Filomat 26:6 (2012), 1247–1256
DOI 10.2298/FIL1206247F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some results on the neutrix composition of the delta function

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Abstract. In this paper we prove that the neutrix composition $\delta^{(s)}\{[\exp_+(x) - H(x)]^{1/r}\}$ exists for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$ and in particular

$$\delta^{(mr-1)}\{[\exp_+(x) - H(x)]^{1/r}\} = \sum_{k=0}^{m-1} \frac{(-1)^{mr+k-1} r(mr-1)! c_{mr-1,k}}{2k!} \delta^{(k)}(x),$$

for $m, r = 1, 2, \dots$.

1. Introduction

Certain operations on smooth functions can be extended without difficulty to arbitrary distributions. Others (such as multiplication, convolution and change of variables) can be defined only for particular distributions. For instance, one may consider stationary Schrdinger operator with singular potential

$$\Delta u + a\delta u,$$

where δ is the Dirac-delta function and a is the so-called coupling constant, gives a model of scattering on a particle, located at the origin see [1]. The fact that the product $\delta.u$ is not defined in the classical theory of distributions. Further consider the interaction two delta waves in a model ruled by Burgers perturbed equation see [19], or the heat equation involving the δ distribution as a coefficient see [15]. Also we refer a reader to [2, 16].

In the theory of distributions, many arguments show that no meaning can be generally given to expressions of the form $F(f(x))$, where F is a distribution and f is a locally summable function.

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from divergent integral is referred to as the Hadamard finite part. In fact his method can be regarded as a particular applications of the neutrix calculus developed by van der Corput see [3].

Using the concepts of a neutrix and neutrix limit, the first author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions, and this has been exploited in the context of distributions, particularly in connection with the composition of distributions, see [4, 5]. By Fisher's definition Koh and Li give a meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \dots$, see [12], and more general form

2010 *Mathematics Subject Classification*. Primary 46F10
Keywords. Distribution, Dirac-delta function, composition of distributions, neutrix, neutrix limit
Received: 26 April 2012; Revised: 08 July 2012; Accepted: 09 July 2012
Communicated by Ljubiša D.R. Kočinac
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$(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [13]. Recently the rth powers of the Dirac function $\delta(x)$ and the Heaviside function $H(x)$ for negative integers have been defined in [17] and [18] respectively.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a,b]$ be the space of distributions defined on $\mathcal{D}[a,b]$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:
(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\Big[\frac{1}{|f'(x)|}\frac{d}{dx}\Big]^r \delta(x - x_0),$$

for $r = 0, 1, 2, \ldots$, see [11].

In order to give a more general definition for the composition of distributions, the following definition for the neutrix composition of distributions was given in [4] and was originally called the composition of distributions.

Definition 1.1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h the open interval (a,b) if

$$\underset{n\to\infty}{\text{N}-\lim} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [3], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n,\ \ln^r n : \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

2. Results

The following theorems were proved in [6], [8], [9], [10], respectively.

Theorem 2.1. *The neutrix composition $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0,1,2,\dots$ and $(s+1)\lambda = 1,3,\dots$ and

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda-1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0,1,2,\dots$ and $(s+1)\lambda = 2,4,\dots$.

Theorem 2.2. *The neutrix composition $\delta^{(s)}(\sinh^{-1} x_+)$ exists and*

$$\delta^{(s)}(\sinh^{-1} x_+) = \sum_{k=0}^{s}\sum_{i=0}^{k}\binom{k}{i}(-1)^{s+i+k}\frac{(k-2i+1)^s+(k-2i-1)^s}{2^k k!}\delta^{(k)}(x)$$

for $s = 0,1,2,\dots$.

Theorem 2.3. *The neutrix composition $\delta^{(s)}[\ln^r(1+|x|)]$ exists and*

$$\delta^{(s)}[\ln^r(1+|x|)] = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}[1+(-1)^k]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x)$$

for $s = 0,1,2,\dots$ and $r = 1,2,\dots$.

In particular, the composition $\delta[\ln(1+|x|)]$ exists and

$$\delta[\ln(1+|x|)] = \delta(x).$$

Theorem 2.4. *The neutrix composition $\delta^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/m})]$ exists and*

$$\delta^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/m})] =$$
$$= \sum_{k=0}^{s-1}\sum_{i=0}^{k}\binom{k}{i}(-1)^{s+i+k}\frac{(k-2i+1)^s+(k-2i-1)^s}{2^k k!}\delta^{(k)}(x)$$

for $s = 0,1,2,\dots$.

In the following, the functions $\exp_+(x)$ and $\exp_-(x)$ are defined by

$$\exp_+(x) = \begin{cases} \exp(x), & x \ge 0, \\ 0, & x < 0 \end{cases} \quad \text{and} \quad \exp_-(x) = \begin{cases} \exp(x), & x \le 0, \\ 0, & x > 0. \end{cases}$$

$H(x)$ denotes Heaviside's function.

The constants $c_{i,k}$ are defined by the expansion

$$\frac{\ln^k(1+x)}{1+x} = \sum_{i=0}^{\infty} c_{i,k}x^i$$

for $i,k = 0,1,2,\dots$.

We also need the following lemma, which can be easily proved by induction:

Lemma 2.5.

$$\int_{-1}^{1} t^i\rho^{(s)}(t)\,dt = \begin{cases} 0, & 0 \le i < s, \\ (-1)^s s!, & i = s \end{cases}$$

and

$$\int_{0}^{1} t^s\rho^{(s)}(t)\,dt = \tfrac{1}{2}(-1)^s s!$$

for $s = 0,1,2,\dots$.

The following theorem was proved in [7].

Theorem 2.6. *The neutrix composition* $\delta^{(s)}\{[\exp_+(x)-1]^r\}$ *exists and*

$$\delta^{(s)}\{[\exp_+(x)-1]^r\} = \sum_{k=0}^{rs+r-1} \frac{(-1)^{s+k}s!c_{rs+r-1,k}}{2rk!}\delta^{(k)}(x),$$

for $r = 1,2,\ldots$ *and* $s = 0,1,2,\ldots$.

In particular

$$\begin{aligned}\delta[\exp_+(x)-1] &= \frac{1}{2}\delta(x),\\ \delta\{[\exp_+(x)-1]^2\} &= -\frac{1}{4}\delta(x)+\frac{1}{4}\delta'(x),\\ \delta'\{[\exp_+(x)-1]^2\} &= \frac{1}{2}\delta(x)-\frac{1}{2}\delta'(x).\end{aligned}$$

We now prove the following theorem.

Theorem 2.7. *The neutrix composition* $\delta^{(s)}\{[\exp_+(x)-H(x)]^{1/r}\}$ *exists for* $r = 1,2,\ldots$ *and* $s = 0,1,2,\ldots$,

$$\delta^{(mr-1)}\{[\exp_+(x)-H(x)]^{1/r}\} = \sum_{k=0}^{m-1} \frac{(-1)^{mr+k-1}r(mr-1)!c_{mr-1,k}}{2k!}\delta^{(k)}(x), \tag{1}$$

for $m, r = 1,2,\ldots$ *and*

$$\delta^{(s)}\{[\exp_+(x)-H(x)]^{1/r}\} = 0, \tag{2}$$

for $s \neq mr-1$ *and* $m, r = 1,2,\ldots$.

In particular

$$\delta^{(r-1)}\{[\exp_+(x)-H(x)]^{1/r}\} = -\frac{(-1)^r r!}{2}\delta(x), \tag{3}$$

$$\delta^{(2r-1))}\{[\exp_+(x)-H(x)]^{1/r}\} = \frac{(2r)!}{4}\delta(x)-\frac{(2r)!}{4}\delta'(x), \tag{4}$$

$$\begin{aligned}\delta^{(3r-1)}\{[\exp_+(x)-H(x)]^{1/r}\} &= -\frac{(-1)^r r(3r-1)!}{2}\delta(x)\\ &\quad -\frac{(-1)^r 3r(3r-1)!}{4}[\delta'(x)+\delta''(x)],\end{aligned} \tag{5}$$

for $r = 1,2,\ldots$.

Proof. We will first of all prove equations (1) and (2) on the interval $[-1,1]$. To do this, we need to evaluate

$$\begin{aligned}\int_{-1}^{1} x^k\delta_n^{(s)}\{[\exp_+(x)-H(x)]^{1/r}\}\,dx &= \\ &= \int_0^1 x^k\delta_n^{(s)}\{[\exp(x)-1]^{1/r}\}\,dx + \int_{-1}^0 x^k\delta_n^{(s)}(0)\,dx\\ &= n^{s+1}\int_0^1 x^k\rho^{(s)}\{n[\exp(x)-1]^{1/r}\}\,dx + n^{s+1}\int_{-1}^0 x^k\rho^{(s)}(0)\,dx\\ &= I_1 + I_2.\end{aligned} \tag{6}$$

It is immediately obvious that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = 0. \tag{7}$$

Making the substitution $n[\exp(x)-1]^{1/r}=t$ or

$$x=\ln[1+(t/n)^r],$$

we have

$$dx=\frac{rt^{r-1}\,dt}{n^r[1+(t/n)^r]}.$$

Then for $n>1$, we have

$$\begin{aligned}I_1 &= rn^{s-r+1}\int_0^1\frac{\ln^k[1+(t/n)^r]t^{r-1}}{1+(t/n)^r}\rho^{(s)}(t)\,dt\\ &= \sum_{i=0}^{\infty}rc_{i,k}\int_0^1\frac{t^{r(i+1)-1}}{n^{r(i+1)-s-1}}\rho^{(s)}(t)\,dt.\end{aligned}$$

When $s=mr-1$, we have

$$I_1=\sum_{i=0}^{\infty}rc_{i,k}\int_0^1\frac{t^{r(i+1)-1}}{n^{r(i+1)-mr}}\rho^{(mr-1)}(t)\,dt$$

and it follows that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}I_1 &= \operatorname*{N-lim}_{n\to\infty}\int_0^1 x^k\delta_n^{(mr-1)}\{[\exp(x)-1]^r\}\,dx\\ &= \frac{(-1)^{mr-1}r(mr-1)!c_{mr-1,k}}{2},\end{aligned}\tag{8}$$

on using the lemma, for $k=0,1,2,\dots,m-1$, $r=1,2,\dots$ and $s=0,1,2,\dots$.

Next, when $k=m$, we have

$$\begin{aligned}&\int_0^1\left|x^m\delta_n^{(mr-1)}\{[\exp(x)-1]^{1/r}\}\right|dx\le\\ &\quad\le rn^{mr-r}\int_0^1\left|\frac{\ln^m[1+(t/n)^r]t^{r-1}}{1+(t/n)^r}\rho^{(mr-1)}(t)\right|dt\\ &\quad=O(n^{-r}),\end{aligned}$$

since $|\ln^m[1+(t/n)^r]|=O(n^{-mr})$. Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^m\delta_n^{(mr-1)}\{[\exp(x)-1]^{1/r}\}\psi(x)\,dx=0,\tag{9}$$

for $m,r=1,2,\dots$.

Further,

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\int_{-1}^0 x^m\delta_n^{(mr-1)}(0)\psi(x)\,dx &= \operatorname*{N-lim}_{n\to\infty}n^{mr}\int_{-1}^0 x^m\rho^{(mr-1)}(0)\psi(x)\,dx\\ &= 0,\end{aligned}\tag{10}$$

for $m,r=1,2,\dots$.

Now let φ be an arbitrary function in $\mathcal{D}[-1,1]$. By Taylor's Theorem we have

$$\varphi(x)=\sum_{k=0}^{m-1}\frac{x^k\varphi^{(k)}(0)}{k!}+\frac{x^m\varphi^{(m)}(\xi x)}{m!},$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty}&\langle \delta_n^{(mr-1)}\{[\exp(x)-H(x)]^{1/r}\},\varphi(x)\rangle = \\
&= \mathrm{N}\!-\!\lim_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1} x^k\delta_n^{(mr-1)}\{[\exp(x)-H(x)]^{1/r}\}\,dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^{1}\frac{x^m}{m!}\delta_n^{(mr-1)}\{[\exp(x)-H(x)]^{1/r}\}\varphi^{(m)}(\xi x)\,dx \\
&= \mathrm{N}\!-\!\lim_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{0}^{1} x^k\delta_n^{(mr-1)}\{[\exp(x)-1]^{1/r}\}\,dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{0} x^k\delta_n^{(mr-1)}(0)\,dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty}\int_{0}^{1}\frac{x^m}{m!}\delta_n^{(mr-1)}\{[\exp(x)-1]^{1/r}\}\varphi^{(m)}(\xi x)\,dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty}\int_{-1}^{0}\frac{x^m}{m!}\delta_n^{(mr-1)}(0)\varphi^{(m)}(\xi x)\,dx \\
&= \sum_{k=0}^{m-1}\frac{(-1)^{mr-1}r(mr-1)!c_{m-1,k}}{2k!}\varphi^{(k)}(0) \\
&= \sum_{k=0}^{m-1}\frac{(-1)^{mr+k-1}r(mr-1)!c_{m-1,k}}{2k!}\langle\delta^{(k)}(x),\varphi(x)\rangle,
\end{aligned}$$

on using equations (6) to (10), for $m, r = 1, 2, \ldots$.

This proves that the neutrix composition $\delta^{(mr-1)}\{[\exp_+(x)-H(x)]^{1/r}\}$ exists and

$$\delta^{(mr-1)}\{[\exp_+(x)-H(x)]^{1/r}\} = \sum_{k=0}^{m-1}\frac{(-1)^{mr+k-1}r(mr-1)!c_{m-1,k}}{2k!}\delta^{(k)}(x),$$

on the interval $[-1, 1]$ for $m, r = 1, 2, \ldots$.

It is obvious that $\delta^{(mr-1)}\{[\exp_+(x)-1]^{1/r}\} = 0$, if $x \neq 0$ and so the neutrix composition $\delta^{(mr-1)}\{[\exp_+(x)-1]^{1/r}\}$ exists on the real line for $m, r = 1, 2, \ldots$.

Equations (2), (3) and (4) follow on noting that $c_{0,0} = c_{2,0} = c_{2,2} = 1$, $c_{1,0} = c_{1,1} = -1$ and $c_{2,1} = -3/2$.

When $s \neq mr - 1$, it is obvious that

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_1 = 0, \tag{11}$$

for $k = 0, 1, 2, \ldots, m-1$, $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ and when $k = s$, we have

$$\begin{aligned}
\int_0^1 &\left|x^s\delta_n^{(s)}\{[\exp(x)-1]^{1/r}\}\right|dx \leq \\
&\leq rn^{s-r+1}\int_0^1\left|\frac{\ln^s[1+(t/n)^r]t^{r-1}}{1+(t/n)^r}\rho^{(s)}(t)\right|dt \\
&= O(n^{s-rs-r+1}),
\end{aligned}$$

since $|\ln^s[1+(t/n)^r]| = O(n^{-rs})$. Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^s\delta_n^{(s)}\{[\exp(x)-1]^{1/r}\}\psi(x)\,dx = 0, \tag{12}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. Equation (2) now follows as above using equations (7), (11) and (12).
This completes the proof of the theorem. □

Corollary 2.8. *The neutrix composition $\delta^{(s)}\{[H(-x) - \exp\_(x)]^{1/r}\}$ exists for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$,*

$$\delta^{(mr-1)}\{[H(-x) - \exp\_(x)]^{1/r}\} = \sum_{k=0}^{m-1} \frac{(-1)^k r(mr-1)! c_{m-1,k}}{2k!} \delta^{(k)}(x), \tag{13}$$

for $m, r = 1, 2, \ldots$ and

$$\delta^{(s)}\{[H(-x) - \exp\_(x)]^{1/r}\} = 0, \tag{14}$$

for $s \neq mr - 1$ and $m, r = 1, 2, \ldots$.
In particular

$$\delta^{(r-1)}\{[H(-x) - \exp\_(x)]^{1/r}\} = \frac{r!}{2}\delta(x), \tag{15}$$

$$\delta^{(2r-1))}\{[H(-x) - \exp\_(x)]^{1/r}\} = -\frac{(2r)!}{4}\delta(x) + \frac{(2r)!}{4}\delta'(x), \tag{16}$$

$$\begin{aligned} \delta^{(3r-1)}\{[H(-x) - \exp\_(x)]^{1/r}\} &= \frac{r(3r-1)!}{2}\delta(x) \\ &\quad + \frac{3r(3r-1)!}{4}[\delta'(x) + \delta''(x)], \end{aligned} \tag{17}$$

for $r = 1, 2, \ldots$.

Proof. We will first of all prove equations (13) and (14) on the interval $[-1, 1]$. We have

$$\begin{aligned} \int_{-1}^{1} x^k \delta_n^{(s)}\{[H(-x) - \exp\_(x)]^{1/r}\}\, dx &= \\ &= \int_{-1}^{0} x^k \delta_n^{(s)}\{[1 - \exp(x)]^{1/r}\}\, dx + \int_0^1 x^k \delta_n^{(s)}(0)\, dx \\ &= n^{s+1} \int_{-1}^{0} x^k \rho^{(s)}\{n[1 - \exp(x)]^{1/r}\}\, dx + n^{s+1} \int_0^1 x^k \rho_n^{(s)}(0)\, dx \\ &= J_1 + J_2. \end{aligned} \tag{18}$$

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} J_2 = 0. \tag{19}$$

Making the substitution $n[1 - \exp(x)]^{1/r} = t$ or

$$x = \ln[1 - (t/n)^r],$$

we have

$$dx = -\frac{rt^{r-1}\, dt}{n^r[1 - (t/n)^r]}.$$

Then for $n > 1$, we have

$$\begin{aligned} J_1 &= rn^{s-r+1} \int_0^1 \frac{\ln^k[1 - (t/n)^r]t^{r-1}}{1 - (t/n)^r} \rho^{(s)}(t)\, dt \\ &= \sum_{i=0}^{\infty} (-1)^i rc_{i,k} \int_{-1}^{0} \frac{t^{r(i+1)-1}}{n^{r(i+1)-s-1}} \rho^{(s)}(t)\, dt. \end{aligned}$$

When $s = mr - 1$, we have

$$J_1 = \sum_{i=0}^{\infty}(-1)^i rc_{i,k}\int_{-1}^{0}\frac{t^{r(i+1)-1}}{n^{r(i+1)-mr}}\rho^{(mr-1)}(t)\,dt$$

and it follows that

$$\begin{aligned} \underset{n\to\infty}{\mathrm{N-lim}}\, J_1 &= \underset{n\to\infty}{\mathrm{N-lim}}\int_{-1}^{0} x^k\delta_n^{(mr-1)}\{[1-\exp(x)]^{1/r}\}\,dx \\ &= \frac{r(mr-1)!c_{mr-1,k}}{2}, \end{aligned} \tag{20}$$

on using the lemma, for $k = 0, 1, 2, \ldots, m-1$, $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$.

Next, when $k = m$, we have

$$\begin{aligned} &\int_{-1}^{0}\left|x^m\delta_n^{(mr-1)}\{[1-\exp(x)]^{1/r}\}\right|dx \le \\ &\quad\le rn^{mr-r}\int_{-1}^{0}\left|\frac{\ln^m[1-(t/n)^r]t^{r-1}}{1-(t/n)^r}\rho^{(mr-1)}(t)\right|dt \\ &\quad= O(n^{-r}). \end{aligned}$$

Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_{-1}^{0} x^m\delta_n^{(mr-1)}\{[1-\exp(x)]^{1/r}\}\psi(x)\,dx = 0, \tag{21}$$

for $m, r = 1, 2, \ldots$.

If now φ is in $\mathcal{D}[-1, 1]$, then, similar to the above, we have

$$\begin{aligned} &\underset{n\to\infty}{\mathrm{N-lim}}\langle\delta_n^{(mr-1)}\{[H(-x)-\exp_-(x)]^{1/r}\}, \varphi(x)\rangle = \\ &\quad= \underset{n\to\infty}{\mathrm{N-lim}}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 x^k\delta_n^{(mr-1)}\{[H(-x)-\exp_-(x)]^{1/r}\}\,dx \\ &\qquad+ \underset{n\to\infty}{\mathrm{N-lim}}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{0} x^k\delta_n^{(mr-1)}(0)\,dx \\ &\qquad+ \underset{n\to\infty}{\mathrm{N-lim}}\int_0^1\frac{x^m}{m!}\delta_n^{(mr-1)}\{[H(-x)-\exp_-(x)]^{1/r}\}\varphi^{(m)}(\xi x)\,dx \\ &\qquad+ \underset{n\to\infty}{\mathrm{N-lim}}\int_{-1}^{0}\frac{x^m}{m!}\delta_n^{(mr-1)}(0)\varphi^{(m)}(\xi x)\,dx \\ &\quad= \sum_{k=0}^{m-1}\frac{r(mr-1)!c_{m-1,k}}{2k!}\varphi^{(k)}(0) \\ &\quad= \sum_{k=0}^{m-1}\frac{(-1)^k r(mr-1)!c_{m-1,k}}{2k!}\langle\delta^{(k)}(x), \varphi(x)\rangle, \end{aligned}$$

on using equations (18) to (21), for $m, r = 1, 2, \ldots$.

The equations (14) to (17) follow as above. This completes the proof of the corollary. □

Corollary 2.9. *The neutrix composition $\delta^{(s)}[|\exp(x)-1|^{1/r}]$ exists for $r=1,2,\dots$ and $s=0,1,2,\dots,$*

$$\delta^{(mr-1)}[|\exp(x)-1|^{1/r}] = \sum_{k=0}^{m-1}\frac{(-1)^{mr+k-1}r(mr-1)!c_{mr-1,k}}{2k!}\delta^{(k)}(x), \tag{22}$$

for $m,r=1,2,\dots$ and

$$\delta^{(s)}\{[\exp(x)-1]^{1/r}\} = 0, \tag{23}$$

for $s\neq mr-1$ and $m,r=1,2,\dots.$

In particular

$$\delta^{(r-1)}[|\exp(x)-1|^{1/r}] = -\frac{(-1)^r r!}{2}\delta(x), \tag{24}$$

$$\delta^{(2r-1))}[|\exp(x)-1|^{1/r}] = \frac{(2r)!}{4}\delta(x)-\frac{(2r)!}{4}\delta'(x), \tag{25}$$

$$\begin{aligned}\delta^{(3r-1)}[|\exp(x)-1|^{1/r}] &= -\frac{(-1)^r r(3r-1)!}{2}\delta(x)\\ &\quad -\frac{(-1)^r 3r(3r-1)!}{4}[\delta'(x)+\delta''(x)],\end{aligned} \tag{26}$$

for $r=1,2,\dots.$

Proof. This time we have

$$\begin{aligned}\int_{-1}^{1} x^k\delta_n^{(s)}[|\exp_+(x)-H(x)|^{1/r}]\,dx &= \\ &= \int_0^1 x^k\delta_n^{(s)}\{[\exp(x)-1]^{1/r}\}\,dx+\int_{-1}^0 x^k\delta_n^{(s)}\{[1-\exp(x)]^{1/r}\}\,dx\\ &= I_1+J_2\end{aligned}$$

and it follows that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle\delta_n^{(mr-1)}[|\exp(x)-1|^{1/r}],\varphi(x)\rangle &= \\ &= \operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 x^k\delta_n^{(mr-1)}\{[\exp(x)-1]^{1/r}\}\,dx\\ &\quad+\operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 x^k\delta_n^{(mr-1)}\{[1-\exp(x)]^{1/r}\}\,dx\\ &= \sum_{k=0}^{m-1}\frac{(-1)^{mr+k-1}r(mr-1)!c_{m-1,k}}{2k!}\langle\delta^{(k)}(x),\varphi(x)\rangle\\ &\quad+\sum_{k=0}^{m-1}\frac{(-1)^k r(mr-1)!c_{m-1,k}}{2k!}\langle\delta^{(k)}(x),\varphi(x)\rangle\end{aligned}$$

and equation (22) follows.

Equations (23) to (26) follow easily, completing the proof of the corollary. □

References

[1] A. Antonevich, On certain high-order partial differantial expressions with δ potential, Integral Transforms Spec. Funct. 22 (2011) 255–261.
[2] A. Bredimas, On the Lamropoulos-Luco's $\delta-$"no go" conjecture in variational flux synthesis, in: Annals Nucl. Energy, Vol. 5, Pergamon Press, 1978.
[3] J.G. van der Corput, Introduction to the neutrix calculus, J. Analyse Math. 7 (1959) 291–398.
[4] B. Fisher, On defining the change of variable in distributions, Rostock. Math. Kolloq. 28 (1985) 75–86.
[5] B. Fisher, On defining the distribution $\delta^{(r)}(f(x))$, Rostock. Math. Kolloq. 23 (1993) 73–80.
[6] B. Fisher, The delta function and the composition of distributions, Dem. Math. 35 (2002) 117–123.
[7] B. Fisher, Results on the composition and neutrix composition of the delta function, submitted.
[8] B. Fisher, A. Kılıçman, On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function, Integral Transforms Spec. Funct. 21 (2010) 935–944.
[9] B. Fisher, T. Kraiweeradechachai, E. Özçağ, Results on the neutrix composition of the delta function, Hacet. J. Math. Stat. 36 (2007) 147–156.
[10] B. Fisher, E. Özçağ, A result on the neutrix composition of the delta function, Adv. Dyn. Syst. Appl. 6 (2011) 49–56.
[11] I.M. Gel'fand, G.E. Shilov, Generalized Functions, Vol. I, Academic Press, 1964.
[12] E.L. Koh, C.K. Li, On distributions δ^k and $(\delta')^k$, Math. Nachr. 157 (1992) 243–248.
[13] H. Kou, B. Fisher, On composition of distributions, Publ. Math. Debrecen 40 (1992) 279–290.
[14] T. Kraiweeradechachai, S. Orankitjaroen, B. Fisher, E. Özçağ, Further results on the neutrix composition of the delta function, East-West J. Math. 11 (2009) 151–164.
[15] D. Mitrović , On the heat equation involving the Dirac distribution as a coefficient, Math. Comp. Model. 50 (2009) 109–115.
[16] M. Nedeljkov, S. Pilipovic, D. Rajter, Heat equation with singular potential and singular data, Proc. Roy. Soc. Edinburgh Sect. A 135 (2005) 863–886.
[17] E. Özçağ, Defining the kth powers of the Dirac Delta distribution for negative integers, Appl. Math. Letters 14 (2001) 419–423.
[18] E. Özçağ, I. Ege, H. Gürçay, On powers of the Heaviside function for negative integers, J. Math. Anal. Appl. 326 (2007) 101–107.
[19] C.O.R. Sarrico, Collision of delta-waves in a turbulent model studied via a distribution product, Nonlinear Anal. 73 (2010) 2868–2875.