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Some generalized equalities for the q-gamma function

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Abstract. The q-analogue of the gamma function is defined by $\Gamma_q(x)$ for $x > 0$, $0 < q < 1$. In this work the neutrix and neutrix limit are used to obtain some equalities of the q-gamma function for all real values of x.

1. Introduction and preliminaries

Since the q-gamma function is very important in the theory of the basic hypergeometric series, its applications is rather extensive in the literature [1, 3, 4, 6]. New equalities and inequalities for the q-gamma function is established by using its q-integral representations [3, 13, 14]. Recently, neutrix calculus, given by van der Corput [2], have been used widely in many applications in mathematics and physics. B. Fisher and Y. Kuribayashi applied the neutrix calculus to give some results on the classical Euler's gamma function [5]. A. Salem used the concepts of the neutrix and neutrix limit to define the q-analogue of the gamma and incomplete gamma function and their derivatives for negative values of x [11, 12]. Y. J. Ng and H. van Dam applied neutrix calculus to quantum field theory, obtaining finite renormalization in the the quantum field theory [7, 8]. In this paper, we aim to generalize some equations of the q-gamma function via the theory of neutrices.

A neutrix $\mathcal{N}$ is defined as a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in $\mathcal{N}$, $\nu(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in $\mathcal{N}$ are called negligible functions.

In this work, we let $\mathcal{N}$ be the neutrix having domain the open interval $N' = (0, (1-q)^{-1})$ and range N'', the real numbers, with the negligible functions being finite linear sums of the functions

$$\epsilon^{\lambda} \ln^{r-1} \epsilon,\ \ln^{r} \epsilon,\ [\epsilon]_q^{\lambda} \qquad \lambda < 0,\ r = 1, 2, \ldots$$

and all being functions $O(\epsilon)$ which converge to zero in the usual sense as ϵ tends to zero.

Let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in \mathcal{N}$, then c is called the neutrix limit of f as ξ tends to b and we write $\mathrm{N-}\lim_{\xi\to b} f(\xi) = c$. Note that if $f(\xi)$ tends to c in the normal sense as ξ tends to zero, it converges to c in the neutrix sense.

For any $x \in \mathbb{C}$ and $n \in \mathbb{N}$, the basic number $[x]_q$ and the q-factorial $[n]_q!$ are defined by

$$[x]_q = \frac{1-q^x}{1-q}$$

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and

$$[n]_q! = \begin{cases} [1]_q[2]_q \dots [n]_q, & n = 1, 2, \dots \\ 1, & n = 0. \end{cases}$$

The q-analogue of the derivative of $f(x)$, called its q-derivative, is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{if } x \neq 0$$

and

$$(D_q f)(0) = f'(0) \quad \text{provided } f'(0) \text{ exists.}$$

The q-integral is defined [9] by

$$\int_0^a f(x) d_q x = (1-q) \sum_{n=0}^{\infty} a q^n f(a q^n).$$

Notice that the series on the right-hand side is guaranteed to be convergent as soon as the function f is such that, for some $C > 0$, $\alpha > -1$, $|f(x)| < Cx^{\alpha}$ in a right neighborhood of $x = 0$.
The formula of q-integration by parts is given for suitable functions f and g by

$$\int_a^b f(x) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) d_q f(x).$$

The reader may find more on q-calculus and its applications in the books [1, 6, 10].

2. Definition of the q-gamma function

The q-analogue of Euler's gamma function $\Gamma(x)$ is defined by the q-integral representations [1, 6]

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q^{-qt}\, d_q t = \int_0^{[\infty]_q} t^{x-1} E_q^{-qt}\, d_q, \quad x > 0 \tag{1}$$

where

$$E_q^x = \sum_{i=0}^{\infty} q^{i(i-1)/2} \frac{x^i}{[i]_q!} = (-(1-q)x; q)_\infty$$

is one of the important q-analogues of the classical exponential function and the q-derivative of E_q^x is $D_q E_q^x = E_q^{qx}$.

Note that $\Gamma_q(x)$ reduces to $\Gamma(x)$ in the limit $q \to 1$ and it satisfies the property that

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad x > 0, \quad \Gamma_q(1) = 1. \tag{2}$$

In this work, we let N be the neutrix having domain the open interval $N' = (0, (1-q)^{-1})$ and range $N'' = \mathbb{R}$, with the negligible functions being finite linear sums of the functions

$$\epsilon^{\lambda} \ln^{r-1} \epsilon, \ \ln^r \epsilon, \qquad \lambda < 0, \ r = 1, 2, \dots$$

and all being functions $O(\epsilon)$ which converge to zero in the usual sense as ϵ tends to zero.
The q-gamma function converges absolutely for all $x > 0$ due to the q-exponential function E_q^{-qt}. The omitting of the first n terms of the series of the q-exponential function allows one to extend the domain of

convergence of the q-integral (1). Using this regularization technique, it has been shown in [11] that the q-gamma function is defined by the neutrix limit

$$\Gamma_q(x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{1-q}} t^{x-1}E_q^{-qt}\,d_qt =$$

$$= \int_0^{\frac{1}{1-q}} t^{x-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\Big]\,d_{qt} + (1-q)^{1-x}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i+x})} \tag{3}$$

for $x > -n$, $n = 1, 2, \ldots$, $x \neq 0, -1, -2, \ldots, -n+1$, and

$$\Gamma_q(-n) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{1-q}} t^{-n-1}E_q^{-qt}\,d_qt = \tag{4}$$

$$= \int_0^{\frac{1}{1-q}} t^{-n-1}\Big[E_q^{-qt} - \sum_{i=0}^{n}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\Big]\,d_{qt} +$$

$$+(1-q)^{n+1}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n})} + \frac{(-1)^n q^{\frac{n(n+1)}{2}}(1-q)^{n+1}\ln(1-q)}{(q,q)_n\ln q} \tag{5}$$

for $n = 1, 2, \ldots$.

3. Main results

Now, we use neutrix calculus as a tool for generalizing equation (2) and the q-analogue of Gauss duplication formula for all real numbers of x. Firstly, we need the following lemma. By the lemma, we will give an alternative equation for the function $\Gamma_q(x)$ for negative integer values of x with using the Heaviside's function $H(x)$, which is equal to zero for $x < 0$ and to 1 for $x > 0$.

Lemma 3.1.

$$\begin{aligned}\Gamma_q(-n) &= \int_0^{\frac{1}{1-q}} t^{-n-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}t^n H(1-t)\Big]d_qt + \\ &+ (1-q)^{n+1}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n})}\end{aligned} \tag{6}$$

for $n = 0, 1, 2, \ldots$.

Proof. We have

$$\begin{aligned}\int_\epsilon^{\frac{1}{1-q}} t^{-n-1}E_q^{-qt}d_qt &= \int_\epsilon^{\frac{1}{1-q}} t^{-n-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}t^n H(1-t)\Big]d_qt + \\ &+\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}\int_\epsilon^{\frac{1}{1-q}} t^{-n+i-1}d_qt + \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}\int_\epsilon^1 t^{-1}d_qt = \\ &= \int_\epsilon^{\frac{1}{1-q}} t^{-n-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}t^n H(1-t)\Big]d_qt + \\ &+\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q![i-n]_q}\Big[(1-q)^{n-i} - \epsilon^{i-n}\Big] - \frac{(-1)^n q^{\frac{n(n+1)}{2}}(q-1)}{[n]_q!\ln q}\ln\epsilon.\end{aligned}$$

Now, taking the neutrix limit of both sides of the last equation as ϵ tends to 0 and using the equation (4) we get the desired result. □

We can now prove the following theorem.

Theorem 3.2.

$$\Gamma_q(x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\Gamma_q(x+\epsilon)$$

for all x.

Proof. Since $\Gamma_q(x)$ is a continuous function for $x \neq 0,-1,-2,...$ its neutrix limit becomes normal limit as ϵ tends to zero and the result follows for $x \neq 0,-1,-2,\ldots$.
Now we will consider $\Gamma_q(x)$ at the point $x=-n$, $n=1,2,\ldots$. For $0<\epsilon<1$, we have from equation (3) that

$$\begin{aligned}\Gamma_q(-n+\epsilon) &= \int_0^{\frac{1}{1-q}} t^{-n+\epsilon-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\Big]\,d_qt + (1-q)^{n-\epsilon+1}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n+\epsilon})} = \\ &= \int_0^{\frac{1}{1-q}} t^{-n+\epsilon-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}t^nH(1-t)\Big]\,d_qt + \\ &+ \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}\int_0^1 t^{\epsilon-1}\,d_qt + (1-q)^{n-\epsilon+1}\sum_{i=0}^{n-1}\frac{(-1)^i . q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n+\epsilon})}.\end{aligned}$$

Now recalling that the neutrix is given in the section 1 and also the property that the neutrix limit is unique and its precisely the same as the ordinary limit, if it exists, we write

$$\begin{aligned}\mathrm{N}\!-\!\lim_{\epsilon\to 0}\Gamma_q(-n+\epsilon) &= \int_0^{\frac{1}{1-q}} t^{-n-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q}t^nH(1-t)\Big]d_qt + \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n})} = \\ &= \Gamma_q(-n)\end{aligned}$$

by using the lemma 3.1.
On the other hand for $0<\epsilon<1$, we have from equation (3) that

$$\begin{aligned}\Gamma_q(-n-\epsilon) &= \int_0^{\frac{1}{1-q}} t^{-n-\epsilon-1}\Big[E_q^{-qt} - \sum_{i=0}^{n}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\Big]\,d_qt + (1-q)^{n+\epsilon+1}\sum_{i=0}^{n}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q,q)_i(1-q^{i-n-\epsilon})} = \\ &= \int_0^{\frac{1}{1-q}} t^{-n-\epsilon-1}\Big[E_q^{-qt} - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i - \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}t^nH(1-t)\Big]\,d_qt - \\ &- \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}\int_1^{\frac{1}{1-q}} t^{-\epsilon-1}d_qt + \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}\frac{(1-q)^{n-i+\epsilon}}{[-n-\epsilon+i]_q} + \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}\frac{(1-q)^{\epsilon}}{[-\epsilon]_q}.\end{aligned}$$

Now taking the neutrix limit of both sides of the equation, using the lemma 3.1 and the fact that $[-\epsilon]_q = -q^{-\epsilon}[\epsilon]_q$, the result follows. □

We know that equation (2) holds for $x>0$. The following theorem does however hold.

Theorem 3.3.

$$\Gamma_q(x+1) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}[x+\epsilon]_q\Gamma_q(x+\epsilon)$$

for all x.

Proof. The result can easily be obtained because of the continuity of $\Gamma_q(x)$ for $x \neq 0, -1, -2, \ldots$. Now we will consider $\Gamma_q(-n)$. For $0 < \epsilon < 1$, we can write by the definition of $\Gamma_q(x)$ given in the equation (3) that

$$\Gamma_q(x+\epsilon+1) = \mathrm{N}-\lim_{\delta\to 0} \int_\delta^{\frac{1}{1-q}} t^{x+\epsilon} E_q^{-qt} d_q t.$$

Then using the q-integration by parts, we have

$$\Gamma_q(x+\epsilon+1) = \mathrm{N}-\lim_{\delta\to 0} \Big[(1-q)^{-x-\epsilon} E_q^{-(1-q)^{-1}} + \delta^{x+\epsilon} E_q^{-\delta} + [x+\epsilon]_q \int_\delta^{\frac{1}{1-q}} t^{x+\epsilon-1} E_q^{-qt} d_q t\Big].$$

Since

$$E_q^{-(1-q)^{-1}} = ((1-q)(1-q)^{-1}; q)_\infty = (1; q)_\infty = 0$$

and $\delta^{x+\epsilon} E_q^{-\delta}$ is a negligible function we have

$$\begin{aligned}\Gamma_q(x+\epsilon+1) &= \mathrm{N}-\lim_{\delta\to 0} [x+\epsilon]_q \int_\delta^{\frac{1}{1-q}} t^{x+\epsilon-1} E_q^{-qt} d_q t = \\ &= [x+\epsilon]_q \Gamma_q(x+\epsilon).\end{aligned}$$

Then using theorem 3.2 we get

$$\begin{aligned}\mathrm{N}-\lim_{\epsilon\to 0} \Gamma_q(x+\epsilon+1) &= \mathrm{N}-\lim_{\epsilon\to 0} [x+\epsilon]_q \Gamma_q(x+\epsilon) = \\ &= \Gamma_q(x+1),\end{aligned}$$

completing the proof of the theorem. □

The Gauss multiplication formula has q-analogue of the form

$$\Gamma_q(nx)\Gamma_r(\frac{1}{n})\Gamma_r(\frac{2}{n})\ldots\Gamma_r(\frac{n-1}{n}) = (1+q\ldots+q^{n-1})^{nx-1}\Gamma_r(x)\Gamma_r(x+\frac{1}{n})\ldots\Gamma_r(x+\frac{n-1}{n}) \quad (7)$$

with $r = q^n$, [6].
Now as an application of theorem 3.3, we will generalize formula (7) for all x.

Theorem 3.4.

$$\Gamma_q(nx)\Gamma_r(\frac{1}{n})\Gamma_r(\frac{2}{n})\ldots\Gamma_r(\frac{n-1}{n}) = \mathrm{N}-\lim_{\epsilon\to 0}(1+q\ldots+q^{n-1})^{nx+n\epsilon-1}\Gamma_r(x+\epsilon)\Gamma_r(x+\epsilon+\frac{1}{n})\ldots\Gamma_r(x+\epsilon+\frac{n-1}{n})$$

for all x.

Proof. Since $\Gamma_q(nx)$ is a continuous function for the values $nx \neq 0, -1, -2, \ldots$ the result follows immediately for $nx \neq 0, -1, -2, \ldots$. For the case $nx = -m$, $m = 1, 2, \ldots$ we have by equation (7) for $0 < |\epsilon| < \frac{1}{n}$ that

$$(1+q\ldots+q^{n-1})^{-m+n\epsilon-1}\Gamma_r(\frac{-m}{n}+\epsilon)\ldots\Gamma_r(\frac{-m}{n}+\epsilon+\frac{n-1}{n}) = \Gamma_q(-m+n\epsilon)\Gamma_r(\frac{1}{n})\ldots\Gamma_r(\frac{n-1}{n}).$$

Then by using theorem 3.2 we have

$$\begin{aligned}\mathrm{N}-\lim_{\epsilon\to 0}(1+q\ldots+q^{n-1})^{-m+n\epsilon-1}\Gamma_r(\frac{-m}{n}+\epsilon)\ldots\Gamma_r(\frac{-m}{n}+\epsilon+\frac{n-1}{n}) &= \Gamma_r(\frac{1}{n})\ldots\Gamma_r(\frac{n-1}{n})\,\mathrm{N}-\lim_{\epsilon\to 0}\Gamma_q(-m+n\epsilon) = \\ &= \Gamma_r(\frac{1}{n})\ldots\Gamma_r(\frac{n-1}{n})\Gamma_q(-m).\end{aligned}$$

Hence the proof is complete. □

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