Filomat 26:6 (2012), 1209–1214
DOI 10.2298/FIL1206209C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The (C, α) integrability of functions by weighted mean methods

İbrahim Çanak[a], Ümit Totur[b]

[a]*Ege University, Department of Mathematics, 35100 Izmir, Turkey*
[b]*Adnan Menderes University, Department of Mathematics, 09010 Aydin, Turkey*

Abstract. Let $p(x)$ be a nondecreasing continuous function on $[0, \infty)$ such that $p(0) = 0$ and $p(t) \to \infty$ as $t \to \infty$. For a continuous function $f(x)$ on $[0, \infty)$, we define

$$s(t) = \int_0^t f(u)du \text{ and } \sigma_\alpha(t) = \int_0^t \left(1 - \frac{p(u)}{p(t)}\right)^\alpha f(u)du.$$

We say that a continuous function $f(x)$ on $[0, \infty)$ is (C, α) integrable to a by the weighted mean method determined by the function $p(x)$ for some $\alpha > -1$ if the limit $\lim_{t\to\infty} \sigma_\alpha(t) = a$ exists.

We prove that if the limit $\lim_{t\to\infty} \sigma_\alpha(t) = a$ exists for some $\alpha > -1$, then the limit $\lim_{t\to\infty} \sigma_{\alpha+h}(t) = a$ exists for all $h > 0$.

Next, we prove that if the limit $\lim_{t\to\infty} \sigma_\alpha(t) = a$ exists for some $\alpha > 0$ and

$$\frac{p(t)}{p'(t)} f(t) = O(1), \quad t \to \infty,$$

then the limit $\lim_{t\to\infty} \sigma_{\alpha-1}(t) = a$ exists.

1. Introduction

Let $p(x)$ be a nondecreasing continuous function on $[0, \infty)$ such that $p(0) = 0$ and $p(t) \to \infty$ as $t \to \infty$. For a continuous function $f(x)$ on $[0, \infty)$, we define

$$s(t) = \int_0^t f(u)du \text{ and } \sigma_\alpha(t) = \int_0^t \left(1 - \frac{p(u)}{p(t)}\right)^\alpha f(u)du.$$

A continuous function $f(x)$ on $[0, \infty)$ is said to be (C, α) integrable to a by the weighted mean method determined by the function $p(x)$ for some $\alpha > -1$ if the limit $\lim_{t\to\infty} \sigma_\alpha(t) = a$ exists.

If we take $p(x) = x$, we have the definition of (C, α) integrability of $f(x)$ on $[0, \infty)$ given by Laforgia [1]. The $(C, 0)$ integrability of $f(x)$ is convergence of the improper integral $\int_0^\infty f(t)dt$.

It will be shown as a corollary of our first result in this paper that convergence of the improper integral $\int_0^\infty f(t)dt$ implies the existence of the limit $\lim_{t\to\infty} \sigma_\alpha(t)$ for $\alpha > 0$. However, there are some (C, α) integrable

2010 *Mathematics Subject Classification*. Primary 40A10; Secondary 40C10, 40D05, 40G05
Keywords. Divergent integrals, Cesàro integrability, (C, α) integrability, Tauberian conditions and theorems
Received: 10 March 2012; Accepted: 18 May 2012
Communicated by Ljubiša D.R. Kočinac
Email addresses: ibrahimcanak@yahoo.com, ibrahim.canak@ege.edu.tr (İbrahim Çanak), utotur@yahoo.com, utotur@adu.edu.tr (Ümit Totur)

functions by the weighted mean method determined by the function $p(x)$ which fail to converge as improper integrals. Adding some suitable condition, which is called a Tauberian condition, one may get the converse. Any theorem which states that convergence of the improper integral follows from the (C,α) integrability of $f(x)$ by the weighted mean method determined by the function $p(x)$ and a Tauberian condition is said to be a Tauberian theorem.

Çanak and Totur [2, 3] have recently proved the generalized Littlewood theorem and Hardy-Littlewood type Tauberian theorems for $(C,1)$ integrability of $f(x)$ on $[0,\infty)$ by using the concept of the general control modulo analogous to the one defined by Dik [4]. Çanak and Totur [5] have also given alternative proofs of some classical type Tauberian theorems for $(C,1)$ integrability of $f(x)$ on $[0,\infty)$.

In this paper we prove that if the limit $\lim_{t\to\infty}\sigma_\alpha(t) = a$ exists for some $\alpha > -1$, then the limit $\lim_{t\to\infty}\sigma_{\alpha+h}(t) = a$ exists for all $h > 0$. As a corollary to this result, we show that if $\int_0^\infty f(t)dt$ is convergent to a, then the limit $\lim_{t\to\infty}\sigma_h(t) = a$ for all $h > 0$. But, the converse of this implication might be true under some condition on p and f. Furthermore, we give conditions under which the limit $\lim_{t\to\infty}\sigma_{\alpha-1}(t) = a$ follows from the existence of the limit $\lim_{t\to\infty}\sigma_\alpha(t) = a$.

2. Results

The next two theorems given for (C,α) integrability of functions by weighted mean methods generalize Theorems 2.1 and 3.2 in Laforgia [1].

The following theorem shows that (C,α) integrability of $f(x)$, where $\alpha > -1$, implies $(C,\alpha+h)$ integrability of $f(x)$, where $h > 0$.

Theorem 2.1. *If the limit* $\lim_{t\to\infty}\sigma_\alpha(t) = a$ *exists for some* $\alpha > -1$, *then the limit* $\lim_{t\to\infty}\sigma_{\alpha+h}(t) = a$ *exists for all* $h > 0$.

Proof. Consider

$$\int_0^t \varphi(u,t)\sigma_\alpha(t)du, \tag{1}$$

where

$$\varphi(u,t) = \frac{1}{B(\alpha+1,h)}\frac{p'(u)}{p(t)}\left(\frac{p(u)}{p(t)}\right)^\alpha\left(1-\frac{p(u)}{p(t)}\right)^{h-1}, \tag{2}$$

where B denotes the Beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x > 0, y > 0.$$

If we let $v = \frac{p(u)}{p(t)}$ in (2), we have

$$\int_0^t \varphi(u,t)du = 1. \tag{3}$$

We need to prove that

$$\lim_{t\to\infty}\int_0^t \varphi(u,t)\sigma_\alpha(t)dt = a. \tag{4}$$

Since

$$\lim_{t\to\infty}\sigma_\alpha(t) = a \tag{5}$$

by the hypothesis, there exists a value t_ε for any given $\varepsilon > 0$ such that

$$|\sigma_\alpha(t) - a| < \varepsilon, \qquad t \geq t_\varepsilon. \tag{6}$$

It follows from (3) that

$$\int_0^t \varphi(u,t)\sigma_\alpha(t)du - a = \int_0^t \varphi(u,t)[\sigma_\alpha(t) - a]du. \tag{7}$$

To prove (4), it suffices to show that

$$\left|\int_0^t \varphi(u,t)\sigma_\alpha(t)du - a\right| < 2\varepsilon, \tag{8}$$

provided that t is large enough.

We notice that by the hypothesis, the function $\sigma_\alpha(t)$ is bounded on $[0,\infty)$, that is,

$$|\sigma_\alpha(t) - a| < K, \quad 0 \leq t < \infty$$

for some constant K. Using the inequalities (3) and (6), we obtain, by (7),

$$\begin{aligned}
\left|\int_0^t \varphi(u,t)[\sigma_\alpha(t) - a]du\right| &\leq \int_0^{t_\varepsilon} \varphi(u,t)|\sigma_\alpha(t) - a|du + \varepsilon \int_{t_\varepsilon}^t \varphi(u,t)du \\
&< K\int_0^{t_\varepsilon} \varphi(u,t)du + \varepsilon \int_0^t \varphi(u,t)du \\
&= K\int_0^{t_\varepsilon} \varphi(u,t)du + \varepsilon.
\end{aligned}$$

By the substitution $v = \frac{p(u)}{p(t)}$ and (2), we have

$$\begin{aligned}
\int_0^{t_\varepsilon} \varphi(u,t)du &= \frac{1}{B(\alpha+1,h)}\int_0^{t_\varepsilon} \frac{p'(u)}{p(t)}\left(\frac{p(u)}{p(t)}\right)^\alpha \left(1 - \frac{p(u)}{p(t)}\right)^{h-1} dt \\
&= \frac{1}{B(\alpha+1,h)}\int_0^{p(t_\varepsilon)/p(t)} v^\alpha(1-v)^{h-1}dv
\end{aligned}$$

which tends to zero when $t \to \infty$ for any fixed t_ε. Thus, there exists a $\widehat{t_\varepsilon}$ such that

$$K\int_0^{t_\varepsilon} \varphi(u,t)dt < \varepsilon, \quad t > \widehat{t_\varepsilon}.$$

Hence, we have (8) for $t > \widehat{t_\varepsilon}$, and this proves (4). We obtain

$$\begin{aligned}
\int_0^t \varphi(u,t)\sigma_\alpha(t)du &= \int_0^t \varphi(u,t)dt \int_0^u \left(1 - \frac{p(s)}{p(u)}\right)^\alpha f(s)ds \\
&= \int_0^t f(s)\int_s^t \varphi(u,t)\left(1 - \frac{p(s)}{p(u)}\right)^\alpha duds \\
&= \int_0^t f(s)I(s,t)ds.
\end{aligned}$$

Here, we write $I(s,t)$ as

$$I(s,t) = \frac{1}{B(\alpha+1,h)} \int_s^t \frac{1}{p(t)} \left(\frac{p(u)}{p(t)}\right)^{\alpha} \left(1-\frac{p(u)}{p(t)}\right)^{h-1} p'(u) \left(1-\frac{p(s)}{p(u)}\right)^{\alpha} du$$
$$= \frac{1}{B(\alpha+1,h)} \frac{1}{(p(t))^{\alpha+1}} \int_s^t \left(1-\frac{p(u)}{p(t)}\right)^{h-1} p'(u)(p(u)-p(s))^{\alpha} du$$

by using (2). Substituting $p(u) = p(t) - (p(t) - p(s))x$ in $I(s,t)$, we have

$$I(s,t) = \frac{1}{B(\alpha+1,h)} \frac{1}{(p(t))^{\alpha+h}} \int_0^1 (p(t)-p(s))^{h-1} x^{h-1} (p(t)-p(s))^{\alpha} (1-x)^{\alpha} (p(t)-p(s)) dx$$
$$= \frac{1}{B(\alpha+1,h)} \frac{(p(t)-p(s))^{\alpha+h}}{(p(t))^{\alpha+h}} \int_0^1 x^{h-1}(1-x)^{\alpha} dx$$
$$= \left(1-\frac{p(s)}{p(t)}\right)^{\alpha+h},$$

which shows that

$$\int_0^t \varphi(u,t)\sigma_\alpha(t)du = \int_0^t \left(1-\frac{p(u)}{p(t)}\right)^{\alpha+h} f(u)du.$$

This completes of the proof of Theorem 2.1. □

Corollary 2.2. *If* $\int_0^\infty f(t)dt$ *converges to* a, *then the limit* $\lim_{t\to\infty} \sigma_h(t) = a$ *for all* $h > 0$.

Proof. Take $\alpha = 0$ in Theorem 2.1. □

The next theorem is a Tauberian theorem for (C,α) integrability of $f(x)$ continuous on $[0,\infty)$ by the weighted mean method determined by the function $p(x)$ for some $\alpha > -1$.

Theorem 2.3. *If the limit* $\lim_{t\to\infty} \sigma_\alpha(t) = a$ *exists for some* $\alpha > 0$ *and*

$$\frac{p(t)}{p'(t)} f(t) = O(1), \quad t \to \infty, \tag{9}$$

then the limit $\lim_{t\to\infty} \sigma_{\alpha-1}(t) = a$ *exists.*

Proof. Let the function $\theta(t)$ be defined by

$$\theta(t) = \frac{1}{p(t)} \int_0^t \left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1} p(u) f(u) du. \tag{10}$$

Then we have

$$\sigma_{\alpha-1}(t) = \sigma_\alpha(t) + \theta(t).$$

To prove Theorem 2.3, it suffices to show that $\theta(t) \to 0$ as $t \to \infty$. By the definition of $\sigma_\alpha(t)$, we obtain

$$(\sigma_\alpha(t))' = \int_0^t \alpha \left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1} \frac{p(u)p'(t)}{(p(t))^2} f(u)du$$
$$= \alpha \frac{p'(t)}{p(t)} \frac{1}{p(t)} \int_0^t \left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1} p(u)f(u)du$$
$$= \alpha \frac{p'(t)}{p(t)} \theta(t).$$

We also have

$$\begin{aligned}\int_{t_1}^{t_2}(\sigma_\alpha(u))'du &= \sigma_\alpha(t_2)-\sigma_\alpha(t_1)\\ &= \int_{t_1}^{t_2}\alpha\frac{p'(t)}{p(t)}\theta(t)dt\\ &= \alpha\int_{\ln p(t_1)}^{\ln p(t_2)}\theta(p^{-1}(\exp(u)))du\\ &= \alpha\int_{\ln p(t_1)}^{\ln p(t_2)}\eta(u)du.\end{aligned}$$

Here, we used the substitution $p(t)=\exp(u)$ and $\eta(u)=\theta(p^{-1}(\exp(u)))$.

We now need to show that $\lim_{u\to\infty}\eta(u)=0$.

By the simple calculation, we have

$$\eta'(u)=\frac{\exp(u)}{p'(p^{-1}(\exp(u)))}\theta'(p^{-1}(\exp(u)))=\frac{p(t)}{p'(t)}\theta'(t).$$

By (10),

$$p(t)\theta(t)=\int_0^t\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1}p(u)f(u)du. \tag{11}$$

Differentiating the both sides of (11) gives

$$\theta(t)+\frac{p(t)}{p'(t)}\theta'(t)=(\alpha-1)\int_0^t\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-2}\left(\frac{p(u)}{p(t)}\right)^2f(u)du. \tag{12}$$

For the first term on the left-hand side of (12), we have

$$\begin{aligned}\theta(t) &= \frac{1}{p(t)}\int_0^t\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1}p(u)f(u)du\\ &\le \frac{K}{p(t)}\int_0^t\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-1}p'(u)du\\ &= -K\int_1^0 v^{\alpha-1}dv\\ &= \frac{K}{\alpha},\end{aligned}$$

where we used the substitution $1-\frac{p(u)}{p(t)}=v$.

By (9), we have

$$\left|\frac{p(t)f(t)}{p'(t)}\right|\le K.$$

For the term on the right-hand side of (12), we have

$$\begin{aligned}(\alpha-1)\frac{1}{(p(t))^2}\int_0^t\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-2}(p(u))^2f(u)du &\le \frac{(\alpha-1)K}{(p(t))^2}\int_0^1\left(1-\frac{p(u)}{p(t)}\right)^{\alpha-2}p'(u)p(u)du\\ &= \frac{K}{\alpha},\end{aligned}$$

where we used the substitution $1 - \frac{p(u)}{p(t)} = v$.

Finally, we have $|\eta'(t)| \leq \frac{2K}{\alpha}$, which shows that $\eta'(t)$ is bounded. Since $\sigma_\alpha(t)$ is convergent, given any $\epsilon > 0$ there exists a t_ϵ such that

$$|\sigma_\alpha(t_1) - \sigma_\alpha(t_2)| < \epsilon, \tag{13}$$

when $t_1, t_2 > t_\epsilon$.

Suppose $\xi \geq \ln p(t_\epsilon)$ and $\eta(\xi) > 0$. Then $\eta(t) > 0$ for $\xi - \psi < t < \xi$ and $\xi < t < \xi + \psi$ and, where $\psi = \frac{\alpha\eta(\xi)}{2K}$. If we integrate $\eta(u)$ between $\xi - \psi$ and $\xi + \psi$, we have

$$\int_{\xi-\psi}^{\xi+\psi} \eta(u)du = \frac{\alpha}{K}\eta^2(\xi).$$

Furthermore, we have, by (13),

$$\frac{\alpha}{K}\eta^2(\xi) = \int_{\xi-\psi}^{\xi+\psi} \eta(u)du = \frac{1}{\alpha}\left(\sigma_\alpha(p^{-1}(\exp(\xi-\psi))) - \sigma_\alpha(p^{-1}(\exp(\xi+\psi)))\right) < \epsilon,$$

which implies that

$$\eta(\xi) < \sqrt{\frac{K}{\alpha}\epsilon}.$$

This completes the proof of Theorem 2.3. □

In the case that α is a positive integer in Theorem 2.3, we have the following corollary.

Corollary 2.4. *If the limit* $\lim_{t\to\infty} \sigma_\alpha(t) = a$ *exists for some positive integer* α *and the condition (9) holds, then the improper integral* $\int_0^\infty f(t)dt$ *converges.*

Proof. Assume that the limit $\lim_{t\to\infty} \sigma_\alpha(t) = a$ exists for some positive integer α . By Theorem 2.3, the limit $\lim_{t\to\infty} \sigma_{\alpha-1}(t) = a$ also exists, provided that the condition (9) is satisfied. Again by Theorem 2.3, the limit $\lim_{t\to\infty} \sigma_{\alpha-2}(t) = a$ exists. Continuing in this way, we obtain the convergence of $\int_0^\infty f(t)dt$. □

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