Filomat 26:4 (2012), 665–672
DOI 10.2298/FIL1204665M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Topological indices of the Kneser graph $KG_{n,k}$

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**Abstract.** In this paper we use transitivity property of the automorphism group of the Kneser graph to calculate its Wiener, Szeged and PI indices.

## 1. Introduction

All graphs in this paper are simple and connected. In general a graph is denoted by $G = (V, E)$, where $V$ is the set of vertices of $G$ and $E$ is the set of edges of $G$. If $z, u, v \in V$ and $e = uv \in E$, then the edge connecting $u$ to $v$ is denoted by $uv \in E$. Also all graphs considered in this paper are finite in a sense that both $V$ and $E$ are finite sets. For vertices $u$ and $v$ in $V$, a path from $u$ to $v$ is a sequence of vertices $u = u_0, u_1, ..., u_n = v$ such that $u_i u_{i+1} \in$ E where $0 \leq i \leq n - 1$. In this case $n$ is called the length of the path form $u$ to $v$. The length of the shortest path from $u$ to $v$ is called the distance between $u$ and $v$ and is denoted by $d(u, v)$ and $d(z, e) := min\{d(z, u),\ d(z, v)\}$. The Wiener index of the graph $G$ is denoted by $W(G)$ and is defined by:

$$W(G) = \sum_{\{u,v\} \subseteq V} d(u, v).$$

If the sum of distances from a vertex $v$ in $V$ is denoted by $d(v)$, i. e.

$$d(v) = \sum_{x \in V} d(v, x),$$

then

$$W(G) = (1/2) \sum_{v \in V} d(v).$$

The Wiener index for the first time was proposed in [14] in connection with the boiling points of chemical substances. The definition of the Wiener index in terms of the distances between vertices of a graph is give by Hosoya in [10]. Because of the above chemical fact about the Wiener index and also the fact that it is an invariant of the graph, i. e. it is invariant under the automorphism group of the graph, various methods have been developed to calculate it, for example one can refer to [4, 8].

2010 *Mathematics Subject Classification*. Primary 05C12; Secondary 05C90, 05C05
*Keywords*. Topological index, Kneser graph, Wiener index, Szeged index, PI-index
Received: 15 July 2011; Accepted: 20 September 2011
Communicated by Dragan Stevanović
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Apart from the Wiener index, there are numerous indices associated to a graph which are invariant under the automorphism group of the graph. Another topological index that we are interested in is called the Szeged index and is defined as follows. Let $G = (V, E)$ be a simple connected graph and $e = uv$ be an edge in $E$. By $n_u(e|G)$ we mean the number of vertices in $V$ lying closer to $u$ than $v$. The quantity $n_v(e|G)$ is defined similarly. Therefore if we define the sets

$$N_u(e|G) = \{w \in V \mid d(w,u) < d(w,v)\} \quad \text{and} \quad N_v(e|G) = \{w \in V \mid d(w,v) < d(w,u)\},$$

then $n_u(e|G) = |N_u(e|G)|$ and $n_v(e|G) = |N_v(e|G)|$.

The Szeged index of the graph $G = (V, E)$ is defined by the formula

$$Sz(G) = \sum_{e=uv\in E} n_u(e|G)n_v(e|G)$$

Because of the importance of the Szeged index its calculation has been studied by several authors. To mention a few, one can refer to [5, 9, 11]. A different method of calculating the topological indices, based on the properties of the automorphism group of the graph, was initiated in [1, 2] and the method was extended and applied to some well-known graphs. Also this method is used in [15] to compute the Szeged index of a symmetric graph.

Since the Szeged index takes into account how the vertices of the graph $G$ are distributed, it is natural to define an index that takes into account the distribution of the edges of $G$. The Padmakar-Ivan PI-index, [12, 13] is an additive index which takes into account the distribution of the edges of the graph and therefore complements the Szeged index in a certain sense. The next topological index that we are interested in is called the PI-index and is defined as follows. Let $G = (V, E)$ be a simple connected graph and $e = uv$ be an edge in $E$. By $n_{eu}(e|G)$ we mean the number of edges in $E$ lying closer to $u$ than $v$. The quantity $n_{ev}(e|G)$ is defined similarly. Therefore if we define the sets

$$N_{eu}(e|G) = \{f \in E \mid d(f,u) < d(f,v)\} \quad \text{and} \quad N_{ev}(e|G) = \{f \in E \mid d(f,v) < d(f,u)\},$$

then $n_{eu}(e|G) = |N_{eu}(e|G)|$ and $n_{ev}(e|G) = |N_{ev}(e|G)|$.

The PI-index of the graph $G = (V, E)$ is defined by the formula

$$PI(G) = \sum_{e=uv\in E} \Big(n_{eu}(e|G) + n_{ev}(e|G)\Big).$$

In this paper our aim is to use this method which applies group theory to graph theory. For materials from the theory of groups and graph theory one can see [3] and [7]. We also use this method to calculate the Wiener, the Szeged and the PI-index of the Kneser graph. It could be applied to the recently introduced $GA_2$ index [6] as well.

## 2. Concepts and results

In this section we will use some definitions and theorem from [1] to calculate the Wiener, Szeged and PI-index of graphs.

**Definition 2.1.** Let $G$ be a group which acts on a set X. Let us denote the action of $\sigma \in G$ on $x \in X$ by $x^\sigma$. Then $G$ is said to *act transitively* on $X$ if for every $x, y \in X$ there is $\sigma \in G$ such that $x^\sigma = y$.

**Definition 2.2.** Let $G = (V, E)$ be a graph. An *automorphism* $\sigma$ of $G$ is a one-to-one mapping from $V$ to $V$ which preserves adjacency, i. e. $e = uv$ is an edge of G if and only if $e^\sigma := u^\sigma v^\sigma$ is also an edge of $G$. The set of all the automorphisms of the graph $G$ is a group under the usual composition of mappings. This group is denoted by $\mathbb{A}ut(G)$ and is a subgroup of the symmetric group on $X$.

From Definition 2.2 it is clear that $\mathbb{A}ut(G)$ acts on the set $V$ of vertices of $G$. This action induces an action on the set $E$ of edges of $G$. In fact if $e = uv$ is an edge of $G$ and $\sigma \in \mathbb{A}ut(G)$ then $e^\sigma = u^\sigma v^\sigma$ is an edge of $G$ and this is a well-defined action of $\mathbb{A}ut(G)$ on $E$.

**Definition 2.3.** Let $G = (V, E)$ be a graph. $G$ is called *vertex-transitive* if $\mathbb{A}ut(G)$ acts transitively on the set $X$ of vertices of $G$. If $\mathbb{A}ut(G)$ acts transitively on the set $E$ of edges of $G$, then $G$ called an edge-transitive graph.

**Theorem 2.4.** *Let $G = (V, E)$ be a simple vertex-transitive graph and let $v \in V$ be a fixed vertex of $G$. Then*

$$W(G) = (1/2)\ |V|\ d(v),$$

*where*

$$d(v) = \sum_{x\in V} d(v, x).$$

**Theorem 2.5.** *Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of $G$. Then the Szeged index of $G$ is as follows:*

$$Sz(G) = |E| n_u(e|G) n_v(e|G).$$

**Theorem 2.6.** *Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of $G$. Then the PI-index of $G$ is as follows:*

$$PI(G) = |E|\Big(n_{eu}(e|G) + n_{ev}(e|G)\Big).$$

## 3. Computing the Wiener, the Szeged and PI-index of the Kneser graph

**Definition 3.1.** *The Kneser graph $KG_{n,k}$* is the graph whose vertices correspond to the $k$-element subsets of a set of $n$ elements, and where two vertices are connected if and only if the two corresponding sets are disjoint. Clearly we must impose the restriction $n \geq 2k$. Kneser graphs are named after Martin Kneser, who first investigated them. Therefore $KG_{n,k}$ has $\binom{n}{k}$ vertices, it is regular of degree $\binom{n-k}{k}$. The number of edges of $KG_{n,k}$ is $(1/2)\binom{n}{k}\binom{n-k}{k}$.

The complete graph on n vertices is the Kneser graph $KG_{n,1}$. The Kneser graph $KG_{2n-1,n-1}$ is known as the odd graph $O_n$, the odd graph $O_3 = KG_{5,2}$ is isomorphic to the Petersen graph.

If $\sigma$ is a permutation of $\Omega$ and $A \subseteq \Omega$ then $A^\sigma$ is defined by $A^\sigma := \{a^\sigma | a \in A\}$ which is again a subset of $\Omega$ of cardinality $|A|$. Therefore each permutation of $\Omega$ induces a permutation on the set of vertices of $KG_{n,k}$. If $AB$ is an edge of $KG_{n,k}$, then $A$ and $B$ are subset of $\Omega$ with cardinality $k$, where $|A \cap B| = \emptyset$ and for any permutation $\sigma$ of $\Omega$ we have $|A^\sigma \cap B^\sigma| = |A \cap B| = \emptyset$, which proves that $\sigma$ is an element of $\mathbb{A}ut(KG_{n,k})$. Therefore we have proved the following theorem:

**Theorem 3.2.** *The automorphism group of the Kneser graph $KG_{n,k}$ contains a subgroup isomorphic to the symmetric group on n letters.*

From the above fact we can show vertex and edge transitivity of the Kneser graph.

**Lemma 3.3.** *The Kneser graph is both vertex and edge transitive.*

*Proof.* Let $\Omega$ be a set of size $n$. Without loss of generality we may assume $\Omega = \{1, 2, ..., n\}$. Let the Kneser graph be defined on $\Omega$. Consider two distinct vertices $A$ and $B$ of $KG_{n,k}$. We may assume $A = \{1, 2, ..., k\}$, $B = \{1', 2', ..., k'\}$.Then we set $\Omega - A = \{k + 1, ..., n\}$ and $\Omega - B = \{(k + 1)', ..., n'\}$ and both are subsets of $\Omega$. Then $\pi : \Omega \to \Omega$ defined by $i \to i'$ is an element of the symmetric group $S_n$ which by Theorem 4 induces an element of $\mathbb{A}ut(KG_{n,k})$ and $A^\pi = B$. This proves that $KG_{n,k}$ is vertex-transitive.

Now assume $AB$ and $CD$ are distinct edges of $KG_{n,k}$. To prove edge-transitivity of $KG_{n,k}$ it is enough to show that there is a permutation $\pi$ on $\Omega$ such that $A^{\pi} = C$ and $B^{\pi} = D$. Without loss of generality we may assume that $A = \{1, 2, ..., k-1, k\}$, $B = \{k+1, ..., 2k-1, 2k\}$, $C = \{1', 2', ..., (k-1)', k'\}$, $D = \{(k+1)', ..., (2k-1)', (2k)'\}$. Then we set $\Omega - (A \cup B) = \{2k + 1, ..., n\}$ and $\Omega - (C \cup D) = \{(2k + 1)', ..., n'\}$ and both subsets $\Omega$. Now the permutation $\pi : \Omega \to \Omega$ defined by $i \to i'$ has the required property and the lemma is proved. □

**Lemma 3.4.** *Let $k \geq 2$ and $n \geq 2k + 2$. Then for any two vertices like $u$ and $v$ in $KG_{n,k}$ we have:*
*(a) $d(u, v) \leq 2$ if $n \geq 3k - 1$,*
*(b) $d(u, v) \leq 3$ if $n < 3k - 1$.*

*Proof. Case (a)*
Let $u, v$ be two distinct vertices in $KG_{n,k}$. We consider two cases:
*(1) $u \cap v = \emptyset$*
In this case we have $d(u, v) = 1$.
*(2) $|u \cap v| = i$, $1 \leq i \leq k - 1$*
Let $\Omega = \{1, 2, ..., n\}$ and $u, v$ be two distinct subset of $\Omega$ each of cardinality $k$. We show that there is a shortest path of length 2 from $u$ to $v$. Without loss of generality we may assume that $u = \{1, 2, ..., i, i+1, ..., k\}$ and $v = \{1, 2, ..., i, k + 1, ..., 2k - i\}$ such that $1 \leq i \leq k - 1$. We consider $c = \{2k, ..., 3k - 1\}$ which is possible because $n \geq 3k - 1$ therefore $ucv$ is a shortest path of length 2 from $u$ to $v$.
*Case (b)*
Let $u, v$ be two distinct vertices in $KG_{n,k}$. We consider two cases:
*(1) $u \cap v = \emptyset$*
In this case we have $d(u, v) = 1$.
*(2) $|u \cap v| = i$, $1 \leq i \leq k - 1$*
Let $\Omega = \{1, 2, ..., n\}$ and $u, v$ be two distinct subset of $\Omega$ each of cardinality $k$. We show that there is a path from $u$ to $v$ such that $d(u, v) = 2$ *or* 3. If $i = k - 1$ without loss of generality we may assume that $u = \{1, 2, ..., k - 1, k\}$, $v = \{1, 2, ..., k - 1, k + 1\}$ and $f = \{k + 2, ..., 2k + 1\}$ therefore $ufv$ is a path from $u$ to $v$ such that $d(u, v) = 2$. Now if $1 \leq i \leq k - 2$ without loss of generality we may assume that $u = \{1, 2, ..., i, i + 1, ..., k\}$ and $v = \{1, 2, ..., i, k + 1, ..., 2k - i\}$. We consider $c = \{k + 1, ..., 2k\}$, $d = \{i + 1, ..., k, 2k + 1, ..., 2k + i\}$ therefore $ucdv$ is a path from $u$ to $v$ such that $d(u, v) = 3$. □

By Lemma 3.4 the following Corollary follows:

**Corollary 3.5.** *For a positive integer $k \geq 2$ and $n \geq 2k + 1$, the Kneser graph $KG_{n,k}$ is connected.*

**Theorem 3.6.** *Let $k \geq 2$ and $n \geq 2k + 2$. The Wiener index of $KG_{n,k}$ is:*
(1) *If $n \geq 3k - 1$ then we have*

$$W(KG_{n,k}) = (1/2)\binom{n}{k}\left(\binom{n-k}{k} + 2\left(\binom{n}{k} - 1 - \binom{n-k}{k}\right)\right).$$

(2) *If $n < 3k - 1$, then we have*

$$W(KG_{n,k}) = (1/2)\binom{n}{k}\left(\binom{n-k}{k} + 2\left(\sum_{j=1}^{n-2k}\binom{k}{k-j}\binom{n-2k}{j}\right) + 3\left(\binom{n}{k} - 1 - \binom{n-k}{k} - \sum_{j=1}^{n-2k}\binom{k}{k-j}\binom{n-2k}{j}\right)\right).$$

*Proof.* By Lemma 3.3, $KG_{n,k}$ is vertex-transitive and by Theorem 2.4:

$$W(KG_{n,k}) = (1/2)\binom{n}{k}d(A)$$

where $A$ is a fixed vertex of $KG_{n,k}$ and $d(A) = \sum_B d(A, B)$, where $B$ is a subset of $\Omega$ with cardinality $k$.

*Case (1)* By Lemma 3.4 for any vertex like $u \in V$, the number of vertices like $v$ such that $d(u,v) = i$, $0 \leq i \leq 2$ is calculated as follows:

if $d(u,v) = 0$, then the number of choices for $v$ is 1, and if $d(u,v) = 1$ then the number of choices for $v$ is $\binom{n-k}{k}$ and if $d(u,v) = 2$ then the number of choices for $v$ is $\left(\binom{n}{k} - 1 - \binom{n-k}{k}\right)$. Therefore we have

$$W(KG_{n,k}) = (1/2)\binom{n}{k}\left(\binom{n-k}{k} + 2\left(\binom{n}{k} - 1 - \binom{n-k}{k}\right)\right).$$

*Case (2)* By Lemma 3.4 for any vertex like $u \in V$, the number of vertices like $v$ such that $d(u,v) = t$, $0 \leq t \leq 3$ is calculated as follows:

if $d(u,v) = 0$, then the number of choices for $v$ is 1, if $d(u,v) = 1$ then the number of choices for $v$ is $\binom{n-k}{k}$, if $d(u,v) = 2$ then the number of choices for $v$ is $\left(\sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j}\right)$ and if $d(u,v) = 3$ then the number of choices for $v$ is $\left(\binom{n}{k} - 1 - \binom{n-k}{k} - \left(\sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j}\right)\right)$. Therefore we have

$$W(KG_{n,k}) = (1/2)\binom{n}{k}\left(\binom{n-k}{k} + 2\left(\sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j}\right) + 3\left(\binom{n}{k} - 1 - \binom{n-k}{k} - \sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j}\right)\right).$$

□

**Lemma 3.7.** *Let $e = uv \in E(KG_{n,k})$.*

*(1) If $n \geq 3k - 1$, to calculate $n_u(e|KG_{n,k})$ it is enough to calculate vertices like $z$ in $V$ such that $d(u,z) \leq 1$ and $d(u,z) < d(v,z)$,*

*(2) If $n < 3k - 1$ ,to calculate $n_u(e|KG_{n,k})$ it is enough to calculate vertices like $z$ in $V$ such that $d(u,z) \leq 2$ and $d(u,z) < d(v,z)$.*

*Proof.* *Case (1)* , $n \geq 3k - 1$, by Lemma 3.4 for vertices like $u, v, z$ such that $u \neq v$ we have three cases:

(*a*) $d(u,z) = 0$, then $u = z$, therefore $z \in N_u(e|KG_{n,k})$.

(*b*) $d(u,z) = 1$, then $d(v,z) = 0$ *or* 1 *or* 2. If $d(v,z) = 0$ *or* 1, then $z \notin N_u(e|KG_{n,k})$ but if $d(v,z) = 2$, then $z \in N_u(e|KG_{n,k})$.

(*c*) $d(u,z) = 2$, then $d(v,z) = 0$ *or* 1 *or* 2, therefore $z \notin N_u(e|KG_{n,k})$.

*Case (2)* , $n < 3k - 1$, by Lemma 3.4 for vertices like $u, v, z$ such that $u \neq v$ we have four cases:

(*a*) $d(u,z) = 0$, then $u = z$, therefore $z \in N_u(e|KG_{n,k})$.

(*b*) $d(u,z) = 1$, then $d(v,z) = 0$ *or* 1 *or* 2 *or* 3. If $d(v,z) = 0$ *or* 1, then $z \notin N_u(e|KG_{n,k})$ but if $d(v,z) = 2$ *or* 3, then $z \in N_u(e|KG_{n,k})$.

(*c*) $d(u,z) = 2$, then $d(v,z) = 0$ *or* 1 *or* 2 *or* 3. If $d(v,z) = 0$ *or* 1 *or* 2, then $z \notin N_u(e|KG_{n,k})$ but if $d(v,z) = 3$, then $z \in N_u(e|KG_{n,k})$.

(*d*) $d(u,z) = 3$, then $d(v,z) = 0$ *or* 1 *or* 2 *or* 3, therefore $z \notin N_u(e|KG_{n,k})$. □

**Theorem 3.8.** *Let $k \geq 2$ and $n \geq 2k + 2$. The Szeged index of $KG_{n,k}$ is:*

*(1) If $n \geq 3k - 1$ then we have*

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\left(\binom{n-k}{k} - \binom{n-2k}{k}\right)^2.$$

*(2) If $n < 3k - 1$, then we have*

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\left(\sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j} + 1\right)^2.$$

*Proof.* Since by Lemma 3.3, $KG_{n,k}$ is edge-transitive, we can use Theorem 2.5 to write

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}n_u(e|KG_{n,k})n_v(e|KG_{n,k}),$$

where $e = uv$ is a fixed edge of $KG_{n,k}$. Since $KG_{n,k}$ is a symmetric graph therefore $n_u(e|KG_{n,k}) = n_v(e|KG_{n,k})$, hence

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\Big(n_u(e|KG_{n,k})\Big)^2.$$

We proceed to calculate $n_u(e|KG_{n,k})$. We define $E_i$, $0 \leq i \leq 2$ to be the number of vertices like $x$ in $V$ such that $d(u, x) = i$ and $d(u, x) < d(v, x)$

*Case (1)* By Lemma 3.7 it is enough to calculate $E_0$ and $E_1$. It is obvious that $E_0 = 1$ and $E_1 = \binom{n-k}{k} - 1 - \binom{n-2k}{k}$. Then we have

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\left(\binom{n-k}{k} - \binom{n-2k}{k}\right)^2.$$

*Case (2)* By Lemma 3.7 it is enough to calculate $E_0$, $E_1$ and $E_2$. It is obvious that $E_0 = 1$, $E_1 = \binom{n-k}{k} - 1$. Now we calculate $E_2 = \left(\sum_{j=1}^{n-2k}\binom{k}{k-j}\binom{n-2k}{j}\right) - E_1$, in fact for vertices like $z$ such that $d(u, z) = 2$ and $d(u, z) < d(v, z)$, we have $|u \cap z| = k - j$ where $1 \leq j \leq n - 2k$. Therefore the number of vertices like $z$ such that $d(u, z) = 2$ and $d(u, z) < d(v, z)$ is $\left(\sum_{j=1}^{n-2k}\binom{k}{k-j}\binom{n-2k}{j}\right) - E_1$ where $E_1$ is the number of vertices like $r$ such that $d(v, r) = 1$. Then we have

$$Sz(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\left(\sum_{j=1}^{n-2k}\binom{k}{k-j}\binom{n-2k}{j} + 1\right)^2.$$

□

**Theorem 3.9.** *Let $k \geq 2$ and $n \geq 2k + 2$. The PI-index of $KG_{n,k}$ is:*

*(1) If $n \geq 3k - 1$, then we have*

$$PI(KG_{n,k}) = \binom{n}{k}\binom{n-k}{k}(E_0 + E_1),$$

*where $E_0 = \binom{n-k}{k} - 1$ and*

$$E_1 = (1/2)\left(\sum_{i=1}^{k-1}\binom{k}{k-i}\binom{n-2k}{i}\left(\binom{n-2k}{k} - \binom{n-2k-i}{k}\right)\right) +$$

$$\sum_{i=1}^{k-1}\binom{k}{k-i}\binom{n-2k}{i}\left(\binom{n-k}{k} - \binom{n-k-i}{k} - \left(\binom{n-2k}{k} - \binom{n-2k-i}{k}\right)\right),$$

*(2) If $n < 3k - 1$, then we have*

$$PI(KG_{n,k}) = \binom{n}{k}\binom{n-k}{k}(S_0 + S_1 + S_2),$$

*where* $S_0 = \binom{n-k}{k} - 1$,

$$S_1 = \sum_{j=1}^{n-2k} \left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\binom{k}{k-j}\binom{n-2k}{j},$$

*and*

$$S_2 = \sum_{j=1}^{n-2k} \left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\left(\binom{n-k}{k} - 1\right).$$

*Proof.* Since by Lemma 3.3, $KG_{n,k}$ is edge-transitive, we can use Theorem 2.6 to write

$$PI(KG_{n,k}) = (1/2)\binom{n}{k}\binom{n-k}{k}\Big(n_{eu}(e|KG_{n,k}) + n_{ev}(e|KG_{n,k})\Big),$$

where $e = uv$ is a fixed edge of $KG_k$. Since $KG_{n,k}$ is a symmetric graph, therefore $n_{eu}(e|KG_{n,k}) = n_{ev}(e|KG_{n,k})$, hence

$$PI(KG_{n,k}) = \binom{n}{k}\binom{n-k}{k} n_{eu}(e|KG_{n,k}).$$

We proceed to calculate $n_{eu}(e|KG_{n,k})$.

*Case (1)* , $n \geq 3k-1$, by Lemma 3.7 we define $E_i$, $0 \leq i \leq 1$ to be the number of edges like $g$ in $E$ such that $d(u, g) = i$ and $d(u, g) < d(v, g)$. $E_0 = \binom{n-k}{k} - 1$, to calculate $E_1$ we consider two cases:

*(a)* Let $w \in V$, where $u, w$ are adjacent and $|w \cap v| = k - i$ where $1 \leq i \leq k-1$. In this case we calculate the number of edges like $f = st$ where $s, t \in V$, $d(u, t) = d(s, u) = 1$ and $d(v, f) > 1$. If $|w \cap v| = k - i$ where $1 \leq i \leq k-1$ then the number of vertices like $r$ such that $d(u, r) = d(v, r) = d(w, r) = 1$ is $\binom{n-2k-i}{k}$ and the number of choices $w$ is $\binom{k}{k-i}\binom{n-2k}{i}$ and the number of vertices that adjacent to $w, u$ but are not adjacent to $v$ is $\binom{k}{k-i}\binom{n-2k}{i}\Big(\binom{n-2k}{k} - \binom{n-2k-i}{k}\Big)$. Therefore the number of all edges in this case is

$$(1/2)\left(\sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\left(\binom{n-2k}{k} - \binom{n-2k-i}{k}\right)\right).$$

*(b)* Let $w \in V$, where $u, w$ are adjacent and $|w \cap v| = k - i$ where $1 \leq i \leq k-1$. In this case we calculate the number of edges like $g = ac$ where $a, c \in V$ and $d(u, a) = 1, d(c, u) = 2$ or $d(u, a) = 2, d(c, u) = 1$. It is enough to calculate the number of vertices like $t$ where $d(w, t) = 1$, $d(u, t) = 2$ and $d(v, t) > 2$. If $|w \cap v| = k - i$ where $1 \leq i \leq k-1$ then the number of choices for $t$ is $\binom{n-k}{k} - \binom{n-k-i}{k} - \Big(\binom{n-2k}{k} - \binom{n-2k-i}{k}\Big)$. In fact $\binom{n-k-i}{k}$ is the number of edges like $h$ such that $d(v, h) = 1$ and $\binom{n-2k}{k} - \binom{n-2k-i}{k}$ is the number of edges that calculated in *case*$(a)$. Therefore the number of all edges in this case is $\sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\Big(\binom{n-k}{k} - \binom{n-k-i}{k} - \Big(\binom{n-2k}{k} - \binom{n-2k-i}{k}\Big)\Big)$, hence the number of all edges in the cases $(a)$, $(b)$ is $(1/2)\Big(\sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\Big(\binom{n-2k}{k} - \binom{n-2k-i}{k}\Big)\Big) + \sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\Big(\binom{n-k}{k} - \binom{n-k-i}{k} - \Big(\binom{n-2k}{k} - \binom{n-2k-i}{k}\Big)\Big)$.

Therefore $PI(KG_{n,k}) = \binom{n}{k}\binom{n-k}{k}(E_0 + E_1)$, where $E_0 = \binom{n-k}{k} - 1$ and

$$E_1 = (1/2)\left(\sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\left(\binom{n-2k}{k} - \binom{n-2k-i}{k}\right)\right) +$$

$$\sum_{i=1}^{k-1} \binom{k}{k-i}\binom{n-2k}{i}\left(\binom{n-k}{k} - \binom{n-k-i}{k} - \left(\binom{n-2k}{k} - \binom{n-2k-i}{k}\right)\right),$$

*Case (2)* , $n < 3k-1$, by Lemma 3.7 we define $S_i$, $0 \leq i \leq 2$ to be the number of edges like $g$ in $E$ such that $d(u,g) = i$ and $d(u,g) < d(v,g)$. $S_0 = \binom{n-k}{k} - 1$, and similar to *case*$(b)$ above $S_1 = \sum_{j=1}^{n-2k} \binom{k}{k-j}\binom{n-2k}{j}\left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)$. Now to calculate $S_2$ it is enough we calculate the number of edges like $X \in E$ such that $d(u,X) = 2$ and $d(u,X) < d(v,X)$, therefore $S_2 = \sum_{j=1}^{n-2k} \left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\left(\binom{n-k}{k} - 1\right)$, because if let $w \in V$, where $u, w$ are adjacent and $|w \cap v| = k - j$ where $1 \leq j \leq i$, then the number of vertices like $r$ such that $d(v,r) = d(r,w) = 1$ is $\binom{n-k-j}{k}$, therefore the number of edges like $f$ such that $d(u,f) = 2$ and $d(u,f) < d(v,f)$ is $\left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\left(\binom{n-k}{k} - 1\right)$ where in fact $\binom{n-k}{k} - \binom{n-k-j}{k}$ is the number of vertices like $s$ such that $d(w,s) = 1$, $d(u,s) = 2$ and $d(u,s) < d(v,s)$ and $\binom{n-k}{k} - 1$ is the number of vertices like $z \in V$ such that $d(s,z) = 1$ and $d(u,z) = 3$. Hence we have $PI(KG_{n,k}) = \binom{n}{k}\binom{n-k}{k}(S_0 + S_1 + S_2)$, where $S_0 = \binom{n-k}{k} - 1$, $S_1 = \sum_{j=1}^{n-2k} \left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\binom{k}{k-j}\binom{n-2k}{j}$, and $S_2 = \sum_{j=1}^{n-2k} \left(\binom{n-k}{k} - \binom{n-k-j}{k}\right)\left(\binom{n-k}{k} - 1\right)$. $\square$

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