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Vertex-removal in α-domination

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Abstract. Let $G = (V, E)$ be any graph without isolated vertices. For some α with $0 < \alpha \leq 1$ and a dominating set S of G, we say that S is an α-dominating set if for any $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The cardinality of a smallest α-dominating set of G is called the *α-domination number* of G and is denoted by $\gamma_\alpha(G)$. In this paper, we study the effect of vertex removal on α-domination.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. The *degree* $\deg(x)$ of a vertex x denotes the number of neighbors of x in G. The *maximum degree* and *minimum degree* of vertices of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A *leaf* is a vertex of degree one and a *support vertex* is one that is adjacent to a leaf. We denote by $S(G)$ the set of all support vertices of G. A set of vertices S in G is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A *subdivided star* is obtained from a star with at least two edges by subdividing every edge exactly once. The *corona cor*(H) of a graph H is that graph obtained from H by adding a pendant edge to each vertex of H. For notation and graph theory terminology in general we follow [7]

Let G be a graph with no isolated vertex. For $0 < \alpha \leq 1$, a set $S \subseteq V$ is said to *α-dominate* a graph G, if for any vertex $v \in V - S$, $|N(v) \cap S| \geq \alpha|N(v)|$. The minimum cardinality of an α-dominating set is the *α-domination number*, denoted $\gamma_\alpha(G)$. We refer an α-dominating set of cardinality $\gamma_\alpha(G)$ as a $\gamma_\alpha(G)$-set. For references on α-domination in graphs see, for example, [2–4, 6]. Dunbar et al. in [4] suggested the study of graphs in which removing of any edge changes the α-domination number.

For a $\gamma_\alpha(G)$-set S in a graph G and a vertex $x \in S$, if $S - \{x\}$ is an α-dominating set for $G - x$, then we denote $pn(x, S) = \{x\}$.

We remark that α-domination could be defined for any graph G. However in the first introductory paper [4], Dunbar et al. defined it only for graphs with no isolated vertex. So we adopt this definition in this paper.

For many graph parameters, criticality is a fundamental question. Much has been written about graphs where a parameter increases or decreases whenever an edge or vertex is removed or added. For the

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domination number, Brigham, Chinn, and Dutton [1] began the study of those graphs where the domination number decreases on the removal of any vertex. They defined a graph G to be *domination vertex critical*, or just γ-vertex critical, if removal of any vertex decreases the domination number. This concept is now well studied in domination theory.

In this paper we study the same concept for α-domination. We call a graph G, *α-domination vertex critical* if removal of any vertex decreases the α-domination number.

Observation 1.1. *For any graph G of order n, $\gamma_\alpha(G) < n$.*

Observation 1.2. *In any graph $G \neq P_2$, there is a $\gamma_\alpha(G)$-set containing all support vertices of G.*

Proposition 1.3. *([4]) If $0 < \alpha \leq \frac{1}{\Delta(G)}$, then $\gamma_\alpha(G) = \gamma(G)$.*

Proposition 1.4. *([4]) If $1 \geq \alpha > 1 - \frac{1}{\Delta(G)}$, then $\gamma_\alpha(G) = \alpha_0(G)$.*

A set $S \subseteq V(G)$ is a 2-*packing* of G if for every two different vertices $x, y \in S$, $N[x] \cap N[y] = \emptyset$.

2. Results

Proposition 2.1. *Let G be a graph without isolated vertices. For any vertex $v \in V(G) - S(G)$, and any $0 < \alpha \leq 1$, $\gamma_\alpha(G) - 1 \leq \gamma_\alpha(G - v) \leq \gamma_\alpha(G) + deg(v) - 1$ and these bounds are sharp.*

Proof. Let G be a graph without isolated vertices and $v \in V(G) - S(G)$. Let S be a $\gamma_\alpha(G)$-set. If $v \notin S$, then S is an α-dominating set for $G - v$, and so $\gamma_\alpha(G - v) \leq \gamma_\alpha(G)$. Thus we assume that $v \in S$. Then $S \cup N_G(v) - \{v\}$ is an α-dominating set for $G - v$, and so $\gamma_\alpha(G - v) \leq \gamma_\alpha(G) + deg(v) - 1$. Thus the upper bound follows.

For the lower bound let D be a $\gamma_\alpha(G - v)$-set. Then $D \cup \{v\}$ is an α-dominating set for G, and so the lower bound follows.

To see the sharpness of the upper bound, let x be the center of a star $K_{1,k}$ for $k \geq 2$, and $\alpha \leq \frac{1}{2}$. Let G be obtained from $K_{1,k}$ by subdividing each edge of $K_{1,k}$ three times. Note the G has $3k$ vertices of degree two, k vertices of degree one, and a vertex of degree k (the vertex x). Now it is easy to see that $\gamma_\alpha(G) = k + 1$, and $\gamma_\alpha(G - x) = 2k$. To see the sharpness of the lower bound consider a cycle C_4. □

We call a graph G, *α-domination vertex critical*, or just *γ_α-vertex critical* if for any $v \in V(G) - S(G)$, $\gamma_\alpha(G - v) < \gamma_\alpha(G)$.

We note that if for a graph G with no isolated vertex, $V(G) - S(G) = \emptyset$, then G is α-domination vertex critical. Thus P_2 is obviously α-domination vertex critical, since $V(P_2) - S(P_2) = \emptyset$.

2.1. γ_α-vertex critical graphs

In this subsection we present our results on γ_α-vertex critical graphs.

Proposition 2.2. *A graph G is γ_α-vertex critical if and only if for any non-support vertex x, there is a $\gamma_\alpha(G)$-set S containing x such that $pn(x, S) = \{x\}$.*

Proof. (⇒) Let G be a γ_α-vertex critical graph and $x \notin S(G)$. Then $\gamma_\alpha(G-x) = \gamma_\alpha(G)-1$. Let S be a $\gamma_\alpha(G-x)$-set. It is obvious that $D = S \cup \{x\}$ is an α-dominating set for G and $pn(x, D) = \{x\}$.

(⇐) Let $x \notin S(G)$ and let S be a $\gamma_\alpha(G)$-set containing x such that $pn(x, S) = \{x\}$. Then $S - \{x\}$ is an α-dominating set for $G - x$ implying that $\gamma_\alpha(G - x) < \gamma_\alpha(G)$. Thus G is γ_α-vertex critical. □

Since $\gamma_\alpha(K_{1,n}) = 1$, we obtain the following.

Lemma 2.3. *$K_{1,n}$ is γ_α-vertex critical if and only if $n = 1$.*

Proposition 2.4. *Every support vertex in a γ_α-vertex critical graph is adjacent to exactly one leaf.*

Proof. Let G be a γ_α-vertex critical. Assume that there is a support vertex x such x is adjacent to two leaves x_1 and x_2. By Lemma 2.3, we may assume that $N(x)$ contains a vertex of degree at least two. Then x is a support vertex in $G - x_1$. By Observation 1.2, let S be a $\gamma_\alpha(G - x_1)$-set such that $x \in S$. Then S is an α-dominating set for G, a contradiction. □

Observation 2.5. *A subdivided star is not γ_α-vertex critical.*

Theorem 2.6. *Let H be a connected graph of order at least two. Then $G = cor(H)$ is γ_α-vertex critical.*

Proof. Since $G = cor(H)$, each vertex x of G is either a leaf a support vertex adjacent to exactly one leaf. We observe that $\gamma_\alpha(G) = |S(G)|$. Let x be a leaf of G. We show that $\gamma_\alpha(G-x) < \gamma_\alpha(G)$. Let y be the support vertex adjacent to x. Since H is connected of order at least two, there is a vertex $z \in N(y)$ such that $deg(z) > 1$. Then z is a support vertex. Now $S(G) - \{y\}$ is an α-dominating set for $G - x$, implying that $\gamma_\alpha(G - x) < \gamma_\alpha(G)$. Thus G is γ_α-vertex critical. □

Let $\mathcal{T}$ be the class of all trees T such that $T \in \mathcal{T}$ if and only if:
(1) $T = P_2$, or
(2) $diam(T) \geq 3$, and for any vertex x of T either x is a leaf or x is a support adjacent to exactly one leaf.

Theorem 2.7. *A tree T is γ_α-vertex critical for $0 < \alpha \leq \frac{1}{\Delta(T)}$, if and only if $T \in \mathcal{T}$.*

Proof. ($\Longleftarrow$) It is obvious that P_2 is γ_α-vertex critical. If $T \neq P_2$ is a tree in $\mathcal{T}$, then Theorem 2.6 implies that T is γ_α-vertex critical.

$\Longrightarrow$ Let T be a γ_α-vertex critical tree. If $diam(T) = 1$, then $T = P_2$ and so $T \in \mathcal{T}$. If $diam(T) = 2$, then by Lemma 2.3, T is not γ_α-domination vertex critical. Thus we assume that $diam(T) \geq 3$. We show that any vertex of T is either a leaf or a support vertex.

Let y be vertex of T such that y is neither a leaf nor a support vertex. If each leaf of T is at distance two from y, then by Proposition 2.4, y is the center of a subdivided star, a contradiction to Observation 2.5. Thus assume that there is a leaf x in T such that $d(x, y) \geq 3$. Let $d(x, y) = t$ and $P : x - x_1 - x_2 - ... - x_t = y$ be the shortest path between x and y.

If x_2 is not a support vertex, then by Proposition 2.2, there is a $\gamma_\alpha(T)$-set S containing x_2 such that $pn(x_2, S) = \{x_2\}$. But then $\{x_1, x\} \cap S \neq \emptyset$. Since $\alpha\Delta(T) \leq 1$, we see that $(S - \{x, x_2\}) \cup \{x_1\}$ is an α-dominating set for T, a contradiction. Thus x_2 is a support vertex. Let y_2 be a leaf adjacent to x_2. If x_3 is not a support vertex, then by Proposition 2.2, there is a $\gamma_\alpha(T)$-set S containing x_3 such that $pn(x_3, S) = \{x_3\}$. But $S \cap \{x_2, y_2\} \neq \emptyset$. Then $S_1 = (S - \{y_2\}) \cup \{x_2\}$ is a $\gamma_\alpha(T)$-set such that $pn(x_3, S_1) = \{x_3\}$ and $x_2 \in S_1$. So $S_1 - \{x_3\}$ is an α-dominating set for T, a contradiction. Thus x_3 is a support vertex. By continuing this process we obtain that $x_i \in S(T)$ for $i = 1, 2, ..., t - 1$. By Proposition 2.2, there is a $\gamma_\alpha(T)$-set D containing y such that $P_n(y, D) = \{y\}$. We may assume that $x_{t-1} \in D$, since $x_{t-1} \in S(T)$. Then $D - \{y\}$ is an α-dominating set for T, a contradiction. □

Problem 2.8. *Characterize γ_α-vertex critical trees for $\alpha > \frac{1}{\Delta(T)}$.*

Proposition 2.9. *([4]) If $\frac{1}{2} < \alpha \leq 1$, then:*
(1) $\gamma_\alpha(P_n) = \lfloor \frac{n}{2} \rfloor$.
(2) $\gamma_\alpha(C_n) = \lceil \frac{n}{2} \rceil$.

Proposition 2.10. *([4]) If $0 < \alpha \leq \frac{1}{2}$, then $\gamma_\alpha(P_n) = \gamma_\alpha(C_n) = \lceil \frac{n}{3} \rceil$.*

Proposition 2.11. *(1) For $0 < \alpha \leq \frac{1}{2}$, the path P_n is γ_α-vertex critical if and only if $n \in \{2, 4\}$.*
(2) For $\frac{1}{2} < \alpha \leq 1$, the path P_n is γ_α-vertex critical if and only if $n = 2k$.

Proof. If $0 < \alpha \leq \frac{1}{2}$, then the result follows from Theorem 2.7.

Assume next that $\frac{1}{2} < \alpha \leq 1$. By Proposition 2.9, $\gamma_\alpha(P_n) = \lfloor \frac{n}{2} \rfloor$. Let $n = 2k$ for some integer $k \geq 1$. It is easy to see that P_2 and P_4 are γ_α-vertex critical. Thus we assume now that $n \geq 6$. Let x be a vertex such that x is not a support vertex. If x is a leaf then by Proposition 2.9,

$$\gamma_\alpha(P_n - x) = \gamma_\alpha(P_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1 < \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus assume now that x is not a leaf. Let $G = P_n - x$. Then G has two components P_{n_1} and P_{n_2}. Clearly we may assume that n_1 is even and n_2 is odd. Then

$$\gamma_\alpha(G) = \gamma_\alpha(P_{n_1}) + \gamma_\alpha(P_{n_2}) = \left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor.$$

A simple calculation shows that $\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor < \lfloor \frac{n}{2} \rfloor$. Thus P_n is γ_α-vertex critical.

Finally, we show that P_n is not γ_α-vertex critical if n is odd. Let n be odd and let x be a leaf. Then $\gamma_\alpha(P_n) = \gamma_\alpha(P_{n-1})$, since $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$, as desired. □

Using Propositions 2.9 and 2.10, we obtain the following proposition similarly.

Proposition 2.12. *(1) For $0 < \alpha \le \frac{1}{2}$, the cycle C_n is γ_α-vertex critical if and only if $n \equiv 1$ (mod 3).*
(2) For $\frac{1}{2} < \alpha \le 1$, the cycle C_n is always γ_α-vertex critical.

Observation 2.13. *([4]) If K_n is the complete graph of order n, then $\gamma_\alpha(K_n) = \lceil \alpha(n-1) \rceil$.*

Proposition 2.14. *A complete graph K_n of order $n \ge 2$ is γ_α-vertex critical if and only if*

$$\alpha > \frac{\lceil \alpha(n-2) \rceil}{n-1}.$$

Proof. By Observation 2.13, the complete graph K_n is γ_α-vertex critical if and only if $\lceil \alpha(n-2) \rceil < \lceil \alpha(n-1) \rceil$. This is equivalent with $\lceil \alpha(n-2) \rceil < \alpha(n-1)$, and this is equivalent with $\alpha > \lceil \alpha(n-2) \rceil/(n-1)$. □

Proposition 2.15. *([4]) If $K_{m,n}$ is a complete bipartite graph with $1 \le m \le n$, then $\gamma_\alpha(K_{m,n}) = \min\{m, \lceil \alpha m \rceil + \lceil \alpha n \rceil\}$.*

Proposition 2.16. *If $2 \le m < n$, then $K_{m,n}$ is γ_α-vertex critical if and only if $m \ge \lceil \alpha m \rceil + \lceil \alpha n \rceil$,*

$$\alpha > \frac{\lceil \alpha(m-1) \rceil}{m} \text{ and } \alpha > \frac{\lceil \alpha(n-1) \rceil}{n}.$$

Proof. Let X and Y be the partite sets of $G = K_{m,n}$ with $|X| = m$ and $|Y| = n$. First assume that $m < \lceil \alpha m \rceil + \lceil \alpha n \rceil$. By Proposition 2.15, $\gamma_\alpha(G) = m$. Let S be a $\gamma_\alpha(G - y)$-set, where $y \in Y$. Then Proposition 2.15 implies

$$\begin{aligned} |S| = \gamma_\alpha(G - y) &= \gamma_\alpha(K_{m,n-1}) \\ &= \min\{m, \lceil \alpha m \rceil + \lceil \alpha(n-1) \rceil\} \\ &\ge \min\{m, \lceil \alpha m \rceil + \lceil \alpha n \rceil - 1\} \ge m, \end{aligned}$$

and therefore G is not γ_α-vertex critical in that case.

Next assume that $m \ge \lceil \alpha m \rceil + \lceil \alpha n \rceil$. Then $\gamma_\alpha(G) = \lceil \alpha m \rceil + \lceil \alpha n \rceil$. Let S_1 be a $\gamma_\alpha(G - y)$-set, where $y \in Y$, and let S_2 be a $\gamma_\alpha(G - x)$-set, where $x \in X$. Then similar to the proof of Proposition 2.14, we observe that G is γ_α-vertex critical if and only if $\alpha > \lceil \alpha(m-1) \rceil/m$ and $\alpha > \lceil \alpha(n-1) \rceil/n$. □

Proposition 2.17. *If $2 \le m$, then $K_{m,m}$ is γ_α-vertex critical if and only if $m \le 2\lceil \alpha m \rceil$ or $m > 2\lceil \alpha m \rceil$ and*

$$\alpha > \frac{\lceil \alpha(m-1) \rceil}{m}.$$

Proof. Let $G = K_{m,m}$. First assume that $m \le 2\lceil \alpha m \rceil$. By Proposition 2.15, $\gamma_\alpha(G) = m$. Let S be a $\gamma_\alpha(G - x)$-set, where $x \in V(G)$. Then Proposition 2.15 implies

$$\begin{aligned} |S| = \gamma_\alpha(G - x) &= \gamma_\alpha(K_{m-1,m}) \\ &= \min\{m-1, \lceil \alpha m \rceil + \lceil \alpha(m-1) \rceil\} \le m-1. \end{aligned}$$

and therefore G is γ_α-vertex critical in that case.

Next assume that $m > 2\lceil \alpha m \rceil$. Then $\gamma_\alpha(G) = 2\lceil \alpha m \rceil$. Let S be a $\gamma_\alpha(G-x)$-set, where $x \in V(G)$. Then similar to the proof of Proposition 2.14, we observe that G is γ_α-vertex critical if and only if $\alpha > \lceil \alpha(m-1) \rceil/m$. □

Proposition 2.18. *There is no induced-subgraph characterization for γ_α-vertex critical graphs.*

Proof. Let G be an arbitrary graph, and $H = cor(G)$. Clearly, G is an induced subgraph of H. Then $\gamma_\alpha(H) = |V(G)|$, and $V(G)$ is a $\gamma_\alpha(H)$-set. Let v be a leaf of H and $u \in N(v)$. Then $V(G) - \{u\}$ is an α-dominating set for $H - v$, implying that $\gamma_\alpha(H - v) < \gamma_\alpha(H)$. Thus H is γ_α-vertex critical. □

Theorem 2.19. *([5]) If G is a γ-vertex critical graph of order $n = (\gamma(G) - 1)(\Delta(G) + 1) + 1$, then G is regular.*

Theorem 2.20. *If G is a γ_α-vertex critical graph of order n, then $n \leq (\gamma_\alpha(G) - 1)(\Delta(G) + 1) + 1$. Furthermore, if $\delta(G) > 1$ and equality holds, then G is regular.*

Proof. Let G be a γ_α-vertex critical graph of order n, and let S be a $\gamma_\alpha(G - v)$-set, where $v \in V(G) - S(G)$. Any vertex of S dominates at most $1 + \Delta(G)$ vertices of G including itself. Thus S dominates at most $(\gamma_\alpha(G) - 1)(\Delta(G) + 1) \geq n - 1$ vertices of G, as desired.

Now assume that $n = (\gamma_\alpha(G) - 1)(\Delta(G) + 1) + 1$. Thus any vertex of S dominates exactly $1 + \Delta(G)$ vertices of G, and so has degree $\Delta(G)$. Furthermore S is a 2-packing. Let $u \in N(v) - S$. Since $\delta(G) > 1$, the vertex $u \notin V(G) - S(G)$. Let D be a $\gamma_\alpha(G - u)$-set. Then $|D| = \gamma_\alpha(G) - 1$, and, as before, we obtain that any vertex of D is of degree $\Delta(G)$, and D is a 2-packing. Since S is a $\gamma_\alpha(G - v)$-set, u is adjacent to a vertex $a \in S$, and now $deg_{G-u}(a) < \Delta(G)$, and so $a \notin D$. We deduce that $D - S \neq \emptyset$. Also clearly $v \notin D$. Let $w \in D - S$. Since S is a $\gamma_\alpha(G - v)$-set, we obtain that $1 = |N(w) \cap S| \geq \alpha deg(w) = \alpha\Delta(G)$, and so $\alpha \leq \frac{1}{\Delta(G)}$. By Proposition 1.3, $\gamma_\alpha(G) = \gamma(G)$, and also $\gamma_\alpha(G - a) = \gamma(G - a)$ for any vertex a. Thus G is γ-vertex critical. By Theorem 2.19, G is regular. □

Fulman et al. [5] proved that if G is a γ-vertex critical graph, then $diam(G) \leq 2(\gamma(G) - 1)$. However with a similar proof we obtain the following.

Proposition 2.21. *If G is a γ_α-vertex critical graph, then $diam(G) \leq 2(\gamma_\alpha(G) - 1)$.*

Theorem 2.22. *If G is a γ_α-vertex critical graph of order n, then for any vertex $v \in V(G) - S(G)$,*

$$\gamma_\alpha(G) \geq \left\lceil \frac{\alpha\delta(G - v)n + \Delta(G)}{\alpha\delta(G - v) + \Delta(G)} \right\rceil.$$

Proof. Let G be a γ_α-vertex critical graph of order n, and let $v \in V(G) - S(G)$. Let $H = G - v$ and let S be a $\gamma_\alpha(H)$-set. Then $|S| = \gamma_\alpha(G) - 1$. Let M be the set of edges between S and $V(H) - S$. By counting the edges from S to $V(H) - S$, we obtain that

$$|M| \leq \sum_{v \in S} deg(v) \leq |S|\Delta(G).$$

On the other hand, since S is an α-dominating set for H, we find that

$$|M| \geq \sum_{v \in V(H)-S} \alpha deg_H(v) \geq \alpha\delta(H)(|V(H)| - |S|).$$

Now we obtain

$$|S|\Delta(G) \geq \alpha\delta(H)(n - 1 - |S|).$$

Since $|S| = \gamma_\alpha(G) - 1$ and $H = G - v$, a simple calculation imply that

$$\gamma_\alpha(G) \geq \frac{\alpha\delta(G - v)n + \Delta(G)}{\alpha\delta(G - v) + \Delta(G)}.$$

□

Proposition 2.23. *([4]) If $0 < \alpha < 1$, then for any graph G, $\gamma_\alpha(G) + \gamma_{1-\alpha}(G) \leq n$.*

Theorem 2.24. *If G is a γ_α-vertex critical graph of order n and size m, then*

$$\gamma_\alpha(G) \geq \left\lceil \frac{2\alpha m - \alpha\Delta(G) + \Delta(G)}{\Delta(G)(\alpha+1)} \right\rceil.$$

Proof. Let G be a γ_α-vertex critical graph of order n and size m. Let $v \in V(G) - S(G)$ and $H = G - v$. Let S be a $\gamma_\alpha(H)$-set. Then $\sum_{v\in S} deg_H(v) \geq \sum_{v\in V(H)-S} \alpha deg_H(v)$. Now

$$\begin{aligned}(\alpha+1)|S|\Delta(G) &\geq \alpha \sum_{v\in S} deg_H(v) + \sum_{v\in S} deg_H(v)\\ &\geq \alpha \sum_{v\in S} deg_H(v) + \sum_{v\in V(H)-S} \alpha deg_H(v)\\ &\geq \alpha \sum_{v\in V(H)} deg_H(v)\\ &= \alpha(2m - 2deg_G(v)) \geq \alpha(2m - 2\Delta(G)).\end{aligned}$$

Since $|S| = \gamma_\alpha(G) - 1$, a simple calculation completes the proof. □

By Proposition 2.23, we have the following.

Corollary 2.25. *Let $0 < \alpha < 1$. If G is a γ_α-vertex critical graph of order n and size m, then*

$$\gamma_{1-\alpha}(G) \leq \left\lfloor \frac{(1+\alpha)\Delta(G)n + \alpha\Delta(G) - 2\alpha m - \Delta(G))}{\Delta(G)(\alpha+1)} \right\rfloor.$$

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