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Some remarks on the brachistochrone problem with Coulomb friction

Marian Mureşan[a]

[a] *"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, M. Kogălniceanu str., RO-400084, Cluj-Napoca, România*

Abstract. This paper introduces some results on the brachistochrone problem with Coulomb friction and nonzero initial velocity. The following boundary conditions are considered hereafter: fixed end points and fixed initial and final velocities.

1. Introduction

1.1. The mathematical statement of the problem

The aim of this section is to introduce the brachistochrone problem. A seminal paper on this topic from the time of Johann Bernoulli till nowdays with modern ideas and results is [21].

The figures of the present paper were drawn by *Mathematica*®.

The first problem formulated in terms that now belongs to what is called the calculus of variations was stated by Johann Bernoulli in June 1696 in Acta Eruditorum, a journal directed by G. W. Leibniz.

The problem stated by J. Bernoulli is also called the brachistochrone problem or the least time sliding, [7, p. 5], [13, pp. 100–102], [14, p. 33], [1, p. 25], and reads as follows. Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time. The term derives from the Greek $\beta\rho\alpha\chi\iota\sigma\tau o\sigma$ (brachistos) "the shortest" and $\chi\rho o\nu o\sigma$ (chronos) "time, delay". So this is a minimum time problem.

In a more modern language this problem is stated as follows. Consider in a vertical plane a rectangular system of axes so that Ou is the horizontal axis and Ox the vertical axis oriented vertically downwards the smooth curves connecting two fixed points $A = (u_0, x_0)$ and $B = (u_1, x_1)$ with $0 \le u_0 \ne u_1$, $0 \le x_0 < x_1$. Find the shape of the smooth curve down which a bead sliding with initial speed $v_0 \ge 0$ downward from A and accelerated by gravity will slip (without friction) to B in the least time. The case with nonzero kinetic friction is stated and discussed only for uniform and for Coulomb friction in Section 2.

We try to rephrase this problem in a mathematical language. Based on the law of conservation of energy we write

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = mg(x - x_0),$$

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Email address: mmarianus24@yahoo.com (Marian Mureşan)

where g is the gravitational acceleration. Thus the movement equation becomes

$$v(u) = \sqrt{2g}\,\sqrt{x - x_a}\,, \text{ with } x_a = x_0 - \frac{v_0^2}{2g}.$$

Because of $v = \mathrm{d}s/\,\mathrm{d}t$, where s is the arc length, [20], we have that $\mathrm{d}t = \dfrac{\mathrm{d}s}{\sqrt{2g}\,\sqrt{x - x_a}}$.

Therefore the time needed to follow the arc is

$$t = \int_{\widehat{AB}} \frac{\mathrm{d}s}{\sqrt{2g}\,\sqrt{x - x_a}} = \frac{1}{\sqrt{2g}} \int_{u_0}^{u_1} \sqrt{\frac{1 + x'^2(u)}{x(u) - x_a}}\,\mathrm{d}u,$$

where $(\cdot)'$ represents the derivative of the quantity $(\cdot)$ with respect to time t.

We conclude that the time t depends on the $\widehat{AB}$ arc, i.e., on the function that represents this arc. We also have two boundary constraints, namely $x(u_0) = x_0$ and $x(u_1) = x_1$. The arcs satisfying the last two constraints are said to be admissible arcs or feasible arcs to this problem. By an arc we understand an absolutely continuous function, [3, p. 4].

A survey on the brachistochrone problem without friction can be found in [17].

1.2. The Euler-Lagrange, Weierstrass-Erdmann, and the transversality necessary conditions

For our purposes we use some notions and notations from [7, Chapter 2]. Euler-Lagrange, Weierstrass-Erdmann, and the transversality necessary conditions are useful to us. To state these necessary conditions we consider the problem

$$\min \Lambda[x], \quad \Lambda[x] = \int_a^b L(t, x(t), x'(t))\,\mathrm{d}t, \quad x(t) = (x_1(t), \ldots, x_n(t)), \tag{1}$$

with phase constraints and boundary constraints

$$(t, x(t)) \in T \subset [a, b] \times \mathbb{R}^n, \text{ for all } t \in [a, b], \quad (a, x(a), b, x(b)) \in B. \tag{2}$$

We suppose that

$$T = \mathrm{cl}\,(\mathrm{int}\,T), \quad B \subset \mathbb{R}^{1+n+1+n} \text{ closed and } L(t, x, v) \in C^1(T \times \mathbb{R}^n; \mathbb{R}). \tag{3}$$

The set $\Omega \subset \mathrm{AC}([a, b]; \mathbb{R}^n)$ of feasible arcs is defined as

$$\Omega = \{x \mid (t, x(t)) \in T, \quad t \in [a, b], \quad L(\cdot, x(\cdot), x'(\cdot)) \in \mathcal{L}^1([a, b]) \text{ and } \quad (a, x(a), b, x(b)) \in B\}. \tag{4}$$

If the set B is a singleton, we do not need any other assumption. Otherwise there are necessary some assumptions on, at least, a neighborhood B_0 of the point $e[x] = (a, x(a), b, x(b)) \in B$. In such cases it is necessary that B_0 be a manifold of class C^1 of dimension k, $0 \le k \le 2n + 1$, and having a tangent hyperplane B' at $e[x]$ whose vectors are denoted

$$h = (\tau_1, \xi_1, \tau_2, \xi_2), \quad \xi_1 = (\xi_1^1, \ldots, \xi_1^1) = (\mathrm{d}x_1^1, \ldots, \mathrm{d}x_1^n), \quad \xi_2 = (\xi_2^1, \ldots, \xi_2^1) = (\mathrm{d}x_2^1, \ldots, \mathrm{d}x_2^n).$$

In what follows $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Theorem 1.1. ([7]) *Consider problem* (1)–(2) *together with conditions* (3)–(4) *and let $x^*(t)$, $a \le t \le b$, be an arc with essential bounded derivative, $(t, x^*(t)) \in \mathrm{int}\,T$, $t \in [a, b]$. Suppose that x^* supplies a strong local minimum of problem* (1)–(2). *Then*

(a) (Euler-Lagrange necessary condition) *The n functions $L_{v_i}(\cdot, x^*(\cdot), x^{*\prime}(\cdot))$ coincide a.e. with the n absolutely continuous functions, let $-\lambda_i(\cdot)$ and $-\lambda_i'(t) = L_{x_i}(t, x^*(t), x^{*\prime}(t))$. Thus we may write*

$$\frac{\mathrm{d}(L_{v_i}(t, x^*(t), x^{*\prime}(t))}{\mathrm{d}t} = L_{x_i}(t, x^*(t), x^{*\prime}(t)), \ \text{a.e. } t \in [a, b], \quad i = 1, \dots, n.$$

In vectorial form the function $\lambda(t) = -L_v(t, x^*(t), x^{*\prime}(t))$ is absolutely continuous and

$$\frac{\mathrm{d}(L_v(t, x^*(t), x^{*\prime}(t))}{\mathrm{d}t} = L_x(t, x^*(t), x^{*\prime}(t)), \ \text{a.e. } t \in [a, b].$$

(b) (Weierstrass-Erdmann necessary condition) *If x^* is continuous with piecewise continuous derivative on the interval $[a, b]$, then at each point t_0 of the first kind of discontinuity of $x^{*\prime}$, we have*

$$\begin{aligned}
&L_{v_i}(t_0, x^*(t_0), x^{*\prime}(t_0 - 0)) = L_{v_i}(t_0, x^*(t_0), x^{*\prime}(t_0 + 0)), \quad i = 1, \dots, n,\\
&L(t_0, x^*(t_0), x^{*\prime}(t_0 - 0)) - \langle x^{*\prime}(t_0 - 0), L_v(t_0, x^*(t_0), x^{*\prime}(t_0 - 0))\rangle\\
=&L(t_0, x^*(t_0), x^{*\prime}(t_0 + 0)) - \langle x^{*\prime}(t_0 + 0), L_v(t_0, x^*(t_0), x^{*\prime}(t_0 + 0))\rangle.
\end{aligned}$$

(c) (Transversality necessary condition) *In the case of free boundary problems let B' be the tangent hyperplane to B at $e[x] = (a, x^*(a), b, x^*(b))$. Consider $h = (\tau_1, \xi_1, \tau_2, \xi_2) = (\mathrm{d}t_1, \mathrm{d}x_1, \mathrm{d}t_2, \mathrm{d}x_2)$ an arbitrary element in B'. Then the following transversality equality*

$$\Delta = \left(\left(L - \sum_{i=1}^{n} x_i^{*\prime} L_{v_i}\right) \mathrm{d}t + \sum_{i=1}^{n} L_{v_i} \,\mathrm{d}x_i\right)\Bigg|_{(a,x^*(a))}^{(b,x^*(b))} = 0$$

is true for all tangent vectors $h = (\mathrm{d}t_1, \mathrm{d}x_1, \mathrm{d}t_2, \mathrm{d}x_2) \in B'$, the tangent hyperplane to B at $(a, x^*(a), b, x^*(b))$. Here the coefficients of $\mathrm{d}t$ and $\mathrm{d}x_i$ are the absolutely continuous functions $M(\cdot)$, respectively, $-\lambda(\cdot)$ evaluated at the points $(a, x^*(a))$ and $(b, x^*(b))$. If B has at $(a, x^*(a), b, x^*(b))$ only a tangent cone B', then $\Delta \geq 0$, for all $h \in B'$.

1.3. The Lagrange multipliers rule for a Bolza problem

The problem to be considered in this subsection is that of finding in the class $C^1[a, b]$ the arcs

$$x_i = x_i(t), \quad i = 1, \dots, n, \quad t \in [a, b], \tag{5}$$

satisfying the differential equations and boundary conditions

$$\varphi_\alpha(t, x, x') = 0, \quad \alpha = 1, \dots, m < n, \tag{6}$$

$$\psi_\mu(a, x(a), b, x(b)) = 0, \quad \mu = 1, \dots, p \leq 2n + 2, \tag{7}$$

on which minimizes a Bolza functional of the form

$$I = G(a, x(a), b, x(b)) + \int_a^b f(t, x, x') \,\mathrm{d}t. \tag{8}$$

Theorem 1.2. ([15], [5]) *Suppose that the functions φ, ψ, G, and f are of class C^2. For every minimizing arc x^* for the problem of Bolza* (5)–(8) *there exist constants c_i and a function*

$$F = \lambda_0 f + \sum_{\alpha=1}^{m} \lambda_\alpha(t)\varphi_\alpha$$

such that the equations

$$F_{x_i'} = \int_a^t F_{x_i} \,\mathrm{d}s + c_i, \quad \varphi_\alpha = 0, \quad t \in [a, b],$$

hold at every point of x^*, furthermore such that the end-points of x^* satisfy, besides the equations $\psi_\mu = 0$, the condition that

$$\left.\left(F - \sum_{i=1}^{n} x_i' F_{x_i'}\right) \mathrm{d}t + \sum_{i=1}^{n} F_{x_i'} \,\mathrm{d}x_i \right|_a^b + \lambda_0 \,\mathrm{d}G = 0$$

for every set of differentials $\mathrm{d}t_1$, $\mathrm{d}x_{i_1}$, $\mathrm{d}t_2$, $\mathrm{d}x_{i_2}$ which satisfy the equations $\mathrm{d}\psi_\mu = 0$. The first multiplier λ_0 is a constant, and the multipliers $\lambda_\alpha(\cdot)$ are continuous except possible at the values of t defining corners of x^*. The elements of the set $\lambda_0, \lambda_\alpha(\cdot)$ do not vanish simultaneously at any point of x^*.

2. Brachistochrone with Coulomb friction

We here introduce the brachistochrone problem with friction. The case of uniform friction with zero initial velocity is discussed in Subsection 2.2. Subsection 2.3 is dedicated to the brachistochrone problem with Coulomb friction and nonzero initial velocity.

2.1. Kinematics of a particle

Let a Cartesian coordinate system uOx coincides with a vertical plane so that Ou is the horizontal axis and Ox is the axis oriented vertically upwards (see Fig. 1). We are looking for the smooth curves connecting two fixed points $A = (u_0, x_0)$ and $B = (u_1, x_1)$ with $u_0, u_1 \geq 0$, $u_0 \neq u_1$, and $0 \leq x_1 < x_0$ so that a bead M sliding with initial speed $v_0 \geq 0$ downward from A and accelerated by gravity will slip with a nonlinear kinetic friction to B in the least time T.

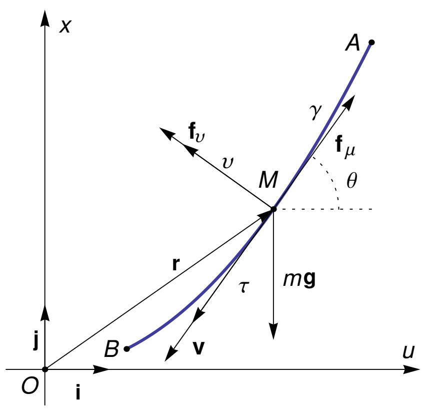

Figure 1: Curvilinear motion

We suppose for the beginning that there exists a solution represented by a sufficiently smooth curve γ and an arbitrary point M lying on γ. Let $\boldsymbol{\tau}$ be the unit tangent vector to γ at M, $\mathbf{v}$ be the velocity vector of the bead M, $\boldsymbol{\nu}$ be the unit normal vector to γ at M, $\mathbf{g}$ be the acceleration gravity vector, $\mathbf{f}_\mu$ be the friction force, $\mathbf{f}_\nu$ be the normal component of the constraint reaction force, θ be the slope angle of the tangent, and $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors of the Cartesian coordinate system uOx.

The position of a particle M relative to the coordinate system uOx is determined by the position vector $\mathbf{r}$ (see Fig. 1). The particle M is moving from A to B, so its position vector $\mathbf{r}$ is a function of time t, i.e.,

$$\mathbf{r} = \mathbf{r}(t) = (u(t), x(t)).$$

The velocity of the particle M at time t is defined as

$$\mathbf{v} = \mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = (u'(t), x'(t)).$$

Denote $v(t) = \|\mathbf{v}(t)\|$ and $g = \|\mathbf{g}\|$. Then $\|\boldsymbol{\tau}\| = \|\boldsymbol{\nu}\| = \|\mathbf{i}\| = \|\mathbf{j}\| = 1$ and $\langle\boldsymbol{\tau}, \boldsymbol{\nu}\rangle = \langle\mathbf{i}, \mathbf{j}\rangle = 0$. From Fig. 1, we have that

$$\begin{aligned} &\boldsymbol{\tau} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}, && \boldsymbol{\nu} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, && \frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j} = -\boldsymbol{\nu},\\ &\mathbf{i} = -\cos\theta\,\boldsymbol{\tau} - \sin\theta\,\boldsymbol{\nu}, && \mathbf{j} = -\sin\theta\,\boldsymbol{\tau} + \cos\theta\,\boldsymbol{\nu}, && \mathbf{v} = -v\cos\theta\mathbf{i} - v\sin\theta\mathbf{j} \end{aligned}$$

and

$$\begin{aligned} &u'(t) = -v(t)\cos\theta, && x'(t) = -v(t)\sin\theta,\\ &v(t) = \sqrt{u'^2(t) + x'^2(t)}, && \theta'(t) = \frac{u'(t)x''(t) - u''(t)x'(t)}{v^2(t)}. \end{aligned} \tag{9}$$

It is obvious that $\|\mathrm{d}\boldsymbol{\tau}/\mathrm{d}\theta\| = 1$ and $\langle\boldsymbol{\tau}, \mathrm{d}\boldsymbol{\tau}/\mathrm{d}\theta\rangle = 0$.

The acceleration of the particle M at instant t is defined and expressed as

$$\mathbf{a} = \mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}(v(t)\boldsymbol{\tau})}{\mathrm{d}t} = \frac{\mathrm{d}v(t)}{\mathrm{d}t}\boldsymbol{\tau} + v(t)\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}.$$

Thus we have

$$\mathbf{a}(t) = v'(t)\,\boldsymbol{\tau} + v(t)\,\theta'(t)\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}. \tag{10}$$

Newton's second law of the motion of the particle M is

$$m\mathbf{a} = \mathbf{w} + \mathbf{f}_\mu + \mathbf{f}_\nu, \tag{11}$$

where $\mathbf{w}$ is the weight of the particle, i.e.,

$$\mathbf{w} = m\mathbf{g} = -mg\mathbf{j} = mg\left(\sin\theta\,\boldsymbol{\tau} + \cos\theta\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}\right), \tag{12}$$

$\mathbf{f}_\mu$ is the friction force,

$$\mathbf{f}_\mu(t) = -f(t)\boldsymbol{\tau}(t), \tag{13}$$

and $\mathbf{f}_\nu$ is the normal component of the constraint reaction force,

$$\mathbf{f}_\nu(t) = n(t)\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}. \tag{14}$$

Taking into account (10) and (12)–(14), Eq. (11) can be written as

$$m\left(v'(t)\,\boldsymbol{\tau} + v\,\theta'(t)\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}\right) = mg\left(\sin\theta(t)\,\boldsymbol{\tau} + \cos\theta(t)\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}\right) - f(t)\,\boldsymbol{\tau} + n(t)\,\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\theta}.$$

Since for each t the unit vectors $\boldsymbol{\tau}$ and $\mathrm{d}\boldsymbol{\tau}/\mathrm{d}\theta$ are linearly independent, from the previous equation we get the following system of differential equations of the motion to the particle

$$\begin{cases} mv'(t) = mg\sin\theta(t) - f(t),\\ mv(t)\theta'(t) = mg\cos\theta(t) + n(t), \end{cases} \tag{15}$$

or, after dividing by m (m is a nonzero constant),

$$\begin{cases} v'(t) = g\sin\theta(t) - \overline{f}(t),\\ v(t)\theta'(t) = g\cos\theta(t) + \overline{n}(t), \end{cases}$$

where $\overline{f} = f/m$ and $\overline{n} = n/m$.

Remark 2.1. For general $\overline{f}$ and $\overline{n}$ the previous system is difficult to integrate getting solutions in closed form.

2.2. Brachistochrone with uniform friction, fixed end points, and zero initial velocity

Problem 2.2. Similar to the statement in Subsection 1.1, consider in a vertical plane a rectangular system of axes so that Ou is the horizontal axis and Ox the vertical axis oriented downwards the smooth curves connecting two fixed points $A = (u_0, x_0)$ and $B = (u_1, x_1)$ with $u_0 \geq 0, x_0, x_1 \geq 0$. Find the shape of the smooth curve down which a bead sliding with null initial velocity downward from A and accelerated by gravity will slip with uniform kinetic friction to B in the least time.

Solution 2.2. Since the kinetic friction denoted μ is involved, the terms corresponding to the normal component of weight and the normal component of the acceleration (present because of path curvature) must be included. Including both terms requires a constrained variational technique ([2], [11]), but including the normal component of weight only gives an approximate solution. The tangent and normal vectors are

$$\boldsymbol{\tau} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \boldsymbol{\nu} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j},$$

weight and friction forces are then

$$\mathbf{w} = mg\mathbf{j}, \quad \mathbf{f}_\mu = -\mu|\langle \mathbf{w}, -\boldsymbol{\nu}\rangle|\boldsymbol{\tau} = -\mu mg\cos\theta\,\boldsymbol{\tau},$$

and the components along the curve are

$$\langle \mathbf{w}, \boldsymbol{\tau}\rangle = mg\sin\theta \text{ and } \left\langle \mathbf{f}_\mu, \boldsymbol{\tau}\right\rangle = -\mu mg\cos\theta.$$

The first equation in (15) gives that

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg\frac{\mathrm{d}x}{\mathrm{d}s} - \mu mg\frac{\mathrm{d}u}{\mathrm{d}s},$$

where $\mathbf{v} = v\tau$, and $\mathbf{v}$ is the velocity vector. But

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}s} \implies v^2 = 2g(x-\mu u) \implies v = \sqrt{2g(x-\mu u)},$$

so

$$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{1+x'^2}}{\sqrt{2g(x-\mu u)}}\,\mathrm{d}u \implies t = \int\sqrt{\frac{1+x'^2}{2g(x-\mu u)}}\,\mathrm{d}u.$$

Using the Euler-Lagrange necessary condition leads to $(1+x'^2)(1+\mu x') + 2(x-\mu u)x'' = 0$.

This can be reduced to

$$\frac{1+x'^2}{(1+\mu x')^2} = \frac{c}{x-\mu u}, \quad c \text{ constant.}$$

Denote $x' = \cot(\theta/2)$ and

$$\mathrm{d}u = \frac{c}{2}\,\frac{\sin\theta + 2\mu\cos\theta - \mu^2\sin\theta}{\cot(\theta/2)-\mu}\,\mathrm{d}\theta,$$
$$\mathrm{d}x = \frac{c}{2}\,\frac{\cos(\theta/2)(\sin\theta + 2\mu\cos\theta - \mu^2\sin\theta)}{\cos(\theta/2)-\mu\sin(\theta/2)}\,\mathrm{d}\theta,$$

the solution follows as

$$\begin{cases} u = u_0 + \dfrac{k^2}{2}\,(\theta - \sin\theta + \mu(1-\cos\theta)), \\ x = x_0 + \dfrac{k^2}{2}\,(1-\cos\theta + \mu(\theta+\sin\theta)), \end{cases} \tag{16}$$

where $k^2 = 1/(2gc^2)$.

Figure 2 contains three particular graphs for (16). □

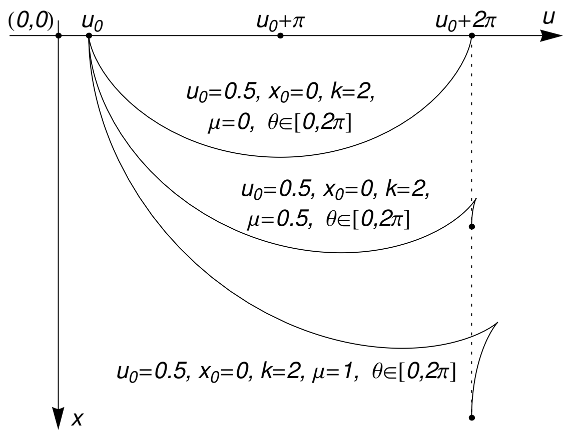

Figure 2: Brachistochrone with uniform friction

Remark 2.3. (a) The above approach can be found in [12] and [22].

(b) We look for the final angle θ_1 as a solution of the nonlinear equation

$$(u_1 - u_0)(1 - \cos\theta_1 + \mu(\theta_1 + \sin\theta_1)) = (x_1 - x_0)(\theta_1 - \sin\theta_1 + \mu(1 - \cos\theta_1)).$$

Then we determine the constant k by

$$k = \sqrt{\frac{2(u_1 - u_0)}{\theta_1 - \sin\theta_1 + \mu(1 - \cos\theta_1)}}\,.$$

Now we can write the parametric equations of the brachistochrone curve joining the points A and B with uniform friction of constant coefficient μ.

(c) The angle of the motion along a trajectory does not depend on the coefficient of friction μ. Indeed,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\sin\theta + \mu(1 + \cos\theta)}{1 - \cos\theta + \mu\sin\theta} = \frac{\cos(\theta/2)(\sin(\theta/2) + \mu\cos(\theta/2))}{\sin(\theta/2)(\sin(\theta/2) + \mu\cos(\theta/2))} = \cot\frac{\theta}{2}\,.$$

(d) The velocity of the motion along a trajectory is given by

$$v = \sqrt{u'^2 + x'^2} = k^2\theta'\left|\sin\frac{\theta}{2} + \mu\cos\frac{\theta}{2}\right|.$$

2.3. Brachistochrone with Coulomb friction and nonzero initial velocity

In the present subsection we consider the brachistochrone problem with Coulomb friction and nonzero initial velocity.

2.3.1. Statement of the problem

We only mention some papers on this subject, namely, [18] and [21].

We recall system (15), namely,

$$\begin{cases} mv'(t) = mg\sin\theta(t) - f(t), \\ mv(t)\theta'(t) = mg\cos\theta(t) + n(t), \end{cases}$$

where $n(t) = \langle \mathbf{f}_\nu,\ \mathrm{d}\tau/\,\mathrm{d}\theta\rangle$ and $f(t) = \mu|n(t)|$.

Similarly to [18], we assume that the normal component of the constraint reaction force is continuously oriented opposite to the gravitational force, that is,

$$n(t) \le 0. \tag{17}$$

From (15), by eliminating $n(t)$, we find the first differential constraint

$$f_1(t) = v'(t) - g\sin\theta(t) + \mu(-v\theta'(t) + g\cos\theta(t)) = 0. \tag{18}$$

We recall (9) under the form of other two differential (nonholonomic) constraints

$$f_2(t) = u'(t) + v(t)\cos\theta(t) = 0, \quad f_3(t) = x'(t) + v(t)\sin\theta(t) = 0. \tag{19}$$

From (17), by the second equation in (15), we are led to

$$v(t)\theta'(t) - g\cos\theta(t) \le 0.$$

We now introduce, based on [18], an unknown function q' so that the previous inequality becomes an equality and this is the next and last differential constraint to our problem

$$f_4(t) = v(t)\theta'(t) - g\cos\theta(t) + {q'}^2(t) = 0. \tag{20}$$

All together we have four differential constraints, namely, $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, and $f_4 = 0$.

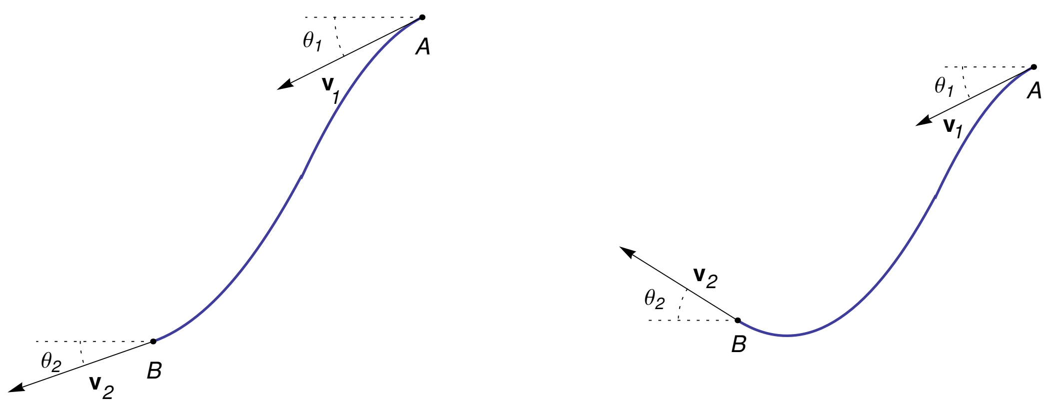

Figure 3: The idea of the case in Section 2.3

Let us introduce the following boundary conditions

$$u(0) = u_0, \quad x(0) = x_0, \quad v(0) = v_0, \tag{21}$$
$$u(T) = u_T, \quad x(T) = x_T, \quad v(T) = v_T. \tag{22}$$

Now, our problem can be expressed as the following Lagrange problem

$$I = \int_0^T \mathrm{d}t \to \min, \tag{23}$$

with the boundary conditions (21) and (22) and the constraints (18)–(20).

2.3.2. Transformation of the problem by the Lagrange multipliers rule

Accordingly to Theorem 1.2, [4], [5], [6, Ch. VII] or [9], we can transform this problem by the Lagrange multipliers rule into an unconstrained Lagrange problem of the form

$$\begin{aligned} J &= \int_0^T \left(1 + \sum_1^4 \lambda_i f_i\right) \mathrm{d}t \\ &= \int_0^T \{1 + \lambda_1[v'(t) - g\sin\theta(t) + \mu(-v(t)\theta'(t) + g\cos\theta(t))] \\ &\qquad + \lambda_2[u'(t) + v(t)\cos\theta(t)] + \lambda_3[x'(t) + v(t)\sin\theta(t)] + \lambda_4[v(t)\theta'(t) - g\cos\theta(t) + q'^2(t)]\}\,\mathrm{d}t. \end{aligned} \tag{24}$$

where $\lambda_1, \dots, \lambda_4$ are functions depending upon t and they have to be found, with the following transversality condition

$$\{\left[F - \left(u'F_{u'} + x'F_{x'} + v'F_{v'} + \theta'F_{\theta'} + q'F_{q'}\right)\right] \mathrm{d}t + F_{u'}\,\mathrm{d}u + F_{x'}\,\mathrm{d}x + F_{v'}\,\mathrm{d}v + F_{\theta'}\,\mathrm{d}\theta + F_{q'}\,\mathrm{d}q\}\Big|_0^T = 0, \tag{25}$$

for every set of differentials $\mathrm{d}u_0$, $\mathrm{d}u_1$, $\mathrm{d}x_0$, $\mathrm{d}x_1$, $\mathrm{d}v_0$, $\mathrm{d}v_1$, $\mathrm{d}\theta_0$, $\mathrm{d}\theta_1$, $\mathrm{d}q_0$, and $\mathrm{d}q_1$ satisfying the boundary conditions (21) and (22). Here F represents the integrand of the functional (24), that is,

$$F = F(u', x', v, v', \theta, \theta', q', \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 1 + \sum_1^4 \lambda_i f_i. \tag{26}$$

The multipliers λ_i are continuous except possibly for the values of t defining the corners of the extremal solutions $x^{*\prime}$.

2.3.3. Transformation of the state variables

Accordingly to [10], introducing the following transformations of the state variables

$$\begin{aligned} &z_1 = u, \quad z_2 = x, \quad z_3 = v, \quad z_4 = \theta, \quad z_5' = q', \\ &z_6' = \lambda_1, \quad z_7' = \lambda_2, \quad z_8' = \lambda_3, \quad z_9' = \lambda_4, \end{aligned}$$

the integrand of the functional (24) can be written as

$$\begin{aligned} &L(z_1', z_2', z_3, z_3', z_4, z_4', z_5', z_6', z_7', z_8', z_9') \\ &= 1 + z_6'\left(z_3' - \mu z_3 z_4' + g(\mu\cos z_4 - \sin z_4)\right) + z_7'(z_1' + z_3\cos z_4) + z_8'(z_2' + z_3\sin z_4) + z_9'(z_3 z_4' - g\cos z_4 + z_5'^2) \end{aligned}$$

and the problem now is

$$J = \int_0^T L(z_1', z_2', z_3, z_3', z_4, z_4', z_5', z_6', z_7', z_8', z_9')\,\mathrm{d}t \ \to \min,$$

with the transversality condition

$$\left[\left(L - \sum_{i=1}^9 z_i' L_{z_i'}\right) \mathrm{d}t + \sum_{i=1}^9 L_{z_i'}\,\mathrm{d}z_i\right]\Bigg|_0^T = 0$$

and the boundary conditions

$$z_1(0) = u_0, \quad z_2(0) = x_0, \quad z_3(0) = v_0, \quad z_i(0) = 0, \quad i = 5, \dots, 9, \tag{27}$$

$$z_1(T) = u_T, \quad z_2(T) = x_T, \quad z_3(T) = v_T. \tag{28}$$

2.3.4. Solving the problem in z variables

The corresponding Euler-Lagrange equations ((a) in Theorem 1.1) are

$$\frac{\partial L}{\partial z_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_i'}\right) = 0, \quad i = 1, \dots, 9, \tag{29}$$

whereas the boundary conditions are

$$\begin{aligned} &\left(\frac{\partial L}{\partial z_i'}\right)\mathrm{d}z_i\Big|_0^T = 0, \quad i = 1, \dots, 9, \\ &\left[\left(L - \sum_{i=1}^{9} \frac{\partial L}{\partial z_i'} z_i'\right)\mathrm{d}t\right]\Big|_0^T = 0. \end{aligned} \tag{30}$$

Since for z_4 there is no initial or final requirement, we have the following natural boundary conditions

$$\begin{aligned} &\left(\frac{\partial L}{\partial z_4'}\right)\Big|_0 = 0 \implies (-\mu v\lambda_1 + v\lambda_4)\big|_0 = 0, \\ &\left(\frac{\partial L}{\partial z_4'}\right)\Big|_T = 0 \implies (-\mu v\lambda_1 + v\lambda_4)\big|_T = 0. \end{aligned} \tag{31}$$

Since $z_5, \dots z_9$ are not given at the final position, we have more natural boundary conditions of the form

$$\left(\frac{\partial L}{\partial z_5'}\right)\Big|_T = 0 \implies (q'\lambda_4)\big|_T = 0, \tag{32}$$

$$\left(\frac{\partial L}{\partial z_6'}\right)\Big|_T = 0 \implies (v' - \mu v\theta' + g(\mu\cos\theta - \sin\theta))\big|_T = 0, \tag{33}$$

$$\left(\frac{\partial L}{\partial z_7'}\right)\Big|_T = 0 \implies (u' + v\cos\theta)|_T = 0, \tag{34}$$

$$\left(\frac{\partial L}{\partial z_8'}\right)\Big|_T = 0 \implies (x' + v\sin\theta)|_T = 0, \tag{35}$$

$$\left(\frac{\partial L}{\partial z_9'}\right)\Big|_T = 0 \implies \left(v\theta' - g\cos\theta + q'^2\right)\big|_T = 0. \tag{36}$$

The final time is not specified, thus the second equation in (30) leads to the following transversality condition

$$\left(L - \sum_{i=1}^{9} \frac{\partial L}{\partial z_i'} z_i'\right)\Big|_T = 0 \tag{37}$$

or in expanded form

$$(1 + \lambda_1 g(\mu\cos\theta - \sin\theta) + v(\lambda_2\cos\theta + \lambda_3\sin\theta) \; -\lambda_4(q'^2 + g\cos\theta))\big|_T = 0. \tag{38}$$

Euler-Lagrange equations (29) explicitly are written as follows

$$\frac{\partial L}{\partial z_3} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_3'}\right) = 0, \tag{39}$$

$$\frac{\partial L}{\partial z_4} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_4'}\right) = 0, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_1'}\right) = 0, \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_2'}\right) = 0, \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_5'}\right) = 0, \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_6'}\right) = 0, \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_7'}\right) = 0, \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_8'}\right) = 0, \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z_9'}\right) = 0. \tag{47}$$

So in order to give an answer to our problem we have to solve the system composed by the differential equations (39)–(47) with the boundary conditions (27)–(28) and (31)–(36), and with the transversality condition (37).

2.3.5. Return to the original variables

From (39)–(47) we get the following system of nonlinear differential equations

$$(41) \implies \lambda_2' = 0 \implies \lambda_2 = c_2^*, \ \text{constant} \tag{48}$$

$$(42) \implies \lambda_3' = 0 \implies \lambda_3 = c_3^*, \ \text{constant} \tag{49}$$

$$(39) \implies \lambda_1' + \mu\theta'\lambda_1 - \lambda_2 \cos\theta - \lambda_3 \sin\theta - \lambda_4\theta' = 0, \tag{50}$$

$$(40) \implies (g\cos\theta + \mu g \sin\theta - \mu v')\lambda_1 + v\sin\theta\,\lambda_2 - v\cos\theta\,\lambda_3 - \mu v\lambda_1' + v'\lambda_4 - g\sin\theta\lambda_4 + v\lambda_4' = 0, \tag{51}$$

$$(44) \implies v' - \mu v\theta' + g(\mu\cos\theta - \sin\theta) = 0, \tag{52}$$

$$(43) \implies (q'\lambda_4)' = 0 \overset{(32)}{\implies} q'\lambda_4 \equiv 0, \tag{53}$$

$$(45) \implies u' + v\cos\theta = 0, \tag{54}$$

$$(46) \implies x' + v\sin\theta = 0, \tag{55}$$

$$(47) \implies v\theta' - g\cos\theta + {q'}^2 = 0. \tag{56}$$

We now discuss the consequences of the (31)–(38) boundary conditions. Because the initial (u_0, x_0) point, the final (u_1, x_1) point, the initial speed $v|_0 = v_0$, and the final speed $v|_T = v_1$, are all fixed, we have that

$$\mathrm{d}u|_0 = \mathrm{d}u|_T = \mathrm{d}x|_0 = \mathrm{d}x|_T = \mathrm{d}v|_0 = \mathrm{d}v|_T = 0.$$

We further have

$$(31) \implies (-\mu v\lambda_1 + v\lambda_4)|_0 = 0 \ \text{ and } \ (-\mu v\lambda_1 + v\lambda_4)|_T = 0, \tag{57}$$

Therefore the first equation in (30) is satisfied for every $i = 1, \ldots, 9$. Taking into account the condition (37) and the fact that $\mathrm{d}t|_0 = 0$, the second equation in (30) is satisfied.

2.3.6. Solving the Euler-Lagrange equations

Now by (53) we have that either $q' \equiv 0$ or $\lambda_4 \equiv 0$. So the cases $q' \equiv 0$ and $\lambda_4 \equiv 0$ do not hold simultaneously. This implies that on the interval $[0, T]$ (T is still unknown) an extremal may consists of curves on which $q' \equiv 0$ and $\lambda_4 \not\equiv 0$ and of curves on which $q' \not\equiv 0$ and $\lambda_4 \equiv 0$. Consequently the functional J may have corner points. If s is the total number of corner points of an extremal with the corresponding time instances t_p, $p \in \{1, \dots, s\}$, then the Weierstrass-Erdmann necessary conditions ((b) of Theorem 1.1) require that

$$\left.\left(\frac{\partial L}{\partial z_i'}\right)\right|_{t_p-0} = \left.\left(\frac{\partial L}{\partial z_i'}\right)\right|_{t_p+0}, \quad i = 1, \dots, 9, \quad p = 1, \dots, s, \tag{58}$$

$$\left.\left(L - \sum_{i=1}^{9} \frac{\partial L}{\partial z_i'} z_i'\right)\right|_{t_p-0} = \left.\left(L - \sum_{i=1}^{9} \frac{\partial L}{\partial z_i'} z_i'\right)\right|_{t_p+0}, \quad p = 1, \dots, s, \tag{59}$$

By (41)–(47) we have that (58) is satisfied identically for $i = 1, 2, 5, \dots, 9$ and $p = 1, \dots, s$. Because the integrand L does not depend explicitly on t, the equations (48)–(56) in regard to (37) have the first integral of the form

$$L - \sum_{i=1}^{9} \frac{\partial L}{\partial z_i'} z_i' = 0 \text{ on } [0, T], \tag{60}$$

the conditions (59) are satisfied. The quantities u, x, and v are continuous at the corner points of the extremal, i.e.,

$$u(t_p - 0) = u(t_p + 0), \quad x(t_p - 0) = x(t_p + 0), \quad v(t_p - 0) = v(t_p + 0). \tag{61}$$

By the last equation in (61), the Weierstrass-Erdmann conditions (58) for $i = 3, 4$ and $p = 1, \dots, s$ are reduced to

$$\lambda_1(t_p - 0) = \lambda_1(t_p + 0), \quad \lambda_4(t_p) = 0, \quad p = 1, \dots, s. \tag{62}$$

Case 1. $q' \equiv 0$ and $\lambda_4 \not\equiv 0$.

From (48) and (49) we have that

$$\lambda_2 = c_2^*, \quad \lambda_3 = c_3^*. \tag{63}$$

According to [18] it follows that

$$v = v(\theta) = \frac{c_1}{\cos\theta}, \quad c_1 \text{ nonzero integration constant} \tag{64}$$

and

$$u = -\frac{c_1^2}{g} \tan\theta + c_2, \tag{65}$$

$$x = -\frac{c_1^2}{g} \tan^2\theta + c_3, \tag{66}$$

$$t = \frac{c_1}{g} \tan\theta + c_4, \tag{67}$$

where c_1, c_2, c_3, and c_4 are constants. Equations (65)–(66) represent the parametric equations of a free-fall parabola in a nonresistant environment.

From (48), (64), and the change of independent variable $\mathrm{d}(\cdot)/\,\mathrm{d}t = \theta'(\,\mathrm{d}(\cdot)/\,\mathrm{d}\theta)$, the equations (50) and (51) yield

$$\frac{\mathrm{d}(\lambda_4 - \mu\lambda_1)}{\mathrm{d}\theta} = -\lambda_1 + \frac{c_1(-\lambda_2 \sin\theta + \lambda_3 \cos\theta)}{g\cos^2\theta}, \tag{68}$$

$$\frac{d\lambda_1}{d\theta} = (\lambda_4 - \mu\lambda_1) + \frac{c_1(\lambda_2\cos\theta + \lambda_3\sin\theta)}{g\cos^2\theta}. \tag{69}$$

This system of linear differential equations in λ_1 and λ_4 gives

$$\lambda_1 = c_5\sin\theta + c_6\cos\theta + \frac{c_1}{g}\lambda_2\sin\theta + \frac{c_1}{g}\lambda_3\sin\theta\tan\theta, \tag{70}$$

$$\lambda_4 = (\cos\theta + \mu\sin\theta)c_5 + (\mu\cos\theta - \sin\theta)c_6 + \frac{c_1}{g}\lambda_3\sin\theta(1+\mu\tan\theta) + \frac{c_1}{g}\lambda_2\sin\theta(\mu - \tan\theta). \tag{71}$$

Case 2. $q' \not\equiv 0$ and $\lambda_4 \equiv 0$.

As before by (48) and (49), we have that

$$\lambda_2 = c_2^{**}, \quad \lambda_3 = c_3^{**}. \tag{72}$$

According to [18] it follows that

$$v = -\frac{1}{B}\frac{\sec\theta}{2\mu(\tan\theta + A) - \sec^2\theta}, \tag{73}$$

$$\lambda_1 = \frac{1}{g}\frac{\sec\theta(\tan\theta + A)}{\sec^2\theta - 2\mu(\tan\theta + A)}, \tag{74}$$

where

$$A = \frac{\mu\lambda_2 + \lambda_3}{\mu\lambda_3 - \lambda_2}, \quad B = \frac{\mu\lambda_3 - \lambda_2}{1+\mu^2} \tag{75}$$

and

$$u = u(\theta) = \frac{1}{2gB^2}\left\{b_1\left[\arctan\left(\frac{\tan\theta - \mu}{a_2}\right) + \frac{a_2(\tan\theta-\mu)}{(\tan\theta-\mu)^2 + a_2^2}\right]\right\} - \frac{2\mu(a_2^2 + a_1(\tan\theta - \mu))}{a_2^2[(\tan\theta-\mu)^2 + a_2^2]^2}\Bigg\} + c_u, \tag{76}$$

$$\begin{aligned} x = x(\theta) = \frac{1}{2gB^2}&\left\{b_2\left[\arctan\left(\frac{\tan\theta - \mu}{a_2}\right) + \frac{a_2(\tan\theta-\mu)}{(\tan\theta-\mu)^2 + a_2^2}\right]\right. \\ &\left. - \frac{1}{(\tan\theta-\mu)^2 + a_2^2} - \frac{2\mu(a_2^2 + \mu a_1)(\tan\theta-\mu) + a_2^2(\mu - a_1)}{a_2^2 + [(\tan\theta-\mu)^2 + a_2^2]^2}\right\} + c_x, \end{aligned} \tag{77}$$

where

$$a_1 = -(A+\mu), \qquad a_2 = \sqrt{1 + \mu^2 + 2\mu a_1},$$

$$b_1 = \frac{a_2^2 - 3\mu a_1}{a_2^5}, \qquad b_2 = \frac{\mu(a_2^2 - 3\mu a_1)}{a_2^5}.$$

We also have the equation of t depending on the angle θ,

$$t = t(\theta) = c_t - \left[b_3\arctan\left(\frac{\tan\theta - \mu}{a_2}\right) - \frac{2\mu(a_2^2 + a_1(\tan\theta-\mu))}{a_2^2[(\tan\theta-\mu)^2 + a_2^2]}\right], \tag{78}$$

where

$$b_3 = \frac{a_2^2 - 2\mu a_1}{a_2^3}.$$

2.4. Arrangement of the line segments on the extremal

Let L_1 and L_2 be the line segments for $q' \equiv 0$, $\lambda_4 \not\equiv 0$ and $q' \not\equiv 0$, $\lambda_4 \equiv 0$, respectively. Let n_{L_1} and n_{L_2} be the total number of line segments L_1 and L_2, respectively, on the extremal of the functional J. By (63), (72), and the conditions (58) ($i = 1, 2$, $p = 1, \dots, s$) we are led to the conclusion that

$$\lambda_2 = c_2^*, \quad \lambda_3 = c_3^*, \quad \text{for all } t \in [0, T]. \tag{79}$$

The arrangement of the line segments on the extremal has to satisfy the condition that the total number of unknown integration constants c_2, c_3, c_1, c_5, c_6 (c_4 is irrelevant regarding the arrangement of the line segments on the brachistochrone) on the line segments L_1, the integration constants c_u and c_x on the line segments L_2, the unknown values of angles $\theta_p = \theta(t_p)$, $p = 1, \dots, s$, and the unknowns c_2^* and c_3^* are equal to the number of available conditions for them to be determined.

If the extremal of the functional J has the beginning and the ending line segment of the type L_1, then in such a case is true that $n_{L_1} = \lfloor s/2 \rfloor + 1$ and $n_{L_2} = \lfloor s/2 \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function, [16, §1.2]. Based on conditions (21)–(22), equations (64) and (65)–(67), the integration constants c_2, c_3, and c_4 may be replaced on the beginning line segment by the functional dependencies on the unknown $\theta_0 = \theta(0)$, and on the ending line segment by the functional dependencies on the unknowns $\theta_f = \theta(T)$ and $v_f = v(T)$. Now, in regard to (79), there is a total of $9 + s + 5(n_{L_1} - 2) + 2n_{L_2} = 4 + 9\lfloor s/2 \rfloor$ unknowns, or in expanded form

$$|\overbrace{\{c_2, c_3, c_1, c_5, c_6, c_2^*, c_3^*, c_5, c_6\}}^{9}| + |\overbrace{\{t_1, \dots, t_s\}}^{s}| + |\overbrace{\{c_2, c_3, c_1, c_5, c_6\}}^{5}|(n_{L_1} - 2) + |\overbrace{\{c_u, c_x\}}^{2}|n_{L_2},$$

and for them to be determined there are available the natural boundary conditions (57) and the conditions (61) and (62) which give an amount of $2 + 5s$ conditions. For this variant, the equation $4 + 9\lfloor s/2 \rfloor = 2 + 5s$ gives that $s = 4$.

Based on the above results we state the following theorem.

Theorem 2.4. *Consider the brachistochrone problem given by the equations* (15) *with the boundary conditions* (21) *and* (22) *and an extremal of it such that the first and the last segment are of type L_1. Then this extremal has four corners points. The parametric equations of the extremal are given by* (65)–(66) *and* (76)–(77).

The other cases can be discussed similarly.

3. Concluding remarks

In [18] Šalinić studied the brachistochrone problem with Coulomb friction and nonzero initial speed. The boundary conditions considered in [18] are: fixed end points and fixed initial velocity. Since here we imposed a final speed, we obtained four corner points whereas in [18] two corner points are found. In [18] the authors also discussed reachability and computation problems.

In spite of the fact that the brachistochrone problem appeared more than 300 years ago it is far from being completely solved. We mention only two developments: the tunnel problem ([19]) and the brachistochrone problem on manifolds ([21], [8]).

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