exponents, that is, $1/p + 1/q = 1$, $p > 1$. One of the earliest variants of the classical Hilbert inequality, that holds for all non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$, is

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}\,dxdy \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p \|g\|_q, \tag{1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible in the sense that it can not be replaced with a smaller constant (see [10]). Throughout this paper $\|\cdot\|_r$ denotes the usual norm in $L^r(\mathbb{R}_+)$, i.e. $\|f\|_r = (\int_{\mathbb{R}_+} f^r(x)dx)^{1/r}$.

The Hilbert inequality is very important in mathematical analysis and its applications and, although classical, is still a field of interest of numerous mathematicians. During decades, it was generalized in many different directions, such as different choices of kernels, sets of integration etc. The resulting inequalities are usually called the Hilbert-type inequalities. For more details about the Hilbert inequality the reader is referred to [9] and [13].

Shortly after discovery of the Hilbert inequality, Hardy, Littlewood and Pólya noted that to every Hilbert-type inequality one can assign its equivalent form. For example, the equivalent inequality assigned to (1) reads

$$\left[\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \frac{f(x)}{x+y}dx\right)^p dy\right]^{\frac{1}{p}} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p, \tag{2}$$

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where $p > 1$ and $f \in L^p(\mathbb{R}_+)$. However, the constant factor included in the right-hand side of (2) is also the best possible. Inequalities related to (2) are usually called the Hardy-Hilbert type inequalities. In this paper, inequalities related to (1) and (2) will simply be referred to as the Hilbert-type inequalities.

As we have already seen, some of the most important contributions in development of the Hilbert inequality are due to Hardy. On the other hand, in 1925, Hardy stated and proved the following integral inequality:

$$\left[\int_{\mathbb{R}_+}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right]^{\frac{1}{p}} \leq \frac{p}{p-1}\|f\|_p, \tag{3}$$

where $p > 1$, and $f \in L^p(\mathbb{R}_+)$ is a non-negative function. This is the original form of the famous Hardy integral inequality, which later on has been extensively studied and used as a model example for investigations of more general integral inequalities. The Hardy inequality includes the integral operator $\mathcal{H}$ defined by

$$\mathcal{H}f(x) = \int_0^x f(t)dt. \tag{4}$$

The integral operator $\mathcal{H}$ is usually called the Hardy operator. For more details about the Hardy inequality, its history and applications, the reader is referred to [12].

In the recent years, several authors considered the Hilbert-type inequalities involving the Hardy operator $\mathcal{H}$. For example, Das and Sahoo [6], obtained the following two inequalities:

$$\int_{\mathbb{R}^2_+}\frac{x^{r_1-\frac{1}{q}-1}y^{r_2-\frac{1}{p}-1}}{(x+y)^s}(\mathcal{H}f)(x)(\mathcal{H}g)(y)dxdy \leq pqB(r_1,r_2)\|f\|_p\|g\|_q, \tag{5}$$

$$\left[\int_{\mathbb{R}_+} y^{r_2p-1}\left(\int_{\mathbb{R}_+}\frac{x^{r_1-\frac{1}{q}-1}}{(x+y)^s}(\mathcal{H}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} \leq qB(r_1,r_2)\|f\|_p, \tag{6}$$

where p and q, $p > 1$ are conjugate exponents, $r_1, r_2 > 0$, $s = r_1 + r_2$, $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$, $f, q \geq 0$. They also showed that the constant factors $pqB(r_1,r_2)$ and $qB(r_1,r_2)$, where $B(\cdot,\cdot)$ denotes the usual Beta function, are the best possible. Very similar inequalities were also studied in the paper [8].

Considering the kernel $K(x,y) = 1/\max\{x^s, y^s\}$, Das and Sahoo [7], established the inequalities

$$\int_{\mathbb{R}^2_+}\frac{x^{r_1-\frac{1}{q}-1}y^{r_2-\frac{1}{p}-1}}{\max\{x^s,y^s\}}(\mathcal{H}f)(x)(\mathcal{H}g)(y)dxdy \leq \frac{pqs}{r_1r_2}\|f\|_p\|g\|_q, \tag{7}$$

and

$$\left[\int_{\mathbb{R}_+} y^{r_2p-1}\left(\int_{\mathbb{R}_+}\frac{x^{r_1-\frac{1}{q}-1}}{\max\{x^s,y^s\}}(\mathcal{H}f)(x)dx\right)^p dy\right]^{\frac{1}{p}} \leq \frac{qs}{r_1r_2}\|f\|_p, \tag{8}$$

under the same assumptions as in (5) and (6). Also, the constant factors involved in the right-hand sides of inequalities (7) and (8) are the best possible.

Considering the kernels $K(x,y) = 1/(x+y)^s$ and $K(x,y) = 1/\max\{x^s,y^s\}$, $s > 0$, we see that they have homogeneity of degree $-s$ in common. The main objective of this paper is to establish an unified treatment of the above inequalities (5), (6), (7) and (8). Namely, we shall deduce the generalizations of these inequalities containing arbitrary homogeneous kernel of negative degree. Also, we shall represent the results in multidimensional setting equipped with non-conjugate exponents.

The paper is organized in the following way: After this Introduction, in Section 2 we introduce definition of non-conjugate exponents in multidimensional setting and indicate some recent results about the Hilbert and the Hardy inequality in such setting. These results will be base in our main results. Further, in Section

3 we derive our main results, i.e. the multidimensional Hilbert-type inequalities involving the Hardy operator in the non-conjugate setting. In Section 4 we analyze our main results in the conjugate setting. In such a way we get the best possible constant factors in some general cases. Finally, Section 5 is dedicated to some particular settings of our main results, which yield multidimensional extensions of some recent results, mentioned in this Introduction.

2. Non-conjugate exponents, Hilbert-type and Hardy-type inequalities

In this section we refer to papers [4] and [14] which provide an unified treatment of multidimensional Hilbert-type inequalities in the setting with non-conjugate exponents. Before we state the appropriate results, we recall the definition of non-conjugate parameters.

Let p_i be the real parameters satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1, \quad p_i > 1,\ i = 1, 2, \dots, n. \tag{9}$$

The parameters p_i' are defined as associated conjugates, that is

$$\frac{1}{p_i} + \frac{1}{p_i'} = 1, \quad i = 1, 2, \dots, n. \tag{10}$$

Since $p_i > 1$, it follows that $p_i' > 1$, $i = 1, 2, \dots, n$. In addition, we define

$$\lambda_n = \frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{p_i'}. \tag{11}$$

Clearly, the relations (9) and (10) imply that $0 < \lambda_n < 1$. Finally, we introduce the parameters q_i defined by

$$\frac{1}{q_i} = \lambda_n - \frac{1}{p_i'}, \quad i = 1, 2, \dots, n, \tag{12}$$

assuming $q_i > 0$, $i = 1, 2, \dots, n$. The above conditions (9)-(12) establish the n-tuple of non-conjugate exponents and were given by Bonsall [3], more than half a century ago. The above conditions also imply relations $\lambda_n = \sum_{i=1}^{n} 1/q_i$ and $1/q_i + 1 - \lambda_n = 1/p_i$, $i = 1, 2, \dots, n$. Of course, if $\lambda_n = 1$, then $\sum_{i=1}^{n} 1/p_i = 1$, which represents the setting with conjugate parameters.

Remark 1. If $n = 2$, then non-conjugate parameters p_1 and p_2 will be denoted with p and q. Also, p' and q' will be their conjugates. Moreover, the parameter λ_2 will simply be denoted as λ.

In this paper we shall be concerned with the Hilbert-type inequalities with homogeneous kernels. Recall, the function $K : \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s}K(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. Furthermore, if $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j} \hat{d}^i\mathbf{u}, \quad i = 1, 2, \dots, n, \tag{13}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{d}^i\mathbf{u} = du_1 \dots du_{i-1} du_{i+1} \dots du_n$ and provided that the above integral converges. Note that the constant factor $k_i(\mathbf{a})$ does not depend on the component a_i. Thus, the component a_i can be replaced with arbitrary real number. This fact will sometimes be used in the sequel, for the sake of simpler notation. Further, in the sequel $d\mathbf{u}$ will denote $du_1 du_2 \dots du_n$.

The following multidimensional Hilbert-type inequalities, in the slightly altered notation, can be found in the paper [14] (see also [4]):

Theorem 2. *Let p_i, p_i', q_i, $i = 1, 2, \dots, n$, and λ_n be as in* (9)–(12), *and let A_{ij}, $i, j = 1, 2, \dots, n$, be the real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$. If $K : \mathbb{R}_+^n \to \mathbb{R}$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, are non-negative measurable functions, then the following two inequalities hold and are equivalent:*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i) d\mathbf{x} \le \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/q_i + \alpha_i} f_i\|_{p_i}, \tag{14}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s) - p_n' \alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} f_i(x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'} \le \prod_{i=1}^{n} k_i^{1/q_i}(q_i \mathbf{A_i}) \prod_{i=1}^{n-1} \|x_i^{(n-1-s)/q_i + \alpha_i} f_i\|_{p_i}, \tag{15}$$

where $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$ and $k_i(q_i \mathbf{A_i}) < \infty$, $i = 1, 2, \dots, n$.

Very recently, Čižmešija et al. investigated in the paper [5] general Hardy-type inequalities in the non-conjugate setting for $n = 2$. As a special consequence, they have obtained the inequality

$$\left[\int_0^\infty y^{-\lambda q'} (\mathcal{H} f)^{q'}(y) dy \right]^{\frac{1}{q'}} \le (p' \lambda)^{\lambda} \|f\|_p, \tag{16}$$

where $\mathcal{H}$ is the Hardy operator (4). This inequality coincides with the earlier Opic's estimate (see [11]). Clearly, for $\lambda = 1$, we obtain the Hardy inequality (3) in the original form.

The results presented in this section will be the base of our further research. Besides, all the notations presented here will be valid throughout the whole paper.

3. Multidimensional Hilbert-type inequalities in the non-conjugate setting

In this section we establish an unified treatment of the multidimensional Hilbert-type inequalities which include the Hardy operator $\mathcal{H}$ and homogeneous kernel. Our first result refers to the setting with non-conjugate exponents defined in the previous section.

Theorem 1. *Suppose p_i, p_i', q_i, $i = 1, 2, \dots, n$, and λ_n are as in* (9)–(12), *and A_{ij}, $i, j = 1, 2, \dots, n$, are the real parameters satisfying $\sum_{i=1}^{n} A_{ij} = 0$. Further, let $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $i = 1, 2, \dots, n$, and let ν_i, μ_i be real parameters satisfying*

$$\alpha_i + \nu_i + \frac{1}{p_i} < \frac{s + 1 - n}{q_i} \le \alpha_i + \nu_i + \mu_i, \quad i = 1, 2, \dots, n. \tag{17}$$

If $K : \mathbb{R}_+^n \to \mathbb{R}$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, are non-negative measurable functions, then

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n} x_i^{\nu_i} (\mathcal{H} f_i)^{\mu_i}(x_i) d\mathbf{x} \le k_n^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \nu) \prod_{i=1}^{n} \left\| f_i^{\frac{q_i \mu_i}{p_i q_i (\alpha_i + \nu_i + \mu_i) + p_i(n-1-s) + q_i}} \right\|_{p_i}^{p_i(\alpha_i + \nu_i + \mu_i) + p_i(n-1-s)/q_i + 1}, \tag{18}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s) - p_n' \alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\nu_i} (\mathcal{H} f_i)^{\mu_i}(x_i) \hat{d}^n \mathbf{x} \right)^{p_n'} dx_n \right]^{1/p_n'} \le k_{n-1}^s(\mathbf{p}, \mathbf{q}, \mathbf{A}, \nu) \times$$

$$\times \prod_{i=1}^{n-1} \left\| f_i^{\frac{q_i \mu_i}{p_i q_i (\alpha_i + \nu_i + \mu_i) + p_i(n-1-s) + q_i}} \right\|_{p_i}^{p_i(\alpha_i + \nu_i + \mu_i) + p_i(n-1-s)/q_i + 1}, \tag{19}$$

where

$$k_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^n \left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i}, \tag{20}$$

$$k_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\boldsymbol{\nu}) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n-1} \left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i}, \tag{21}$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \dots, A_{in})$, $k_i(q_i\mathbf{A_i}) < \infty$, $i = 1, 2, \dots, n$.

Proof. The result follows easily from the relations (14) and (15) for the appropriate choice of non-negative measurable functions f_i, $i = 1, 2, \dots, n$.

Namely, if the functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, are replaced with $x_i^{\nu_i}(\mathcal{H}f_i)^{\mu_i}(x_i)$, then the terms on the right-hand side of inequality (14) become

$$\begin{aligned} \|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i}^{p_i} &= \|x_i^{(n-1-s)/q_i+\alpha_i+\nu_i}(\mathcal{H}f_i)^{\mu_i}(x_i)\|_{p_i}^{p_i} \\ &= \int_{\mathbb{R}_+} x_i^{p_i\mu_i\left[\frac{n-1-s}{q_i\mu_i}+\frac{\alpha_i+\nu_i}{\mu_i}\right]} (\mathcal{H}f_i)^{p_i\mu_i}(x_i)dx_i \\ &= \int_{\mathbb{R}_+} x_i^{-\lambda q'} (\mathcal{H}f_i)^{q'}(x_i)dx_i = \|x_i^{-\lambda}(\mathcal{H}f_i)(x_i)\|_{q'}^{q'}, \end{aligned} \tag{22}$$

where $q' = p_i\mu_i$ and

$$\lambda = -\frac{q_i(\alpha_i+\nu_i)+n-1-s}{q_i\mu_i}. \tag{23}$$

Moreover, considering the two-dimensional setting with non-conjugate exponents, the expression $\|x_i^{-\lambda}(\mathcal{H}f_i)(x_i)\|_{q'}$ represents the left-hand side of the Hardy-type inequality (16), that is, we have inequality

$$\|x_i^{-\lambda}(\mathcal{H}f_i)(x_i)\|_{q'}^{q'} \le (p'\lambda)^{q'\lambda}\|f_i\|_p^{q'}, \tag{24}$$

with abbreviated

$$p = \frac{p_iq_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}$$

and

$$p' = -\frac{p_iq_i\mu_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}.$$

In other words, the right-hand side of inequality (24) reads

$$\left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{p_i(\alpha_i+\nu_i)+p_i(n-1-s)/q_i} \left\| f^{\frac{q_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}} \right\|_{p_i}^{p_i^2(\alpha_i+\nu_i+\mu_i)+p_i^2(n-1-s)/q_i+p_i}. \tag{25}$$

Hence, relations (22), (24) and (25) yield the series of inequalities

$$\|x_i^{(n-1-s)/q_i+\alpha_i} f_i\|_{p_i} \le \left[\frac{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)+q_i}{p_iq_i(\alpha_i+\nu_i)+p_i(n-1-s)}\right]^{\alpha_i+\nu_i+(n-1-s)/q_i} \left\| f^{\frac{q_i\mu_i}{p_iq_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)+q_i}} \right\|_{p_i}^{p_i(\alpha_i+\nu_i+\mu_i)+p_i(n-1-s)/q_i+1},$$

where $i = 1, 2, \dots, n$, so the inequality (18) follows immediately from (14).

Obviously the same reasoning is used to establish inequality (19) from (15), which completes the proof. □

Note that in the proof of the previous theorem we have used the two-dimensional Hardy inequality with non-conjugate exponents. Now, we are going to consider the special case of Theorem 1 in which the Hardy inequality appears in the classical conjugate setting. In that case, the parameter λ, defined by (23) must be equal to 1, i.e. $\lambda = 1$. Hence, if $\lambda = 1$, then $\nu_i = (s+1-n)/q_i - \alpha_i - \mu_i$, $i = 1,2,\ldots,n$, that is, we have the equalities in the set of conditions (17). In other words, we can eliminate the parameters ν_i, $i = 1,2,\ldots,n$, and the set of conditions (17) reduces to

$$p_i\mu_i > 1, \quad i = 1,2,\ldots,n. \tag{26}$$

In the following corollary it is more convenient to use the classical Hardy operator $\mathcal{H}'$ defined by

$$\mathcal{H}' f(x) = \frac{1}{x}\int_0^x f(t)dt. \tag{27}$$

The classical Hardy operator is also known in the literature as the Cesàro operator.

Corollary 2. *Suppose p_i, p_i', q_i, $i = 1,2,\ldots,n$, and λ_n are as in* (9)–(12), *and A_{ij}, $i,j = 1,2,\ldots,n$, are the real parameters satisfying $\sum_{i=1}^n A_{ij} = 0$. Further, let $\alpha_i = \sum_{j=1}^n A_{ij}$, $i = 1,2,\ldots,n$, and let μ_i, $i = 1,2,\ldots,n$, be real parameters satisfying the conditions* (26). *If $K : \mathbb{R}_+^n \to \mathbb{R}$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1,2,\ldots,n$, are non-negative measurable functions, then*

$$\int_{\mathbb{R}_+^n} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^n x_i^{\frac{s+1-n}{q_i}-\alpha_i} \left(\mathcal{H}' f_i\right)^{\mu_i}(x_i) d\mathbf{x} \le l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\mu) \prod_{i=1}^n \|f_i^{\mu_i}\|_{p_i}, \tag{28}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{(1-\lambda_n p_n')(n-1-s)-p_n'\alpha_n} \left(\int_{\mathbb{R}_+^{n-1}} K^{\lambda_n}(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{s+1-n}{q_i}-\alpha_i} \left(\mathcal{H}' f_i\right)^{\mu_i}(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \le l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\mu) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i}, \tag{29}$$

where

$$l_n^s(\mathbf{p},\mathbf{q},\mathbf{A},\mu) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^n \left(\frac{p_i\mu_i}{p_i\mu_i - 1}\right)^{\mu_i}, \tag{30}$$

$$l_{n-1}^s(\mathbf{p},\mathbf{q},\mathbf{A},\mu) = \prod_{i=1}^n k_i^{1/q_i}(q_i\mathbf{A_i}) \prod_{i=1}^{n-1} \left(\frac{p_i\mu_i}{p_i\mu_i - 1}\right)^{\mu_i}, \tag{31}$$

$\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $k_i(q_i\mathbf{A_i}) < \infty$, $i = 1,2,\ldots,n$.

4. Reduction to conjugate case and the best possible constant factors

Generally speaking, the problem of the best possible constant factors for the Hilbert-type inequalities in the setting with non-conjugate exponents seems to be very hard problem and remains still open.

Hence, in order to obtain the best possible constant factors in the inequalities (28) and (29), we shall consider here their conjugate forms. Namely, if $\{p_i;\ i = 1,2,\ldots,n\}$ is the set of conjugate exponents, then inequality (28) takes form

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\frac{s+1-n}{p_i}-\alpha_i} \left(\mathcal{H}' f_i\right)^{\mu_i}(x_i) d\mathbf{x} \le \bar{l}_n^s(\mathbf{p},\mathbf{A},\mu) \prod_{i=1}^n \|f_i^{\mu_i}\|_{p_i}, \tag{32}$$

where

$$\bar{l}_n^s(\mathbf{p},\mathbf{A},\mu) = \prod_{i=1}^n k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^n \left(\frac{p_i\mu_i}{p_i\mu_i - 1}\right)^{\mu_i}. \tag{33}$$

Similarly, the conjugate form of inequality (29) reads

$$\left[\int_{\mathbb{R}_+} x_n^{(1-p_n')(n-1-s)-p_n'\alpha_n}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\frac{s+1-n}{p_i}-\alpha_i}(\mathcal{H}' f_i)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \leq \bar{l}_{n-1}^s(\mathbf{p},\mathbf{A},\mu)\prod_{i=1}^{n-1}\|f_i^{\mu_i}\|_{p_i}, \quad (34)$$

including the constant factor

$$\bar{l}_{n-1}^s(\mathbf{p},\mathbf{A},\mu) = \prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i})\prod_{i=1}^{n-1}\left(\frac{p_i\mu_i}{p_i\mu_i-1}\right)^{\mu_i}. \quad (35)$$

In the sequel we consider the problem of the best possible constant factors involved in both inequalities (32) and (34). In order to obtain the best possible constant factors, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, defined by (13). More precisely, we assume that

$$k_1(\mathbf{a}) < \infty \text{ for } a_2, \ldots, a_n > -1, \sum_{i=2}^{n} a_i < s-n+1, n \in \mathbb{N}, n \geq 2. \quad (36)$$

By the similar reasoning as in some recent results known from the literature (see papers [1], [2], [15]), the best possible constant factors can be obtained if they don't contain conjugate parameters p_i in the exponents. For that sake, we assume

$$k_1(p_1\mathbf{A_1}) = k_2(p_2\mathbf{A_2}) = \cdots = k_n(p_n\mathbf{A_n}). \quad (37)$$

If we use the change of variables $u_1 = 1/t_2, u_3 = t_3/t_2, u_4 = t_4/t_2, \ldots, u_n = t_n/t_2$, which provides the Jacobian of the transformation

$$\left|\frac{\partial(u_1, u_3, \ldots, u_n)}{\partial(t_2, t_3, \ldots, t_n)}\right| = t_2^{-n},$$

we have

$$k_2(p_2\mathbf{A_2}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1)t_2^{s-n-p_2(\alpha_2-A_{22})}\prod_{j=3}^{n} t_j^{p_2A_{2j}}\hat{d}^1\mathbf{t} = k_1(p_1A_{11}, s-n-p_2(\alpha_2-A_{22}), p_2A_{23}, \ldots, p_2A_{2n}).$$

According to (37), we have $p_1A_{12} = s-n-p_2(\alpha_2-A_{22}), p_1A_{13} = p_2A_{23}, \ldots, p_1A_{1n} = p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A_i})$, $i = 3, \ldots, n$, in the terms of $k_1(\cdot)$. In such a way we see that (37) is fulfilled if

$$p_jA_{ji} = s-n-p_i(\alpha_i-A_{ii}), i, j = 1, 2, \ldots, n, \ i \neq j. \quad (38)$$

The above set of conditions also implies that $p_iA_{ik} = p_jA_{jk}$, when $k \neq i, j$. Hence, we use abbreviations $\widetilde{A_1} = p_nA_{n1}$ and $\widetilde{A_i} = p_1A_{1i}$, $i \neq 1$. Since $\sum_{i=1}^{n} A_{ij} = 0$, one easily obtains that $p_jA_{jj} = \widetilde{A_j}(1-p_j)$. Moreover, $\sum_{i=1}^{n}\widetilde{A_i} = s-n$ (see also paper [15]).

Now if the set of conditions (38) is satisfied, then, by using the above mentioned abbreviations, inequalities (32) and (34) become respectively

$$\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} x_i^{\frac{1}{p_i}+\widetilde{A_i}}(\mathcal{H}' f_i)^{\mu_i}(x_i)d\mathbf{x} \leq \bar{m}_n^s(\mathbf{p},\widetilde{\mathbf{A}},\mu)\prod_{i=1}^{n}\|f_i^{\mu_i}\|_{p_i}, \quad (39)$$

$$\left[\int_{\mathbb{R}_+} x_n^{(p_n'-1)(1+p_n\widetilde{A_n})}\left(\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x})\prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\widetilde{A_i}}(\mathcal{H}' f_i)^{\mu_i}(x_i)\hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \leq \bar{m}_{n-1}^s(\mathbf{p},\widetilde{\mathbf{A}},\mu)\prod_{i=1}^{n-1}\|f_i^{\mu_i}\|_{p_i}, \quad (40)$$

where

$$\bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \mu) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n} \left(\frac{p_i \mu_i}{p_i \mu_i - 1} \right)^{\mu_i}, \tag{41}$$

$$\bar{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \mu) = k_1(\widetilde{\mathbf{A}}) \prod_{i=1}^{n-1} \left(\frac{p_i \mu_i}{p_i \mu_i - 1} \right)^{\mu_i}, \tag{42}$$

and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$.

In order to obtain the best possible constant factors in the inequalities (39) and (40) we need the following two auxiliary results.

Lemma 1. *If $y \geq 1$ and $0 < r \leq 1$ then $(y-1)^r \geq y^r - 1$.*

Proof. Let us define the function $h : [1, \infty\rangle \to \mathbb{R}$ by $h(y) = (y-1)^r - y^r + 1$, where $0 < r \leq 1$. By taking its derivative we get $h'(y) = r\left[(y-1)^{r-1} - y^{r-1}\right]$. Clearly, since $0 < r \leq 1$ we conclude that $h'(y) \geq 0$, i.e. h is increasing function on $[1, \infty\rangle$. In other words, $h(y) \geq h(1)$, that is, $(y-1)^r \geq y^r - 1$, as required. □

Lemma 2. *If $y_1, y_2, \ldots, y_n$ are non-negative real numbers, then*

$$\prod_{i=1}^{n} (y_i - 1) \geq \prod_{i=1}^{n} y_i - \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} y_j. \tag{43}$$

Proof. The result follows easily by the mathematical induction principle. Namely, if $n = 2$ then $(y_1 - 1)(y_2 - 1) = y_1 y_2 - y_1 - y_2 + 1 \geq y_1 y_2 - y_1 - y_2$.

Now, if we suppose that (43) holds, we have

$$\prod_{i=1}^{n+1} (y_i - 1) \geq (y_{n+1} - 1) \left(\prod_{i=1}^{n} y_i - \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} y_j \right) \geq \prod_{i=1}^{n+1} y_i - \sum_{i=1}^{n+1} \prod_{\substack{j=1 \\ j \neq i}}^{n+1} y_j,$$

and the proof is complete. □

Now we are ready to establish the best possible constant factors in the inequalities (39) and (40).

Theorem 3. *Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be non-negative measurable homogeneous function of degree $-s$, $s > 0$, such that for every $i = 2, 3, \ldots, n$*

$$K(1, t_2, \ldots, t_i, \ldots, t_n) \leq C_K K(1, t_2, \ldots, 0, \ldots, t_n),\ 0 \leq t_i \leq 1, \tag{44}$$

where C_K is a positive constant. Further, let $1/p_i < \mu_i \leq 1$, $i = 1, 2, \ldots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \ldots, n$ satisfy conditions as in (36). Then, the constant factor $\bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$ is the best possible in inequality (39).

Proof. Suppose that there exist a positive constant C_n, $0 < C_n < \bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$, such that inequality

$$\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\frac{1}{p_i} + \widetilde{A}_i} (\mathcal{H}' f_i)^{\mu_i} (x_i) d\mathbf{x} \leq C_n \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{p_i} \tag{45}$$

holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$. For this purpose, let's substitute the functions

$$f_i^{\varepsilon}(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ x_i^{-\frac{1+\varepsilon}{p_i \mu_i}}, & x \geq 1, \end{cases} \tag{46}$$

where $0 < \varepsilon < \min\{\min_{1\leq i\leq n}\{p_i\mu_i\} - 1, \min_{1\leq i\leq n}\{p_i + p_i\widetilde{A}_i\}\}$, in the previous inequality.

Since $\|(f_i^\varepsilon)^{\mu_i}\|_{p_i} = (1/\varepsilon)^{1/p_i}$, the right-hand side of inequality (45) becomes C_n/ε. On the other hand we easily get

$$\mathcal{H}' f_i^\varepsilon(x_i) = \begin{cases} 0, & 0 < x_i < 1, \\ \frac{p_i\mu_i}{p_i\mu_i - 1 - \varepsilon} \cdot \frac{x_i^{\frac{p_i\mu_i - 1 - \varepsilon}{p_i\mu_i}} - 1}{x_i}, & x \geq 1, \end{cases} \tag{47}$$

and the left-hand side of the inequality (45), denoted here with L, reads

$$L = \varphi(\varepsilon) \int_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\frac{1}{p_i} + \widetilde{A}_i - \mu_i} \left(x_i^{\frac{p_i\mu_i - 1 - \varepsilon}{p_i\mu_i}} - 1 \right)^{\mu_i} d\mathbf{x},$$

where

$$\varphi(\varepsilon) = \prod_{i=1}^n \left(\frac{p_i\mu_i}{p_i\mu_i - 1 - \varepsilon} \right)^{\mu_i}.$$

Further, the following inequality follows immediately from Lemma 1 and Lemma 2:

$$\prod_{i=1}^n \left(x_i^{\frac{p_i\mu_i - 1 - \varepsilon}{p_i\mu_i}} - 1 \right)^{\mu_i} \geq \prod_{i=1}^n x_i^{\mu_i - \frac{1+\varepsilon}{p_i}} - \sum_{i=1}^n \prod_{\substack{j=1 \\ j\neq i}}^n x_i^{\mu_i - \frac{1+\varepsilon}{p_i}}.$$

Therefore, we have inequality

$$L \geq \varphi(\varepsilon) I - \varphi(\varepsilon) \sum_{i=1}^n I_i \tag{48}$$

where

$$I = \int_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} d\mathbf{x}, \qquad I_i = \int_{[1,\infty\rangle^n} K(\mathbf{x}) x_i^{\frac{1}{p_i} + \widetilde{A}_i - \mu_i} \prod_{\substack{j=1 \\ j\neq i}}^n x_j^{\widetilde{A}_j - \frac{\varepsilon}{p_j}}.$$

In the sequel, we are going to estimate the integrals I and I_i, $i = 1, 2, \ldots, n$. Obviously, the integral I can be rewritten as

$$I = \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{d}^1\mathbf{u} \right] dx_1,$$

providing the inequality

$$\begin{aligned} I &\geq \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{d}^1\mathbf{u} \right] dx_1 - \int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1 \\ &= \frac{1}{\varepsilon} k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - \int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1, \end{aligned} \tag{49}$$

where $\mathbb{D}_i = \{(u_2, u_3, \ldots, u_n); 0 < u_i \leq 1/x_1, u_j > 0, j \neq i\}$ and $\mathbf{1}/\mathbf{p} = (1/p_1, \ldots, 1/p_n)$.

Without losing generality, it is enough to find the upper bound for the integral $\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u}$. Regarding (44), we have

$$\begin{aligned} \int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} &\leq C_K \left[\int_{\mathbb{R}_+^{n-2}} K(1, 0, u_3, \ldots, u_n) \prod_{j=3}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} du_3 \ldots du_n \right] \int_0^{1/x_1} u_2^{\widetilde{A}_2 - \varepsilon/p_2} du_2 \\ &= C_K (1 - \varepsilon/p_2 + \widetilde{A}_2)^{-1} x_1^{\varepsilon/p_2 - \widetilde{A}_2 - 1} k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \ldots, \widetilde{A}_n - \varepsilon/p_n), \end{aligned}$$

where $k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \ldots, \widetilde{A}_n - \varepsilon/p_n)$ is well defined since obviously $\sum_{i=3}^n \widetilde{A}_i < s - n + 2$. Hence, we have

$$\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} = x_1^{\varepsilon/p_i - \widetilde{A}_i - 1} O(1),\ i = 2, 3, \ldots, n,$$

and consequently

$$\int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \hat{d}^1\mathbf{u} \right] dx_1 = O(1).$$

It remains to estimate integrals I_i, $i = 1, 2, \ldots, n$. We have

$$\begin{aligned} I_i = \int_{[1,\infty)^n} K(\mathbf{x}) x_i^{\frac{1}{p_i} + \widetilde{A}_i - \mu_i} \prod_{\substack{j=1 \\ j \neq i}}^n x_j^{\widetilde{A}_j - \frac{\varepsilon}{p_j}} d\mathbf{x} \quad &= \int_{[1,\infty)} x_i^{\frac{1}{p_i} - \mu_i - \frac{\varepsilon}{p_i'} - 1} \left[\int_{[1/x_1,\infty)^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{\substack{j=1 \\ j \neq i}}^n u_j^{\widetilde{A}_j - \frac{\varepsilon}{p_j}} \hat{d}^i\mathbf{u} \right] dx_i \\ &\leq \int_{[1,\infty)} x_i^{\frac{1}{p_i} - \mu_i - \frac{\varepsilon}{p_i'} - 1} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{\substack{j=1 \\ j \neq i}}^n u_j^{\widetilde{A}_j - \frac{\varepsilon}{p_j}} \hat{d}^i\mathbf{u} \right] dx_i \\ &= k_i\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) \int_{[1,\infty)} x_i^{\frac{1}{p_i} - \mu_i - \frac{\varepsilon}{p_i'} - 1} dx_i \\ &= \frac{1}{\mu_i - \frac{1}{p_i} + \frac{\varepsilon}{p_i'}} k_i\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) = O(1), \end{aligned}$$

so inequality (48) yields relation

$$L \geq \frac{\varphi(\varepsilon)}{\varepsilon} k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - \varphi(\varepsilon) O(1).$$

Furthermore, since the right-hand side of inequality (45) is equal to C_n/ε in the setting with functions (46), the above inequality implies

$$\frac{C_n}{\varepsilon} \geq \frac{\varphi(\varepsilon)}{\varepsilon} k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - \varphi(\varepsilon) O(1),$$

that is

$$C_n \geq \varphi(\varepsilon) k_1\left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p}\right) - \varphi(\varepsilon) o(1). \tag{50}$$

Obviously, if $\varepsilon \to 0^+$, then

$$\varphi(\varepsilon) \to \prod_{i=1}^n \left(\frac{p_i \mu_i}{p_i \mu_i - 1} \right)^{\mu_i},$$

thus, by letting $\varepsilon \to 0^+$, the inequality (50) yields $C_n \geq \bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$, which contradicts with our assumption $0 < C_n < \bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$. Hence, $\bar{m}_n^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is the best possible constant in inequality (39). □

With the help of Theorem 3, we also get the best possible constant factor in inequality (40).

Theorem 4. *Let $K : \mathbb{R}_+^n \to \mathbb{R}$ be non-negative measurable homogeneous function of degree $-s$, $s > 0$, fulfilling the condition* (44). *Further, let $1/p_i < \mu_i \leq 1$, $i = 1, 2, \ldots, n$, and let the parameters $\widetilde{A}_i$, $i = 2, \ldots, n$ satisfy conditions as in* (36). *Then, the constant factor $\bar{m}_{n-1}^s(\mathbf{p}, \widetilde{\mathbf{A}}, \boldsymbol{\mu})$ is the best possible in inequality* (40).

Proof. Suppose, on the contrary, that there exist a positive constant C_{n-1}, $0 < C_{n-1} < \bar{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$ such that the inequality (40) holds for all non-negative measurable functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, if we replace $\bar{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$ with C_{n-1}.

In that case, the left hand side of inequality (39), denoted here with L, can be rewritten in the following form:

$$L = \int_{\mathbb{R}_+} \left(x_n^{\frac{1}{p_n}+\widetilde{A}_n} \int_{\mathbb{R}^{n-1}_+} K(\mathbf{x}) \prod_{i=1}^{n-1} x_i^{\frac{1}{p_i}+\widetilde{A}_i} (\mathcal{H}' f_i)^{\mu_i} (x_i) d\hat{}^n\mathbf{x} \right) (\mathcal{H}' f_n)^{\mu_n} (x_n) dx_n.$$

Now, the application of the well-known Hölder's inequality with conjugate exponents p_n and p_n' yields inequality

$$L \le L' \| (\mathcal{H}' f_n)^{\mu_n} \|_{p_n}, \tag{51}$$

where L' denotes the left-hand side of inequality (40).

Furthermore, $L' \le C_{n-1} \prod_{i=1}^{n-1} \| f_i^{\mu_i} \|_{p_i}$, while the Hardy inequality yields inequality

$$\| (\mathcal{H}' f_n)^{\mu_n} \|_{p_n} \le \left(\frac{p_n \mu_n}{p_n \mu_n - 1} \right)^{\mu_n} \| f_n{}^{\mu_n} \|_{p_n}.$$

Hence, the relation (51) yields inequality

$$L \le C_{n-1} \left(\frac{p_n \mu_n}{p_n \mu_n - 1} \right)^{\mu_n} \prod_{i=1}^{n} \| f_i^{\mu_i} \|_{p_i}. \tag{52}$$

Finally, taking into account our assumption $0 < C_{n-1} < \bar{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$, we have

$$0 < C_{n-1} \left(\frac{p_n \mu_n}{p_n \mu_n - 1} \right)^{\mu_n} < \bar{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu) \left(\frac{p_n \mu_n}{p_n \mu_n - 1} \right)^{\mu_n} = \bar{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \mu).$$

Hence, inequality (52) contradicts with the fact that $\bar{m}^s_n(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$ is the best possible constant factor in inequality (39).

Thus the assumption that $\bar{m}^s_{n-1}(\mathbf{p}, \widetilde{\mathbf{A}}, \mu)$ is not the best possible was false. That completes the proof. □

Remark 5. Since $\sum_{i=1}^n \widetilde{A}_i = s - n$, the requirement (36) in the setting with the above mentioned parameters reads: $k_1(\widetilde{\mathbf{A}}) < \infty$ if $\widetilde{A}_i > -1$, $i = 1, 2, \ldots, n$, $n \ge 2$.

5. Two examples and concluding remarks

This section is devoted to the results from the previous two sections in some particular settings. In such a way we shall obtain generalizations of some recent results, mentioned in the Introduction. More precisely, we shall consider two particular homogeneous kernels discussed in the Introduction.

5.1. *First example*

A typical example of the homogeneous kernel with the negative degree of homogeneity is the function $K : \mathbb{R}^n_+ \to \mathbb{R}$ defined by

$$K(\mathbf{x}) = \frac{1}{\left(\sum_{i=1}^n x_i\right)^s}, \; s > 0. \tag{53}$$

Clearly, K is homogeneous function of degree $-s$, and the constant factors $k_i(p_i\mathbf{A_i})$ can be expressed in the terms of the usual Gamma function Γ. For that sake, we use the well-known formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n a_i}} \hat{d}^n\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^n a_i\right)}, \tag{54}$$

which holds for $a_i > 0$, $i = 1,2,\dots n$ (see, for instance [2]). In such a way, the constant factors $k_i(p_i\mathbf{A_i})$, $i = 1,2,\dots,n$, involved in the inequalities (32) and (34) become

$$k_i(p_i\mathbf{A_i}) = \frac{\Gamma(s-n+1-p_i\alpha_i+p_iA_{ii})}{\Gamma(s)} \prod_{j=1, j\neq i}^{n} \Gamma(1+p_iA_{ij}),\ i = 1,2,\dots,n,$$

provided that $A_{ij} > -1/p_i$, $i \neq j$ and $A_{ii} - \alpha_i > (n-s-1)/p_i$.

Moreover, if the parameters A_{ij}, $i, j = 1,2,\dots,n$, satisfy the set of conditions (37), then the above constant factor (taking into account the abbreviations $\widetilde{A}_i$, $i = 1,2,\dots,n$) reduces to

$$k_i(p_i\mathbf{A_i}) = \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma(1+\widetilde{A}_i), \quad i = 1,2,\dots,n,$$

provided that $\widetilde{A}_i > -1$, $i = 1,2,\dots,n$. Now, if we substitute the parameters $\widetilde{A}_i = r_i - 1$, $r_i > 0$ and $\mu_i = 1$, $i = 1,2,\dots,n$, in the inequalities (39) and (40), we get

$$\int_{\mathbb{R}_+^n} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n} x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{H}' f_i\right)(x_i) d\mathbf{x} \leq \frac{\prod_{i=1}^n p_i'}{\Gamma(s)} \prod_{i=1}^{n} \Gamma(r_i) \prod_{i=1}^{n} \|f_i\|_{p_i}, \tag{55}$$

and

$$\left[\int_{\mathbb{R}_+} x_n{}^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\left(\sum_{i=1}^n x_i\right)^s} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} \left(\mathcal{H}' f_i\right)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \leq \frac{\prod_{i=1}^{n-1} p_i'}{\Gamma(s)} \prod_{i=1}^{n} \Gamma(r_i) \prod_{i=1}^{n-1} \|f_i^{\mu_i}\|_{p_i}. \tag{56}$$

Clearly, the kernel (53) satisfies the relation (44) equipped with the positive constant $C_K = 1$. Therefore, in accordance with Theorems 3 and 4 inequalities (55) and (56) contain the best possible constant factors on the right-hand sides. Finally, if $n = 2$ inequalities (55) and (56) become inequalities (5) and (6) in the slightly altered form, due to the well-known relationship between the Gamma and Beta function, i.e. $B(r_1, r_2) = \Gamma(r_1)\Gamma(r_2)/\Gamma(r_1 + r_2)$, $r_1, r_2 > 0$.

5.2. Second example

We conclude this paper with yet another homogeneous kernel of degree $-s$, that is

$$K(\mathbf{x}) = \frac{1}{\max\{x_1^s, \dots, x_n^s\}}, \quad s > 0. \tag{57}$$

It is easy to show the integral formula

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i}}{\max\{1, x_1^s, \dots, x_{n-1}^s\}} \hat{d}^n\mathbf{u} = \frac{s}{\prod_{i=1}^n (1+a_i)}, \tag{58}$$

where $a_i > -1$ and $\sum_{i=1}^n a_i = s - n$. Namely, the previous integral can be represented as

$$\int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{a_i}}{\max\{1, x_1^s, \dots, x_{n-1}^s\}} \hat{d}^n\mathbf{u} = \int_{\mathbb{D}_0} \prod_{i=1}^{n-1} u_i^{a_i} \hat{d}^n\mathbf{u} + \sum_{i=1}^{n-1} \int_{\mathbb{D}_i} \frac{\prod_{k=1}^{n-1} u_k^{a_k}}{x_i^s} \hat{d}^n\mathbf{u},$$

with the regions $\mathbb{D}_0 = \{(u_1, u_2, \ldots, u_{n-1}); u_k \le 1,\ k = 1, 2, \ldots, n-1\}$ and $\mathbb{D}_i = \{(u_1, u_2, \ldots, u_{n-1}); u_i \ge 1,\ u_k \le u_i,\ k \ne i\}$, $i = 1, 2, \ldots, n-1$. By using the well-known Fubini's theorem we get formulas

$$\int_{\mathbb{D}_0} \prod_{i=1}^{n-1} u_i^{a_i} \hat{d}^n\mathbf{u} = \frac{1 + a_n}{\prod_{k=1}^{n}(1 + a_k)}$$

$$\int_{\mathbb{D}_i} \frac{\prod_{k=1}^{n-1} u_k^{a_k}}{x_i^s} \hat{d}^n\mathbf{u} = \frac{1 + a_i}{\prod_{k=1}^{n}(1 + a_k)},$$

that is, we get (58) since $\sum_{i=1}^{n} a_i = s - n$.

Finally, the inequalities (39) and (40) including the parameters $\widetilde{A}_i = r_i - 1$, $r_i > 0$, $\mu_i = 1$, $i = 1, 2, \ldots, n$, and the homogeneous kernel (57) reduce respectively to

$$\int_{\mathbb{R}_+^n} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^{n} x_i^{r_i - \frac{1}{p_i'}} (\mathcal{H}' f_i)(x_i) d\mathbf{x} \le s \prod_{i=1}^{n} \frac{p_i'}{r_i} \prod_{i=1}^{n} \|f_i\|_{p_i}, \tag{59}$$

and

$$\left[\int_{\mathbb{R}_+} x_n^{r_n p_n' - 1} \left(\int_{\mathbb{R}_+^{n-1}} \frac{1}{\max\{x_1^s, \ldots, x_n^s\}} \prod_{i=1}^{n-1} x_i^{r_i - \frac{1}{p_i'}} (\mathcal{H}' f_i)(x_i) \hat{d}^n\mathbf{x}\right)^{p_n'} dx_n\right]^{1/p_n'} \le \frac{s}{p_n'} \prod_{i=1}^{n} \frac{p_i'}{r_i} \prod_{i=1}^{n-1} \|f_i\|_{p_i}. \tag{60}$$

Obviously, the kernel (57) fulfill condition (44), thus according to Theorems 3 and 4 inequalities (59) and (60) involve the best possible constant factors. Moreover, inequalities (59) and (60) are multidimensional extensions of inequalities (7) and (8), presented in the Introduction.

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References

[1] Y. Bicheng, I. Brnetić, M. Krnić, J. Pečarić, Generalization of Hilbert and Hardy-Hilbert integral inequalities, Math. Inequal. Appl. 8 (2) (2005) 259-272.
[2] Y. Bicheng, T.M. Rassias, On the way of weight coefficients and research for the Hilbert-type inequalities, Math. Inequal. Appl. 6 (2003) 625–658.
[3] F. F. Bonsall, Inequalities with non-conjugate parameters, Quart. J. Math. Oxford Ser. 2(2) (1951) 135–150.
[4] I. Brnetić, M. Krnić, J. Pečarić, Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters, Bull. Austral. Math. Soc. 71 (2005) 447–457.
[5] A. Čižmešija, M. Krnić, J. Pečarić, General Hardy-Type Inequalities with Non-conjugate Exponents, Bull. Iran. Math. Soc. in press.
[6] N. Das, S. Sahoo, New inequalities similar to Hardy-Hilbert's inequality, Turk. J. Math. 34 (2010) 153-165.
[7] N. Das, S. Sahoo, New inequalities of Hardy-Hilbert type, Bul. Acad. Stiinte Repub. Mold. Mat. (2) (2010) 109-120.
[8] H. Du, Y. Miao, Several new Hardy-Hilbert's inequalities, FILOMAT 25 (2011) 153–162.
[9] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, 2nd edition, Cambridge University Press, Cambridge, 1967.
[10] G. H. Hardy, Note on a theorem of Hilbert concerning series of positive term, Proc. of London Math. Soc. 23(2) (1925) Records of Proc. xlv–xlvi.
[11] A. Kufner, B. Opic, Hardy-type inequalities, Pitman Researchs Notes in Mathematics Series, 219, Longman Scientifics & Tehnical, Harlow, 1990.
[12] A. Kufner, L. Maligranda, L. E. Persson, The Hardy inequality – About its history and some related results, Vydavatelský servis, Pilsen, 2007.
[13] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[14] I. Perić, P. Vuković, Hardy-Hilbert's inequality with general homogeneous kernel, Math. Inequal. Appl. 12 (2009) 525–536.
[15] P. Vuković, Note on Hilbert-type inequalities, Turk. J. Math. 35 (2011) 1–10.