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A new application of certain generalized power increasing sequences

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Abstract. The main object of this paper is to prove two general theorems by using a two-parameter quasi-$f^{(\beta,\sigma)}$-power increasing sequence instead of a quasi-β-power increasing sequence. The first result (Theorem 2.1) in this paper covers the case when $0 < \beta < 1$ and $\sigma \geqq 0$. The second main result (Theorem 2.3) in this paper covers the exceptional case when $\beta = 1$ and $\sigma \leqq 0$. Each of these theorems also includes several new or known results as their special cases and consequences.

1. Introduction, definitions and preliminaries

A positive sequence $\{b_n\}_{n\in\mathbb{N}}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}_{n\in\mathbb{N}}$ and two positive constants A and B such that (see [1])

$$Ac_n \leqq b_n \leqq Bc_n \qquad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

We write

$$\mathcal{BV}_O = \mathcal{BV} \cap C_O,$$

where

$$C_O := \left\{x : x = \{x_n\}_{n\in\mathbb{N}} \in \Omega \quad \text{and} \quad \lim_{n\to\infty} |x_n| = 0\right\}$$

and

$$\mathcal{BV} := \left\{x : x = \{x_n\}_{n\in\mathbb{N}} \in \Omega \quad \text{and} \quad \sum_{n=1}^{\infty} |x_n - x_{n+1}| < \infty\right\},$$

Ω being the space of all *real-valued* sequences.

Definition 1.1. A positive sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is said to be a quasi-β-power increasing sequence if there exists a constant $K := K(\gamma;\beta) \geqq 1$ such that the following inequality holds true (see [10]):

$$Kn^\beta \gamma_n \geqq m^\beta \gamma_m \qquad (n \geqq m \geqq 1;\ m, n \in \mathbb{N}).$$

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It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative real number β, but the converse may not be true as can be seen by considering the following example:

$$\gamma_n = n^{-\beta} \qquad (n \in \mathbb{N};\ \beta > 0).$$

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers and let $\sum_{n=1}^{\infty} a_n$ be a given infinite series with the associated sequence $\{s_n\}_{n\in\mathbb{N}}$ of partial sums. We denote by z_n^α and t_n^α the nth Cesàro means of order α of the sequences $\{s_n\}_{n\in\mathbb{N}}$ and $\{na_n\}_{n\in\mathbb{N}}$, respectively, that is,

$$z_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1}\, s_v, \tag{1}$$

and

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1}\, va_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad A_0^\alpha = 1 \qquad \text{and} \qquad A_{-n}^\alpha = 0 \qquad (n \in \mathbb{N};\ \alpha > -1). \tag{3}$$

By definition, the following series $\sum_{n=1}^{\infty} a_n$ is said to be summable as follows:

$$\varphi\text{-}|C,\alpha|_k \quad (k \geqq 1;\ \alpha > -1)$$

if (see [2])

$$\sum_{n=1}^{\infty} |\varphi_n (z_n^\alpha - z_{n-1}^\alpha)|^k < \infty. \tag{4}$$

But, since (see [9])

$$t_n^\alpha = n(z_n^\alpha - z_{n-1}^\alpha),$$

where z_n^α is given by (1), the condition (4) can also be written as follows:

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty. \tag{5}$$

In the special case when

$$\varphi_n = n^{1-\frac{1}{k}} \qquad \text{and} \qquad \varphi_n = n^{\delta+1-\frac{1}{k}} \qquad (n \in \mathbb{N};\ k \geqq 1),$$

the (φ-$|C,\alpha|_k$)-summability is the same as the relatively more familiar summabilities:

$$|C,\alpha|_k \qquad \text{and} \qquad |C,\alpha;\delta|_k,$$

respectively.

Recently, by making use of Definition 1.1, Bor and Özarslan [7] proved the following theorem.

Theorem 1.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a quasi-β-power increasing sequence for some real parameter β $(0 < \beta < 1)$, where*

$$X_n := X_n(\beta) \qquad (n \in \mathbb{N};\ 0 < \beta < 1).$$

Suppose also that there exit sequences $\{\kappa_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathcal{BV}_O$ such that

$$|\Delta\lambda_n| \leqq \kappa_n, \tag{6}$$

$$\kappa_n \to 0 \qquad (n \to \infty), \tag{7}$$

$$\sum_{n=1}^{\infty} n|\Delta\kappa_n|X_n < \infty \tag{8}$$

and

$$|\lambda_n|X_n = O(1) \qquad (n \to \infty). \tag{9}$$

If there exists an $\epsilon > 0$ *such that the sequence* $\left\{n^{\epsilon-k}|\varphi_n|^k\right\}_{n\in\mathbb{N}}$ *is non-increasing and if the sequence* $\{w_n^\alpha\}_{n\in\mathbb{N}}$*, defined by* (*see* [11])

$$w_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \\ \max\limits_{1\leqq v\leqq n} \{|t_v^\alpha|\} & (0 < \alpha < 1), \end{cases} \tag{10}$$

satisfies the following condition:

$$\sum_{n=1}^{m} n^{-k} \left(|\varphi_n|w_n^\alpha\right)^k = O(X_m) \qquad (m \to \infty), \tag{11}$$

then the series $\sum\limits_{n=1}^{\infty} a_n\lambda_n$ *is summable as follows:*

$$\varphi\text{-}|C,\alpha|_k \qquad (k \geqq 1;\ 0 < \alpha \leqq 1;\ k\alpha + \epsilon > 1).$$

Remark 1.3. Here, in the hypothesis of Theorem 1.2, we have added the following condition:

$$\{\lambda_n\}_{n\in\mathbb{N}} \in \mathcal{BV}_O.$$

The aim of this paper is to derive extensions of Theorem 1.2 by using a new class of power increasing sequences. For this purpose, we need the concept of the *two-parameter* quasi-$f^{(\beta,\sigma)}$-power increasing sequences given by Definition 1.4 below.

Definition 1.4. A positive sequence $\mathfrak{c} = \{\mathfrak{c}_n\}_{n\in\mathbb{N}}$ is said to be a *two-parameter* quasi-$f^{(\beta,\sigma)}$-power increasing sequence if there exists a constant $\mathcal{K}$ given by

$$\mathcal{K} := \mathcal{K}\left(\mathfrak{c}; f^{(\beta,\sigma)}\right) \geqq 1$$

such that the following inequality holds true (see, for details, [13] and [12, p. 703] for the case when $0 < \beta < 1$ and $\sigma \geqq 0$):

$$\mathcal{K} f_n^{(\beta,\sigma)}\ \mathfrak{c}_n \geqq f_m^{(\beta,\sigma)}\ \mathfrak{c}_m \qquad (n \geqq m \geqq 1;\ m, n \in \mathbb{N}),$$

where

$$f^{(\beta,\sigma)} = \left\{f_n^{(\beta,\sigma)}\right\}_{n\in\mathbb{N}} = \left\{n^\beta(\log n)^\sigma\right\}_{n\in\mathbb{N}} \qquad (\sigma \in \mathbb{R};\ 0 < \beta \leqq 1).$$

Clearly, if we choose $\sigma = 0$, then a quasi-$f^{(\beta,\sigma)}$-power increasing sequence is precisely the same as the above-defined quasi-β-power increasing sequence.

2. A set of main results

Our first main result (Theorem 2.1 below) is based essentially upon Definition 1.4 for the case when $0 < \beta < 1$ and $\sigma \geqq 0$. It provides one of our proposed extensions of Theorem 1.2 of the preceding section.

Theorem 2.1. *Let* $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathcal{BV}_O$ *and let* $\{X_n\}_{n\in\mathbb{N}}$ *be a quasi-*$f^{(\beta,\sigma)}$*-power increasing sequence for some real parameter* β $(0<\beta<1)$ *and some real parameter* σ $(\sigma \geqq 0)$, *where*

$$X_n := X_n(\beta,\sigma) \qquad (n\in\mathbb{N};\ \sigma\geqq 0;\ 0<\beta<1).$$

If the conditions involved in (6) *to* (9) *and* (11) *are satisfied, then the series* $\sum_{n=1}^{\infty} a_n\lambda_n$ *is summable as follows*:

$$\varphi\text{-}|C,\alpha|_k \qquad (k\geqq 1;\ 0<\alpha\leqq 1;\ \sigma\geqq 0;\ k\alpha+\epsilon>1).$$

Remark 2.2. If, as in Theorem 2.1, we assume that $\{X_n\}_{n\in\mathbb{N}}$ is a quasi-$f^{(\beta,\sigma)}$-power increasing sequence, where

$$f^{(\beta,\sigma)} = \left\{f_n^{(\beta,\sigma)}\right\}_{n\in\mathbb{N}} = \left\{n^{\beta}(\log n)^{\sigma}\right\}_{n\in\mathbb{N}} \qquad (\sigma\geqq 0;\ 0<\beta<1), \tag{12}$$

then the sequence $\left\{n^{\beta}(\log n)^{\sigma}\ X_n\right\}_{n\in\mathbb{N}}$ is *non-decreasing*. So, as a special case, we can take

$$X_n = n^{-\beta}(\log n)^{-\sigma} \qquad (n\in\mathbb{N};\ \sigma\geqq 0;\ 0<\beta<1). \tag{13}$$

Under this assumption, we find that

$$\sum_{n=1}^{\infty} n\,|\Delta\kappa_n|\,X_n < \infty \Longrightarrow \sum_{n=1}^{\infty} \kappa_n X_n < \infty \qquad \text{and} \qquad n\kappa_n X_n = O(1) \qquad (n\to\infty), \tag{14}$$

which holds true for all sequences $\{X_n\}_{n\in\mathbb{N}}$ for which the sequence $\left\{f_n^{(\beta,\sigma)}X_n\right\}_{n\in\mathbb{N}}$ is at least non-decreasing. But, if we assume that $\{X_n\}_{n\in\mathbb{N}}$ is a quasi-β-power increasing sequence, that is, if we assume that $\sigma=0$ in (12), then the sequence $\{X_n\}_{n\in\mathbb{N}}$ in (13) would no more imply the assertion in (14), because the sequence $\left\{n^{\beta}X_n\right\}_{n\in\mathbb{N}}$ is decreasing (that is, not necessarily non-decreasing) and, therefore, the assertion in (14) is no longer satisfied. Thus, in general, Theorem 1.2 does not imply Theorem 2.1.

We next consider the seemingly *exceptional* case of Definition 1.4 and Theorem 2.1 when $\beta=1$ and $\sigma\leqq 0$. In this exceptional case, our proposed extension of Theorem 1.2 is given by Theorem 2.3 below.

Theorem 2.3. *Let* $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathcal{BV}_O$ *and let* $\{X_n\}_{n\in\mathbb{N}}$ *be a quasi-*$f^{(1,-\sigma)}$*-power increasing sequence for some real parameter* σ $(\sigma\geqq 0)$, *where*

$$X_n := X_n(1,-\sigma) \qquad (n\in\mathbb{N};\ \sigma\geqq 0).$$

Suppose also that all of the conditions of Theorem 2 *are satisfied with the condition* (8) *replaced by the following condition*:

$$\sum_{n=1}^{\infty}(n\log n)X_n\,|\Delta\kappa_n| < \infty. \tag{15}$$

Then the series $\sum_{n=1}^{\infty} a_n\lambda_n$ *is summable as follows:*

$$\varphi - |C,\alpha|_k \qquad (k\geqq 1;\ 0<\alpha\leqq 1;\ k\alpha+\epsilon>1).$$

We need each of the following lemmas in our proofs of Theorem 2.1 and Theorem 2.3.

Lemma 2.4. ([8]) *If* $0<\alpha\leqq 1$ *and* $1\leqq v\leqq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1}\, a_p\right| \leqq \max_{1\leqq m\leqq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1}\, a_p\right|. \tag{16}$$

Lemma 2.5. *Except for the condition $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathcal{BV}_O$, under the conditions on the sequences $\{X_n\}_{n\in\mathbb{N}}$, $\{\kappa_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ as expressed in the statement of Theorem* 2*, the following assertions hold true*:

$$n\kappa_n X_n = O(1) \qquad (n \to \infty) \tag{17}$$

and

$$\sum_{n=1}^{\infty} \kappa_n X_n < \infty. \tag{18}$$

Proof. First of all, we observe that

$$\begin{aligned}
n\kappa_n X_n &= nX_n \sum_{v=n}^{\infty} \Delta\kappa_v \leqq nX_n \sum_{v=n}^{\infty} |\Delta\kappa_v| \\
&= n^{1-\beta}(\log n)^{-\sigma} n^{\beta}(\log n)^{\sigma} X_n \sum_{v=n}^{\infty} |\Delta\kappa_v| \\
&\leqq n^{1-\beta}(\log n)^{-\sigma} \sum_{v=n}^{\infty} v^{\beta}(\log v)^{\sigma} X_v |\Delta\kappa_v| \\
&\leqq \sum_{v=n}^{\infty} v^{1-\beta}(\log v)^{-\sigma} X_v v^{\beta}(\log v)^{\sigma} |\Delta\kappa_v| \\
&= \sum_{v=n}^{\infty} vX_v|\Delta\kappa_v| = O(1) \qquad (n \to \infty).
\end{aligned}$$

Now, since $\kappa_n \to 0$, we have $\Delta\kappa_n \to 0$. Consequently, for a given positive number ϵ such that

$$0 < \epsilon < \beta + \epsilon < 1,$$

we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \kappa_n X_n &\leqq \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} |\Delta\kappa_v| = \sum_{v=1}^{\infty} |\Delta\kappa_v| \sum_{n=1}^{v} X_n \\
&= \sum_{v=1}^{\infty} |\Delta\kappa_v| \sum_{n=1}^{v} n^{\beta}(\log n)^{\sigma} X_n n^{-\beta}(\log n)^{-\sigma} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| v^{\beta}(\log v)^{\sigma} X_v \sum_{n=1}^{v} n^{-\beta}(\log n)^{-\sigma} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| v^{\beta}(\log v)^{\sigma} X_v \sum_{n=1}^{v} n^{\epsilon}(\log n)^{-\sigma} n^{-\beta-\epsilon} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| v^{\beta} X_v (\log v)^{\sigma} v^{\epsilon}(\log v)^{-\sigma} \sum_{n=1}^{v} n^{-\beta-\epsilon} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| v^{\beta+\epsilon} X_v \int_0^v x^{-\beta-\epsilon} dx \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| v^{\beta+\epsilon} X_v v^{1-\beta-\epsilon} \\
&= O(1) \sum_{v=1}^{\infty} v|\Delta\kappa_v| X_v = O(1).
\end{aligned}$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *Suppose that all of the conditions of Theorem 2.3 are satisfied. Then the assertions* (17) *and* (18) *of Lemma 2.5 are also satisfied.*

Proof. Under the hypotheses of Lemma 2.6, it is easily observed that

$$\begin{aligned} n\kappa_n X_n &= nX_n \sum_{v=n}^{\infty} \Delta\kappa_v \leqq nX_n \sum_{v=n}^{\infty} |\Delta\kappa_v| \\ &= (\log n)^{\sigma} n(\log n)^{-\sigma} X_n \sum_{v=n}^{\infty} |\Delta\kappa_v| \\ &\leqq (\log n)^{\sigma} \sum_{v=n}^{\infty} v(\log v)^{-\sigma} X_v \, |\Delta\kappa_v| \\ &\leqq \sum_{v=n}^{\infty} (\log v)^{\sigma} v(\log v)^{-\sigma} X_v \, |\Delta\kappa_v| \\ &= O(1) \sum_{v=1}^{\infty} (v \log v) X_v \, |\Delta\kappa_v| = O(1). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \kappa_n X_n &\leqq \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} |\Delta\kappa_v| = \sum_{v=1}^{\infty} |\Delta\kappa_v| \sum_{n=1}^{v} X_n \\ &= \sum_{v=1}^{\infty} |\Delta\kappa_v| \sum_{n=1}^{v} n(\log n)^{-\sigma} X_n n^{-1} (\log n)^{\sigma} \\ &\leqq \sum_{v=1}^{\infty} |\Delta\kappa_v| \, v(\log v)^{-\sigma} X_v \sum_{n=1}^{v} n^{-1} (\log n)^{\sigma} \\ &= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| \, v(\log v)^{-\sigma} X_v \int_1^v (\log x)^{\sigma} x^{-1} dx \\ &= O(1) \sum_{v=1}^{\infty} |\Delta\kappa_v| \, v(\log v)^{-\sigma} X_v (\log v)^{\sigma+1} \\ &= O(1) \sum_{v=1}^{\infty} (v \log v) X_v \, |\Delta\kappa_v| = O(1). \end{aligned}$$

This evidently completes the proof of Lemma 2.6. □

3. Proofs of Theorems 2.1 and 2.3

Proof of Theorem 2.1. Let T_n^{α} be the nth (C, α)-mean of the sequence $\{na_n\lambda_n\}_{n\in\mathbb{N}}$ with $0 < \alpha \leqq 1$. Then, by means of (2), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} \, va_v\lambda_v. \tag{19}$$

Thus, by first applying Abel's transformation and then using Lemma 2.4, we find that

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} \, pa_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} \, va_v$$

and

$$
\begin{aligned}
|T_n^\alpha| &\leqq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} |\Delta\lambda_v| \cdot \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1}\, p a_p \right| + \frac{|\lambda_n|}{A_n^\alpha} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right| \\
&\leqq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha\, w_v^\alpha |\Delta\lambda_v| + |\lambda_n| w_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha.
\end{aligned}
$$

Since

$$
|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leqq 2^k (|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k),
$$

in order to complete the proof of Theorem 2.1, by using (5), it is sufficient to show that

$$
\sum_{n=1}^{\infty} n^{-k} |\varphi_n T_{n,r}^\alpha|^k < \infty \qquad (r = 1, 2).
$$

Next, when $k > 1$, by applying Hölder's inequality with indices

$$
k \quad \text{and} \quad k' \quad \left(\frac{1}{k} + \frac{1}{k'} = 1 \right),
$$

we get

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n,1}^\alpha|^k &\leqq \sum_{n=2}^{m+1} n^{-k} (A_n^\alpha)^{-k} |\varphi_n|^k \left(\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v| \right)^k \\
&\leqq \sum_{n=2}^{m+1} n^{-k} n^{-\alpha k} |\varphi_n|^k \left(\sum_{v=1}^{n-1} v^{\alpha k} (w_v^\alpha)^k |\Delta\lambda_v| \right) \left(\sum_{v=1}^{n-1} |\Delta\lambda_v| \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \kappa_v \sum_{n=v+1}^{m+1} \frac{n^{-k} |\varphi_n|^k}{n^{\alpha k}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \kappa_v \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} |\varphi_n|^k}{n^{\alpha k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \kappa_v v^{\epsilon-k} |\varphi_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \kappa_v v^{\epsilon-k} |\varphi_v|^k \int_v^\infty \frac{dx}{x^{\alpha k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v \kappa_v v^{-k} (w_v^\alpha |\varphi_v|)^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\kappa_v) \sum_{r=1}^{v} r^{-k} (w_r^\alpha |\varphi_r|)^k + O(1) m\kappa_m \sum_{v=1}^{m} v^{-k} (w_v^\alpha |\varphi_v|)^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\kappa_v)| X_v + O(1) m\kappa_m X_m
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} |(v+1)\Delta\kappa_v - \kappa_v| X_v + O(1) m\kappa_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\kappa_v| X_v + O(1)\sum_{v=1}^{m-1} \kappa_v X_v + O(1) m\kappa_m X_m \\
&= O(1) \qquad (m \to \infty),
\end{aligned}
$$

by virtue of the hypotheses of Theorem 2 and Lemma 2. Thus, finally, we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-k} |\varphi_n T_{n,2}^{\alpha}|^k &= O(1)\sum_{n=1}^{m} |\lambda_n| n^{-k} (w_n^{\alpha} |\varphi_n|)^k \\
&= O(1)\sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} v^{-k} (w_v^{\alpha}|\varphi_v|)^k + O(1)|\lambda_m| \sum_{n=1}^{m} n^{-k} (w_n^{\alpha}|\varphi_n|)^k \\
&= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1)\sum_{n=1}^{m-1} \kappa_n X_n + O(1)|\lambda_m| X_m = O(1) \qquad (m \to \infty),
\end{aligned}
$$

again by virtue of the hypotheses of Theorem 2.1 and Lemma 2.5. Therefore, we obtain

$$
\sum_{n=1}^{m} n^{-k} |\varphi_n T_{n,r}^{\alpha}|^k = O(1) \qquad (m \to \infty;\ r = 1, 2).
$$

This evidently completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. Our proof of Theorem 2.3 is much akin to that of Theorem 2.1 which we have detailed above fairly adequately. Indeed, in place of Lemma 2.5, Theorem 2.3 is proven by appealing instead to Lemma 2.6. We choose to omit the details involved. □

4. Special cases and consequences

Each of the following special cases and consequences of one of our main results (Theorem 2.1 of the preceding section) is worthy of mention here. Theorem 2.3 can similarly be applied in order to derive its various (known or new) corollaries and consequences.

1. If, in Theorem 2.1, we take $\{X_n\}_{n\in\mathbb{N}}$ as a positive *non-decreasing* sequence and let

$$
\epsilon = 1, \quad \sigma = 0 \qquad \text{and} \qquad \beta = \delta + 1 - \frac{1}{k} \qquad (n \in \mathbb{N};\ 0 \leqq \delta < 1;\ k \geqq 1),
$$

then we get a result of Bor [3].

2. By setting

$$
\epsilon = 1, \quad \sigma = 0 \qquad \text{and} \qquad \beta = 1 - \frac{1}{k} \qquad (n \in \mathbb{N};\ k \geqq 1)
$$

or

$$
\epsilon = 1, \quad \sigma = 0, \quad \alpha = 1 \qquad \text{and} \qquad \beta = 1 - \frac{1}{k} \qquad (n \in \mathbb{N};\ k \geqq 1),
$$

Theorem 2.1 would yield a new result dealing with the summability factor

$$
|C, \alpha|_k \qquad \text{or} \qquad |C, 1|_k,
$$

as the case may be.

3. If the sequence $\{X_n\}_{n\in\mathbb{N}}$ is assumed to be an almost increasing sequence, and if

$$\epsilon = 1, \quad \sigma = 0 \qquad \text{and} \qquad \beta = 1 - \frac{1}{k} \qquad (n \in \mathbb{N};\ k \geqq 1),$$

then Theorem 2.1 reduces to a known result due to Bor and Srivastava [5].

4. If, in Theorem 2.1, we set

$$\epsilon = 1, \quad \sigma = 0 \qquad \text{and} \qquad \beta = \delta + 1 - \frac{1}{k} \qquad (n \in \mathbb{N};\ 0 \leqq \delta < 1;\ k \geqq 1),$$

then we get a result due to Bor [6] involving the summability factor $|C, \alpha; \delta|_k$.

5. If the sequence $\{X_n\}_{n\in\mathbb{N}}$ is taken to be an almost increasing sequence and $\sigma = 0$, then Theorem 2.1 would lead us to a result of Bor and Seyhan [4].

6. By setting $\sigma = 0$ in Theorem 2.1, we readily obtain Theorem 1.2 proven earlier by Bor and Özarslan [7].

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