Filomat 26:6 (2012), 1081–1089
DOI 10.2298/FIL1206081J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the edge monophonic number of a graph

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Abstract. For a connected graph $G = (V, E)$, an edge monophonic set of G is a set $M \subseteq V(G)$ such that every edge of G is contained in a monophonic path joining some pair of vertices in M. The edge monophonic number $m_1(G)$ of G is the minimum order of its edge monophonic sets and any edge monophonic set of order $m_1(G)$ is a minimum edge monophonic set of G. Connected graphs of order p with edge monophonic number p are characterized. Necessary condition for edge monophonic number to be $p-1$ is given. It is shown that for every two integers a and b such that $2 \leq a \leq b$, there exists a connected graph G with $m(G) = a$ and $m_1(G) = b$, where $m(G)$ is the monophonic number of G.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [2]. A chord of a path $u_0, u_1, u_2, ..., u_h$ is an edge u_iu_j, with $j \geq i+2$. An $u - v$ path is called a monophonic path if it is a chordless path. The monophonic path in a connected graph is introduced in [8]. A monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set of G. The monophonic number of a graph G is studied in [3–6]. It was shown that in [7] that determining the monophonic number of a graph is NP-complete. The edge geodetic number of a graph is introduced in [1] and further studied in [9]. An edge monophonic set of G is a set $M \subseteq V(G)$ such that every edge of G is contained in a monophonic path joining some pair of vertices in M. The edge monophonic number $m_1(G)$ of G is the minimum order of its edge monophonic sets and any edge monophonic set of order $m_1(G)$ is a minimum edge monophonic set of G. The maximum degree of G, denoted by $\Delta(G)$, is given by $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex v in G. For any set S of vertices of G, the induced subgraph $< S >$ is the maximal subgraph of G with vertex set S. A vertex v is a simplicial vertex of a graph G if $< N(v) >$ is complete. A vertex v is an universal vertex of a graph G, if it is a full degree vertex of G. A graph G is geodetic if each pair of vertices in G is joined by a unique shortest path. The join of graphs G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. For the graph G given in Figure 1.1, $M = \{v_2, v_4\}$ is a monophonic set of G so that $m(G) = 2$ and $S = \{v_1, v_3, v_6, v_7\}$ is the minimum edge monophonic set for

2010 *Mathematics Subject Classification.* 05C05
Keywords. Monophonic path; Monophonic number; Edge monophonic number
Received: 07 Feb 2011; Accepted: 07 March 2012
Communicated by Dragan Stevanović
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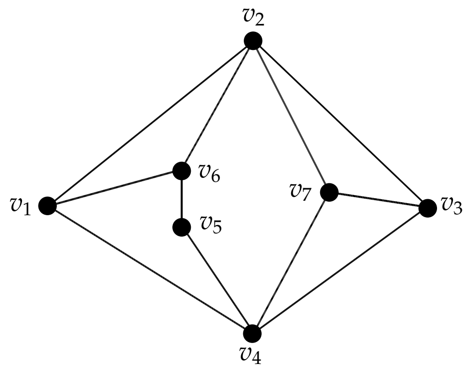
G so that $m_1(G) = 4$.

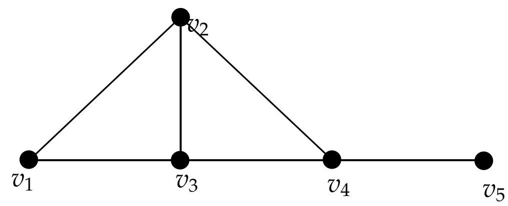


G
Figure 1.1

2. Some results on edge monophonic number of a graph

Definition 2.1. *A vertex v in a connected graph G is said to be a semi-simplicial vertex of G if $\Delta(< N(v) >) = |N(v)| - 1$.*

Remark 2.2. *Every simplicial vertex of G is a semi-simplicial vertex of G but the converse is not true. For the graph G given in Figure 2.1, v_1 and v_5 are semi-simplicial vertices of G and also they are simplicial vertices of G. Now, v_2 and v_3 are semi-simplicial vertices of G but not simplicial vertices of G.*



G
Figure 2.1

Theorem 2.3. *Each semi-simplicial vertex of G belongs to every edge monophonic set of G.*

Proof. Let M be an edge monophonic set of G. Let v be a semi-simplicial vertex of G. Suppose that $v \notin M$. Let u be a vertex of $< N(v) >$ such that $deg_{<N(v)>}(u) = |N(v)| - 1$. Let $u_1, u_2, ..., u_k (k \geq 2)$ be the neighbors of u in $< N(v) >$. Since M is an edge monophonic set of G, the edge uv lies on the monophonic path $P : x, x_1, ..., u_i, u, v, u_j, ..., y$, where $x, y \in M$. Since v is a semi-simplicial vertex of G, u and u_j are adjacent in G and so P is not a monophonic path of G, which is a contradiction. □

Corollary 2.4. *Each simplicial vertex of G belongs to every edge monophonic set of G.*

Proof. Since every simplicial vertex of G is a semi-simplicial vertex of G, the result follows from Theorem 2.3. □

Theorem 2.5. *Let G be a connected graph, v be a cut vertex of G and let M be an edge monophonic set of G. Then every component of $G - v$ contains an element of M.*

Proof. Let v be a cut vertex of G and M be an edge monophonic set of G. Suppose there exists a component, say G_1 of $G - v$ such that G_1 contains no vertex of M. By Corollary 2.4, M contains all the simplicial vertices of G and hence it follows that G_1 does not contains any simplicial vertex of G. Thus G_1 contains at least one edge, say xy. Since M is an edge monophonic set, xy lies on the $u - w$ monophonic path $P : u, u_1, u_2, ..., v, ..., x, y, ..., v_1, ..., v..., w$. Since v is a cut vertex of G, the $u - x$ and $y - w$ sub paths of P both contains v and so P is not a path, which is a contradiction. □

Theorem 2.6. *No cut vertex of a connected graph G belongs to any minimum edge monophonic set of G.*

Proof. Let M be a minimum edge monophonic set of G and $v \in M$ be any vertex. We claim that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, ..., G_r, (r \geq 2)$ be the components of $G - v$. By Theorem 2.5, each component $G_i (1 \leq i \leq r)$ contains an element of M. We claim that $M_1 = M - \{v\}$ is also an edge monophonic set of G. Let xy be an edge of G. Since M is an edge monophonic set, xy lies on a monophonic path P joining a pair of vertices u and v of M. Assume without loss of generality that $u \in G_1$. Since v is adjacent to at least one vertex of each $G_i (1 \leq i \leq r)$, assume that v is adjacent to z in $G_k, k \neq 1$. Since M is an edge monophonic set, vz lies on a monophonic path Q joining v and a vertex w of M such that w must necessarily belongs to G_k. Thus $w \neq v$. Now, since v is a cut vertex of G, the union $P \cup Q$ is a path joining u and w in M and thus the edge xy lies on this monophonic path joining two vertices u and w of M_1. Thus we have proved that every edge that lies on a monophonic path joining a pair of vertices u and v of M also lies on a monophonic path joining two vertices of M_1. Hence it follows that every edge of G lies on a monophonic path joining two vertices of M_1, which shows that M_1 is an edge monophonic set of G. Since $|M_1| = |M| - 1$, this contradicts the fact that M is a minimum edge monophonic set of G. Hence $v \notin M$ so that no cut vertex of G belongs to any minimum edge monophonic set of G. □

Corollary 2.7. *For any non trivial tree T, the edge monophonic number $m_1(G)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique minimum edge monophonic set of T.*

Proof. This follows from Corollary 2.4 and Theorem 2.6. □

Corollary 2.8. *For the complete graph $K_p (p \geq 2), m_1(K_p) = p$.*

Proof. Since every vertex of the complete graph $K_p (p \geq 2)$ is a simplicial vertex, by Corollary 2.4, the vertex set of K_p is the unique edge monophonic set of K_p. Thus $m_1(K_p) = p$. □

Corollary 2.9. *For every pair k, p of integers with $2 \leq k \leq p$, there exists a connected graph G of order p such that $m_1(G) = k$.*

Proof. For $k = p$, the result follows from Corollary 2.8. Also, for each pair of integers with $2 \leq k \leq p$, there exists a tree of order p with k end vertices. Hence the result follows from Corollary 2.7. □

Theorem 2.10. *For the cycle $C_p (p \geq 4), m_1(C_p) = 2$.*

Proof. Let $C_p : v_1, v_2, ..., v_p, v_1$ be the cycle. Let x, y be two non adjacent vertices of C_p. Then it is clear that $\{x, y\}$ is an edge monophonic set of C_p so that $m_1(C_p) = 2$. □

Theorem 2.11. *For the complete bipartite graph $G = K_{m,n}$*
(i) $m_1(G) = 2$ if $m = n = 1$
(ii) $m_1(G) = n$ if $n \geq 2, m = 1$
(iii) $m_1(G) = min\{m, n\}$ if $m, n \geq 2$.

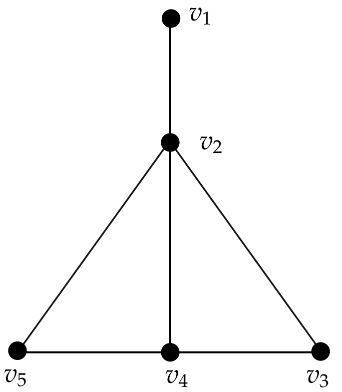
Proof. (i) This follows from Corollary 2.8.
(ii) This follows from Corollary 2.7.
(iii) Let $m, n \geq 2$. First assume that $m < n$.

Let $U = \{u_1, u_2, ..., u_m\}$ and $W = \{w_1, w_2, ..., w_n\}$ be a bipartition of G.Let $M = U$. We prove that M is a minimum edge monophonic set of G. Any edge $u_iw_j (1 \leq i \leq m, 1 \leq j \leq n)$ lies on the monophonic path u_i, w_j, u_k for any $k \neq i$ so that M is an edge monophonic set of G. Let T be any set of vertices such that $|T| < |M|$. If $T \subseteq U$, there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then for any edge $u_i\, w_j (1 \leq j < n)$, the only monophonic path containing u_iw_j are $u_i, w_j, u_k (k \neq i)$ and $w_j, u_i, w_l (l \neq j)$ and so u_iw_j cannot lie in a monophonic path joining two vertices of T. Thus T is not an edge monophonic set of G. If $T \subseteq W$, again T is not an edge monophonic set of G by a similar argument. If $T \subseteq U \cup W$ such that T contains at least one vertex from each of U and W, then, since $|T| < |M|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge u_iw_j does not lie on a monophonic path connecting two vertices of T so that T is not an edge monophonic set. Thus in any case T is not an edge monophonic set of G. Hence M is a minimum edge monophonic set so that $m_1(G) = |M| = m$. Now, if $m = n$, we can prove similarly that $M = U$ or W is a minimum edge monophonic set of G. Thus the theorem follows. □

Remark 2.12. *For any connected graph G of order* $p, 2 \leq m(G) \leq m_1(G) \leq p$.

Proof. A monophonic set needs at least two vertices and therefore $m(G) \geq 2$. Also every edge monophonic set is a monophonic set of G and then $m(G) \leq m_1(G)$. Clearly the set of all vertices of G is an edge monophonic set of G so that $m_1(G) \leq p$. Thus $2 \leq m(G) \leq m_1(G) \leq p$. □

Remark 2.13. *The bounds in Remark 2.12 are sharp. The set of the two end vertices of a path* $P_p (p \geq 2)$ *is its unique edge monophonic set so that* $m_1(P_p) = 2$. *For any non trivial tree T,* $m(T) = m_1(T) =$ *number of end vertices of T. For the complete graph* $G = K_p (p \geq 2), m_1(G) = p$. *Also, the inequalities in the remark can be strict. For the graph G given in Figure 2.2,* $m(G) = 3, m_1(G) = 4, p = 5$ *so that* $2 < m(G) < m_1(G) < p$.



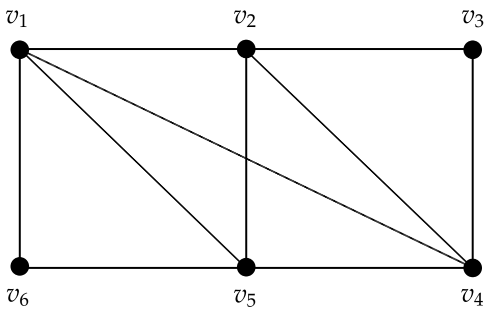
G
Figure 2.2

Corollary 2.14. *Let G be a connected graph with k semi-simplicial vertices. Then* $max(2, k) \leq m_1(G) \leq p$.

Proof. This follows from Theorem 2.3 and Remark 2.12. □

Definition 2.15. *A graph G is said to be a semi-simplicial graph if every vertices of G is a semi-simplicial vertex of G.*

Remark 2.16. *Complete graphs are semi-simplicial graphs. A graph with at least two universal vertex is also semi-simplicial graph. In fact, there are certain semi-simplicial graphs without any universal vertex as the following example shows.*



A semi-complete graph G without any universal vertex
Figure 2.3

Theorem 2.17. *For a semi-simplicial graph $G, m_1(G) = p$.*

Proof. This follows from Theorem 2.3. □

The following Theorem characterizes graphs for which the edge monophonic number is p.

Theorem 2.18. *Let G be a connected graph of order p. Then $m_1(G) = p$ if and only if G is a semi-simplicial graph.*

Proof. If G is a semi-simplicial graph, then by Theorem 2.17, $m_1(G) = p$. Conversely, let $m_1(G) = p$. We claim that G is a semi-simplicial graph. If not, let there exists a vertex v in G such that v is not a semi-simplicial vertex of G. Then for each $w \in N(v)$, there exists $z_w \in [N(v) - \{w\}]$ such that $wz_w \notin E(G)$. Let $M = V(G) - \{v\}$. Consider the edge wv. Since $w, z_w \in M$, the edge wv lies on the monophonic path w, v, z_w. Then M is an edge monophonic set of G with $|M| = p - 1$, which is a contradiction. Therefore, G is a semi-simplicial graph. □

We give below necessary conditions on a graph G for which $m_1(G) = p - 1$.

Theorem 2.19. *Let G be a connected graph of order p. If there exists a unique vertex $v \in V(G)$ such that v is not a semi-simplicial vertex of G, then $m_1(G) = p - 1$.*

Proof. Suppose that there exists a unique vertex $v \in V(G)$ such that v is not a semi-simplicial vertex of G. Then by Theorem 2.3, $m_1(G) \geq p - 1$. Let $M = V(G) - v$. Let $f, h \in V(G)$ such that $e = fh \in E(G)$. If $f, h \in M$, then the edge e lies on the monophonic path fh itself. Therefore, any one of f or h is v, say $f = v$. Since v is not a semi-simplicial vertex of G, there exists $a \in N(v)$ such that $ha \notin E(G)$. Therefore, $e = fh$ is an edge of the monophonic path a, f, h. Hence M is an edge monophonic set of G and so $m_1(G) \leq p - 1$. Therefore, $m_1(G) = p - 1$. Hence the result. □

Corollary 2.20. *Let G be a connected graph of order $p \geq 3$. If G contains exactly one universal vertex, then $m_1(G) = p - 1$.*

Corollary 2.21. *For the wheel $W_{1,p-1} (p \geq 4)$, $m_1(W_{1,p-1}) = p - 1$.*

Theorem 2.22. *Let G be a connected graph of order p_1 with exactly one universal vertex and H be a connected graph of order p_2 with exactly one universal vertex. Then $m_1(G + H) = p_1 + p_2$.*

Proof. Let $u \in V(G)$ and $v \in V(H)$ such that $deg_G(u) = p_1 - 1$ and $deg_H(v) = p_2 - 1$. Now, it is clear that $deg_{G+H}(u) = p_1 + p_2 - 1$ and $deg_{G+H}(v) = p_1 + p_2 - 1$. Then by Theorem 2.18, $m_1(G + H) = p_1 + p_2$. □

For the graph G given Figure 2.1 and in Corollaries 2.20 and 2.21, we see that $m_1(G) = p - 1$. Also it is to be noted that G has unique non semi-simplicial vertex. So we have the following conjecture.

Conjecture 2.23. *Let G be a connected graph of order $p \geq 3$ with $m_1(G) = p - 1$. Then there exists a unique vertex $v \in V(G)$ such that v is not a semi-simplicial vertex of G.*

3. Edge monophonic number of a geodetic graph

Theorem 3.1. *If G is a non complete connected graph such that it has a minimum cutset of G consisting of i independent vertices, then $m_1(G) \leq p - i$.*

Proof. Since G is non complete, it is clear that $1 \leq i \leq p-2$. Let $U = \{v_1, v_2, ..., v_i\}$ be a minimum independent cutset of vertices of G. Let $G_1, G_2, ..., G_m$ $(m \geq 2)$ be the components of $G - U$ and let $M = V(G) - U$. Then every vertex $v_j (1 \leq j \leq i)$ is adjacent to at least one vertex of G_t for every $t (1 \leq t \leq m)$. Let uv be an edge of G. If uv lies in one of G_t for any $t(1 \leq t \leq m)$ then clearly uv lies on the monophonic path (uv itself) joining two vertices u and v of M. Otherwise, uv is of the form $v_j u (1 \leq j \leq i)$, where $u \in G_t$ for some t such that $1 \leq t \leq m$. As $m \geq 2, v_j$ is adjacent to some w in G_s for some $s \neq t$ such that $1 \leq s \leq m$. Thus $v_j u$ lies on the monophonic path u, v_j, w. Thus M is an edge monophonic set of G so that $m_1(G) \leq |V(G) - U| = p - i$. □

Corollary 3.2. *If G is a connected non complete graph such that it has a minimum cutset of G consisting of i independent vertices, then $m_1(G) \leq p - \kappa$, where κ is the vertex connectivity of G.*

Proof. By Theorem 3.1, $m_1(G) \leq p - i$. Since $\kappa \leq i$, it follows that $m_1(G) \leq p - \kappa$. □

Theorem 3.3. *If G is a non complete connected geodetic graph such that U a minimum cutset, then every element of U are independent.*

Proof. Let $U = \{u_1, u_2, ..., u_k\}$ be a cut set of G. Let $G_1, G_2, ..., G_r, (r \geq 2)$ be the components of $G - U$. Suppose that u_1 and u_2 are adjacent. Let x, y be the vertices of G_1 which are adjacent to u_1 and u_2 respectively. Let x_1, y_1 be the vertices of G_2 which are adjacent to u_1 and u_2 respectively.

Case 1. $x_1 = y_1$.
Subcase 1a. $x = y$. Then x, u_2, x_1, u_1, x is an even cycle of length four, which is a contradiction to G is a geodetic graph.
Subcase1b. xy is an edge. Then u_1, u_2, y, x, u_1 is an even cycle of length four, which is a contradiction to G is a geodetic graph.
Subase 1c. $x - y$ is a path of length at least two in G_1. Let the $x - y$ path be $P : x, w_1, w_2, ..., w, y$. Then either $x_1, u_1, x, w_1, w_2, ..., w_n, y, u_2, x_1$ or $u_1, x, w_1, w_2, ..., w_n, y, u_2, u_1$ is an even cycle, which is a contradiction.

Case 2. $x - y$ is a path of length at least two in G_1 and $x_1 - y_1$ is a path of length at least two in G_2. Then by similar argument we get a contradiction. In all cases we get a contradiction. Therefore every element of U are independent. □

Theorem 3.4. *If G is a connected non complete geodetic graph, then $m_1(G) \leq p - \kappa$.*

Proof. This follows from Theorems 3.2 and 3.3. □

The following theorem shows that in a geodetic graph only the complete graph has the edge monophonic number p.

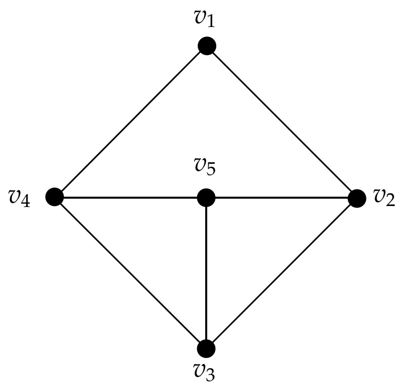
Theorem 3.5. *If G is a geodetic graph. Then $m_1(G) = p$ if and only if $G = K_p$.*

Proof. Let G be a geodetic graph and let $G = K_p$. Then it is clear that $m_1(G) = p$. Now, let $m_1(G) = p$. If $G \neq K_p$, then by Theorem 3.4, $m_1(G) \leq p - \kappa$, which is a contradiction. Therefore $G = K_p$. □

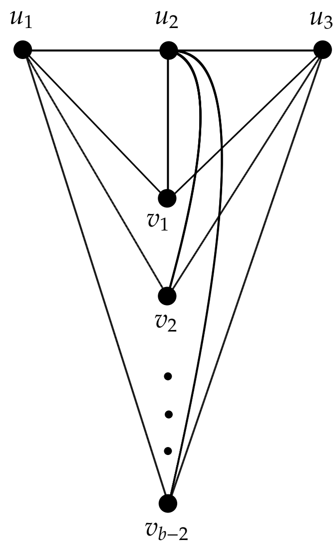
In view of Remark 2.12, we have the following realization theorem.

Theorem 3.6. *For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $m(G) = a$ and $m_1(G) = b$.*

Proof. If $a = b$, take $G = K_{1,a}$. Then it is clear that the set of end vertices of G is the unique monophonic set of G so that $m(G) = a$. By Corollary 2.7, $m_1(G) = a$. If $a = 2, b = 3$, then for the graph G given in Figure 3.1, $m(G) = 2$ and $m_1(G) = 3$. If $a = 2, b \geq 4$, let G be the graph given in Figure 3.2 obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding $b - 2$ new vertices $v_1, v_2, ..., v_{b-2}$ and joining each $v_i (1 \leq i \leq b - 2)$ with u_1, u_2, u_3. It is clear that u_1, u_3 is a monophonic set of G so that $m(G) = 2 = a$. Since u_2 is the only universal vertex of G, it follows from Corollary 2.20 that $m_1(G) = b - 2 + 3 - 1 = b$.



G
Figure 3.1



G
Figure 3.2

If $a \geq 3, b \geq 4, b \neq a + 1$, let G be the graph given in Figure 3.3 obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding the new vertices $v_1, v_2, ..., v_{b-a-1}$ and $w_1, w_2, ..., w_{a-1}$ and joining each $v_i (1 \leq i \leq b-a-1)$ with u_1, u_2, u_3 and also joining each $w_i (1 \leq i \leq a - 1)$ with u_1 and u_2. First we show that $m(G) = a$. Since each $w_i (1 \leq i \leq a - 1)$ is a simplicial vertex of G, it is clear that each $w_i (1 \leq i \leq a - 1)$ belongs to every monophonic set of G. Let $W = \{w_1, w_2, ..., w_{a-1}\}$. Then W is not a monophonic set of G. However, $W \cup \{u_3\}$ is a monophonic set of G and so $m(G) = a$. Next we show that $m_1(G) = b$. Since u_2 is the only universal vertex of G, it follows from Corollary 2.20 that $m_1(G) = b - a - 1 + a - 1 + 3 - 1 = b$.

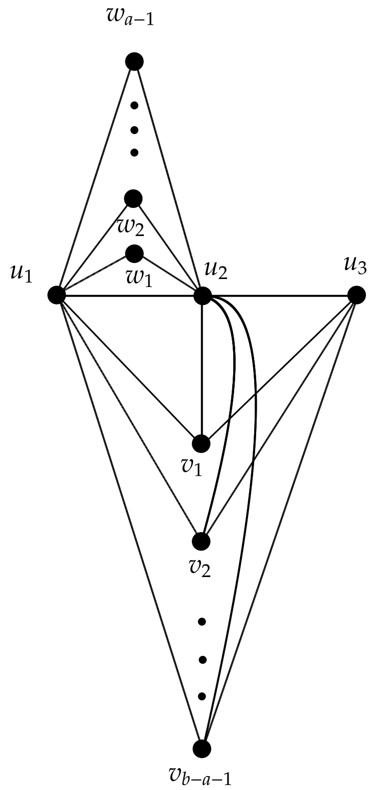


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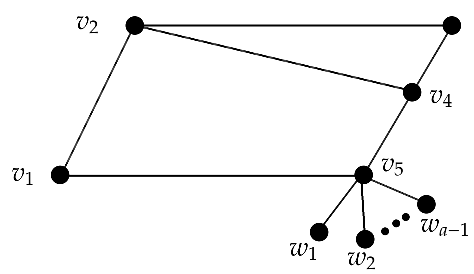
Figure 3.3

If $a \geq 3, b \geq 4$ and $b = a + 1$, consider the graph G given in Figure 3.4. Let $W = \{w_1, w_2, ..., w_{a-1}, v_3\}$ be the set of simplicial vertices of G. It is clear that W is contained in every monophonic set of G. It is easily seen that W is a monophonic set of G and so $m(G) = a$. By Theorem 2.3, W is contained in every edge monophonic set of G. But W is not an edge monophonic set of G. However, $W \cup \{v_2\}$ is an edge monophonic set of G so that $m_1(G) = b = a + 1$. □



G
Figure 3.4

References

[1] M. Atici, On the edge geodetic number of a graph. International Journal of Computer Mathematics, 80(2003), 853-861.
[2] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA, 1990.
[3] Carmen Hernando, Tao Jiang, Merce Mora, Ignacio. M. Pelayo and Carlos Seara, On the Steiner, geodetic and hull number of graphs, Discrete Mathematics, 293 (2005), 139-154.
[4] Esamel M. Paluga, Sergio R. Canoy, Jr., Monophonic numbers of the join and Composition of connected graphs, Discrete Mathematics, 307 (2007) 1146 - 1154.
[5] J. John and S. Panchali, The upper monophonic number of a graph, International J. Math. Combin., 4 (2010), 46-52.
[6] Mitre C. Dourado, Fabio protti and Jayme. L. Szwarcfiter, Algorithmic Aspects of Monophonic Convexity, Electronic Notes in Discrete Mathematics, 30 (2008) 177-182.
[7] Mitre C. Dourado, Fabio Protti, Jame L. Szwarcfiter, Complexity results related to monophonic complexity, Discrete Applied Mathematics, 158(12)(2010), 1268-1274.
[8] Pierre Duchet, Convex sets in graphs, II. Minimal Path Convexity, Journal of Combinatorial Theory Series B, 44(3)(1987), 307-316.
[9] A. P. Santhakumaran and J. John, Edge Geodetic Number of a Graph, Journal of Discrete Mathematical Sciences and Cryptography, 10(3)(2007), 415-432.