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Bounded operators on topological vector spaces and their spectral radii

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Abstract. In this paper, we consider three classes of bounded linear operators on a topological vector space with respect to three different topologies which are introduced by Troitsky. We obtain some properties for the spectral radii of a linear operator on a topological vector space. We find some sufficient conditions for the completeness of these classes of operators. Finally, as a special application, we deduce some sufficient conditions for invertibility of a bounded linear operator.

1. Introduction and preliminaries

Troitsky in [9], presented some various types of bounded linear operators on a topological vector space, see Definition 1.1 below. Also, he endowed each class of them with an appropriate natural operator topology and developed a spectral theory for these classes of linear operators.

Definition 1.1. Let X and Y be topological vector spaces. A linear operator $T : X \to Y$ is said to be:

i. nb-bounded if there exists some zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded in Y;

ii. bb-bounded if for every bounded subset $B \subseteq X$, $T(B)$ is bounded in Y.

The definition of Edwards in [3] corresponds with the notion of bb-boundedness, while the definition of Schaefer in [8] corresponds with the notion of nb-boundedness. However, these definitions are far from being equivalent (see [7, 9]). The most famous examples of topological vector spaces are normed linear spaces. Nevertheless, there are topological vector spaces whose topology does not arise from a norm but are still of interest in analysis. For example, the space of holomorphic functions on an open domain, spaces of infinitely differentiable functions, the Schwartz spaces, and spaces of test functions and the spaces of distributions on them. So, it is natural to investigate bounded operators on general topological vector spaces and consider some known results for bounded (continuous) operators on a normed linear space for different types of bounded operators on a topological vector space. The class of all nb-bounded linear operators from X into Y is denoted by $B_n(X, Y)$. This linear space is equipped with the topology of uniform

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convergence on some zero neighborhood, that means a net (T_α) of *nb*-bounded operators converges to zero in this topology if there exists a zero neighborhood $U \subseteq X$ such that for each zero neighborhood $V \subseteq Y$ there is an α_0 with $T_\alpha(U) \subseteq V$ for every $\alpha \geq \alpha_0$. The class of all *bb*-bounded operators from X into Y is denoted by $B_b(X, Y)$ and is endowed with the topology of uniform convergence on bounded sets. Recall that a net (T_α) of *bb*-bounded operators converges uniformly to zero on a bounded set $B \subseteq X$ if for each zero neighborhood $V \subseteq Y$ there exists an α_0 with $T_\alpha(B) \subseteq V$ for all $\alpha \geq \alpha_0$. The class of all continuous operators from X into Y is denoted by $B_c(X, Y)$ and is assigned with the topology of equicontinuous convergence, namely, a net (T_α) of continuous operators converges equicontinuously to zero if for each zero neighborhood $V \subseteq Y$ there is a zero neighborhood $U \subseteq X$ such that for every $\varepsilon > 0$ there is an α_0 with $T_\alpha(U) \subseteq \varepsilon V$ for all $\alpha \geq \alpha_0$. The symbols $B_n(X)$, $B_b(X)$, and $B_c(X)$ are given for $B_n(X, X)$, $B_b(X, X)$, and $B_c(X, X)$, respectively. In [9], it is shown that $B_n(X) \subseteq B_c(X) \subseteq B_b(X)$. Note that the above inclusions become equalities when X is locally bounded,[9]. In [11], it has been proved that each class of bounded linear operators, with respect to the assumed topology, forms a topological algebra. Troitsky in [9], by using the canonical topology of each class of bounded linear operators, introduced some different aspects of spectral radii for a linear operator on a topological vector space and deduced some relations between them. In particular, he showed that for a continuous linear operator on a sequentially complete locally convex topological vector space, each of the defined spectral radii is greater or equal to the corresponding geometrical radius of the spectrum in each of the topological algebras $B_n(X)$, $B_b(X)$, and $B_c(X)$, respectively [9, Sections 3, 4, 5]. As a main result, we develop some known properties for the spectral radius of a bounded operator on a normed linear space to these spectral radii of a linear operator on a topological vector space. Also, we show that each of the algebras $B_n(X)$, $B_b(X)$, and $B_c(X)$ on a locally convex topological vector space X, with respect to its given topology, is complete if and only if so is X. It is well known that for a bounded linear operator T on a Banach space, $(I - T)$ is invertible whenever $r(T) < 1$, where $r(.)$ denotes the spectral radius and I is the identity operator. Here, by assuming the corresponding spectral radius, we generalize this result to each class of bounded linear operators on a complete locally convex topological vector space. For more about these classes of linear operators, their corresponding operator topologies, and different spectral radii, see [2, 4, 5, 9, 10]. Also, for further information about topological vector spaces and the related notions, the reader is referred to [1, 3, 5, 7–9, 11].

Throughout the paper, the scalar field for every vector space is either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$.

2. Spectral radii

Troitsky in [9], introduced different types of spectral radii for a linear operator on a topological vector space, see Definition 2.1 below. What follows, we investigate some properties for these spectral radii.

Definition 2.1. For a linear operator T on a topological vector space X, consider the following spectral radii.

(i) $r_{nb}(T) = \inf\{\nu > 0 : \frac{T^n}{\nu^n} \to 0$ uniformly on some zero neighborhood $\}$;

(ii) $r_{bb}(T) = \inf\{\nu > 0 : \frac{T^n}{\nu^n} \to 0$ uniformly on every bounded set $\}$;

(iii) $r_c(T) = \inf\{\nu > 0 : \frac{T^n}{\nu^n} \to 0$ equicontinuously $\}$.

In [9], it has been proved that for a linear operator T on a topological vector space X, $r_{bb}(T) \leq r_c(T) \leq r_{nb}(T)$. In general, these numbers are far from being equal. Since *nb*-boundedness is the strongest of the boundedness conditions for a linear operator on a general topological vector space, some special results can be obtained for *nb*-bounded linear operators while these results do not hold for common continuous operators. An interesting result is that for an *nb*-bounded linear operator T on a sequentially complete locally convex topological vector space, $r_{nb}(T)$ is equal to the usual geometrical radius of the spectrum, [9, Section 6].

In the following theorem, part (*iii*) is [9, Lemma 4.8] which is proved in a different way.

Theorem 2.2. *If T and S are two commuting linear operators on a topological vector space X, then*

(i) $r_{nb}(TS) \leq r_{nb}(T)r_{nb}(S)$;

(ii) $r_{bb}(TS) \leq r_{bb}(T)r_{bb}(S)$;

(iii) $r_c(TS) \leq r_c(T)r_c(S)$.

Proof. (*i*) Let $W \subseteq X$ be an arbitrary zero neighborhood. Suppose $\nu > r_{nb}(T)$ and $\mu > r_{nb}(S)$. There is a zero neighborhood U_0 such that the sequences $(\frac{T^n}{\nu^n})$ and $(\frac{S^n}{\mu^n})$ converge to zero uniformly on U_0. Find $n_0 \in \mathbb{N}$ with $\frac{S^n}{\mu^n}(U_0) \subseteq U_0$ for all $n > n_0$. Choose $n_1 \in \mathbb{N}$ such that $\frac{T^n}{\nu^n}(U_0) \subseteq W$ for all $n > n_1$. Therefore for sufficiently large $n \in \mathbb{N}$,

$$\frac{(TS)^n}{(\nu\mu)^n}(U_0) = \frac{T^n S^n}{\nu^n \mu^n}(U_0) \subseteq \frac{T^n}{\nu^n}(U_0) \subseteq W.$$

It follows that $\nu\mu > r_{nb}(TS)$ and so $r_{nb}(TS) \leq r_{nb}(T)r_{nb}(S)$.

(*ii*) Fix a bounded set $B \subseteq X$. Suppose $\nu > r_{bb}(T)$ and $\mu > r_{bb}(S)$. Since the sequence $(\frac{S^n}{\mu^n})$ converges to zero uniformly on B, it is uniformly bounded and so $E = \cup_{n=1}^{\infty} \frac{S^n}{\mu^n}(B)$ is a bounded set. Therefore, there is $n_2 \in \mathbb{N}$ with $\frac{T^n}{\nu^n}(E) \subseteq W$ for all $n > n_2$. Thus

$$\frac{(TS)^n}{(\nu\mu)^n}(B) = \frac{T^n S^n}{\nu^n \mu^n}(B) \subseteq \frac{T^n}{\nu^n}(E) \subseteq W.$$

This shows that $\nu\mu > r_{bb}(TS)$ and so $r_{bb}(TS) \leq r_{bb}(T)r_{bb}(S)$.

(*iii*) Suppose $\nu > r_c(T)$ and $\mu > r_c(S)$. There exists some zero neighborhood U_1 such that for a given $\varepsilon > 0$ there is $n_3 \in \mathbb{N}$ with $\frac{T^n}{\nu^n}(U_1) \subseteq \varepsilon W$ for all $n > n_3$. Find a zero neighborhood U_2 and $n_4 \in \mathbb{N}$ such that $\frac{S^n}{\mu^n}(U_2) \subseteq U_1$ for every $n > n_4$. So, for sufficiently large $n \in \mathbb{N}$, we have

$$\frac{(TS)^n}{(\nu\mu)^n}(U_2) = \frac{T^n S^n}{\nu^n \mu^n}(U_2) \subseteq \frac{T^n}{\nu^n}(U_1) \subseteq \varepsilon W.$$

This implies that $\nu\mu > r_c(TS)$ and so $r_c(TS) \leq r_c(T)r_c(S)$. □

We know that for a bounded linear operator T on a Banach space X, $r(T^n) = r(T)^n$, where $r(.)$ denotes the usual spectral radius for bounded operators. We show that the same result holds for different spectral radii when T is assumed to be continuous and each spectral radius is assumed to be finite.

Theorem 2.3. *Suppose T is a continuous linear operator on a topological vector space X. Then for each $k \in \mathbb{N}$,*

(i) $r_{nb}(T)^k \leq r_{nb}(T^k)$;

(ii) $r_{bb}(T)^k \leq r_{bb}(T^k)$;

(iii) $r_c(T)^k \leq r_c(T^k)$.

Proof. (*i*) Let $W \subseteq X$ be an arbitrary zero neighborhood. For each $m > k$, we can find positive integers p, q with $m = pk + q$ that $0 \leq q < k$. Suppose $\nu > r_{nb}(T^k)$. There exists a zero neighborhood $U \subseteq X$ such that $\frac{(T^k)^n}{\nu^n}$ converges to zero uniformly on U. There is a zero neighborhood U_0 with $T^q(U_0) \subseteq U$ for all $0 \leq q < k$. Choose positive scalar $\alpha_{\nu,nb}$ such that $\alpha_{\nu,nb} > \max\{\frac{1}{\nu^{\frac{q}{k}}}; 0 \leq q < k\}$. Find a zero neighborhood W_1 with $\alpha_{\nu,nb} W_1 \subseteq W$. There is $n_0 \in \mathbb{N}$ such that $\frac{T^{kn}}{\nu^n}(U) \subseteq W_1$ for all $n > n_0$. So,

$$\frac{T^m}{\nu^{\frac{m}{k}}}(U_0) = \frac{T^{pk+q}}{\nu^{p+\frac{q}{k}}}(U_0) \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(T^q(U_0))}{\nu^p} \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(U)}{\nu^p} \subseteq \alpha_{\nu,nb} W_1 \subseteq W,$$

for sufficiently large m, p. Therefore, $\nu^{\frac{1}{k}} > r_{nb}(T)$ and this proves (i).

(ii) Suppose $\nu > r_{bb}(T^k)$. Fix a bounded set $B \subseteq X$. Choose positive scalar $\alpha_{\nu,bb}$ with $\alpha_{\nu,bb} > \max\{\frac{1}{\nu^{\frac{q}{k}}}; 0 \leq q < k\}$. Take a zero neighborhood W_2 such that $\alpha_{\nu,bb} W_2 \subseteq W$. Since for every $0 \leq q < k$, $T^q(B)$ is bounded, $E = \cup_{i=0}^{k} T^i(B)$ is bounded in X. There is $n_1 \in \mathbb{N}$ with $\frac{T^{kn}}{\nu^n}(E) \subseteq W_2$ for all $n > n_1$. Thus

$$\frac{T^m}{\nu^{\frac{m}{k}}}(B) = \frac{T^{pk+q}}{\nu^{p+\frac{q}{k}}}(B) \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(T^q(B))}{\nu^p} \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(E)}{\nu^p} \subseteq \alpha_{\nu,bb} W_2 \subseteq W,$$

for sufficiently large m, p. Therefore, $\nu^{\frac{1}{k}} > r_{bb}(T)$ and so $r_{bb}(T)^k \leq r_{bb}(T^k)$.

(iii) Suppose $\nu > r_c(T^k)$. Choose positive scalar $\alpha_{\nu,c}$ with $\alpha_{\nu,c} > \max\{\frac{1}{\nu^{\frac{q}{k}}}; 0 \leq q < k\}$. Take a zero neighborhood W_3 such that $\alpha_{\nu,c} W_3 \subseteq W$. There exists a zero neighborhood U_1 such that for a given $\varepsilon > 0$ there is $n_2 \in \mathbb{N}$ with $\frac{T^{kn}}{\nu^n}(U_1) \subseteq \varepsilon W_3$ for each $n > n_2$. Choose a zero neighborhood U_2 such that $T^q(U_2) \subseteq U_1$ for all $0 \leq q < k$. Therefore

$$\frac{T^m}{\nu^{\frac{m}{k}}}(U_2) = \frac{T^{pk+q}}{\nu^{p+\frac{q}{k}}}(U_2) \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(T^q(U_2))}{\nu^p} \subseteq \frac{1}{\nu^{\frac{q}{k}}} \frac{T^{pk}(U_1)}{\nu^p} \subseteq \alpha_{\nu,c} \varepsilon W_3 \subseteq \varepsilon W,$$

for sufficiently large m, p. $\square$

Corollary 2.4. *Suppose T is a continuous operator on a topological vector space X. Then for each $k \in \mathbb{N}$,*

(i) $r_{nb}(T^k) = r_{nb}(T)^k$;

(ii) $r_{bb}(T^k) = r_{bb}(T)^k$;

(iii) $r_c(T^k) = r_c(T)^k$.

In [9], it has been proved that for two commuting continuous linear operators T and S on a locally convex space X, $r_c(T + S) \leq r_c(T) + r_c(S)$ (see Theorem 4.9). On the other hand, by Proposition 6.7 in [9], for an *nb*-bounded linear operator on a topological vector space, all of the spectral radii will be equal. So, we have the following.

Proposition 2.5. *Suppose T and S are two commuting nb-bounded linear operators on a locally convex space. Then $r_{nb}(T + S) \leq r_{nb}(T) + r_{nb}(S)$.*

The proof of the following proposition follows the same line as in [9, Theorem 4.9]. We give the details for the sake of convenience.

Proposition 2.6. *Suppose T and S are two commuting bb-bounded linear operators on a locally convex space. Then $r_{bb}(T + S) \leq r_{bb}(T) + r_{bb}(S)$.*

Proof. Without loss of generality, we may assume that $r_{bb}(T)$ and $r_{bb}(S)$ are finite. Suppose that $\eta > r_{bb}(T) + r_{bb}(S)$ and take $\mu > r_{bb}(T)$ and $\nu > r_{bb}(S)$ such that $\eta > \mu + \nu$. Fix a bounded set $B \subseteq X$. Since the sequence $(\frac{S^n}{\mu^n})$ converges to zero uniformly on B, it is uniformly bounded. Thus for a fixed seminorm p, we can find $n_0 \in \mathbb{N}$ with $p(T^n S^m(B)) < \mu^n \nu^m$ for all $n, m > n_0$. Split η into a product of two terms $\eta = \eta_1 \eta_2$ with $\eta_1 > 1$ while still $\eta_2 > \mu + \nu$. If $n > 2n_0$, we have

$$p(\frac{1}{\eta^n}(T + S)^n(B)) \leq \frac{1}{\eta^n} \sum_{k=0}^{n_0} C(n,k) p(T^k S^{n-k}(B)) + \frac{1}{\eta^n} \sum_{k=n_0+1}^{n-n_0} C(n,k) p(T^k S^{n-k}(B))$$

$$+ \sum_{k=n-n_0+1}^{n} C(n,k) p(T^k S^{n-k}(B)).$$

Since $C(n,k) = \frac{(n-k+1)...(n-1).n}{1.2...(k-1).k} \le n^k$ and $\sum_{k=0}^{n} C(n,k)\mu^k\nu^{n-k} = (\mu+\nu)^n$, we have

$$p(\frac{1}{\eta^n}(T+S)^n(B)) \le \frac{n_0^n}{\eta^n}\sum_{k=0}^{n_0} p(T^kS^{n-k}(B)) + \frac{1}{\eta^n}\sum_{k=n_0+1}^{n-n_0} C(n,k)\mu^k\nu^{n-k} + \frac{n_0^n}{\eta^n}\sum_{k=n-n_0+1}^{n} p(T^kS^{n-k}(B))$$

$$\le \frac{n_0^n}{{\eta_1}^n}.\frac{1}{{\eta_2}^n}\sum_{k=0}^{n_0}(p(S^{n-k}T^k(B)) + p(T^{n-k}S^k(B))) + \frac{(\mu+\nu)^n}{\eta^n}.$$

Notice that $\lim \frac{(\mu+\nu)^n}{\eta^n} = 0$ and $\lim \frac{n_0^n}{\eta_1^n} = 0$. Since T is bb-bounded, $(T^k(B))$ is bounded for each fixed k, so that $\lim \frac{1}{\eta_2^{n-k}} S^{n-k}(T^k(B)) = 0$. It follows that the expression $\frac{1}{\eta_2^n} p(S^{n-k}(T^k(B)))$ is uniformly bounded for sufficiently large n. Similarly, for every k between 0 and n_0, the expression $\frac{1}{\eta_2^n} p(T^{n-k}(S^k(B)))$ is uniformly bounded for sufficiently large $n \in \mathbb{N}$. So, there is $n_1 \in \mathbb{N}$ with

$$\frac{1}{\eta_2^n}\sum_{k=0}^{n_0}(p(S^{n-k}T^k(B)) + p(T^{n-k}S^k(B)))$$

is uniformly bounded for all $n > n_1$. This shows that $\lim p(\frac{1}{\eta^n}(T+S)^n(B)) = 0$, so that $\eta > r_{bb}(T+S)$. □

Theorem 2.7. *Suppose T and S are two continuous linear operators on a topological vector space X. Then,*

(i) $r_{nb}(TS) = r_{nb}(ST)$;

(ii) $r_{bb}(TS) = r_{bb}(ST)$;

(iii) $r_c(TS) = r_c(ST)$.

Proof. (*i*) Let $W \subseteq X$ be an arbitrary zero neighborhood and $\nu > r_{nb}(ST)$. There is a zero neighborhood $U_0 \subseteq X$ such that $\frac{(ST)^n}{\nu^n}$ converges to zero uniformly on U_0. Choose a zero neighborhood U_1 such that $S(U_1) \subseteq U_0$. Find a zero neighborhood $U_2 \subseteq X$ with $\frac{T}{\nu}(U_2) \subseteq W$. Choose $n_0 \in \mathbb{N}$ such that $\frac{(ST)^{n-1}}{\nu^{n-1}}(U_0) \subseteq U_2$ for all $n > n_0$. Thus,

$$\frac{(TS)^n}{\nu^n}(U_1) = \frac{T(ST)^{n-1}S}{\nu\nu^{n-1}}(U_1) \subseteq \frac{T(ST)^{n-1}}{\nu\nu^{n-1}}(U_0) \subseteq \frac{T}{\nu}(U_2) \subseteq W.$$

It follows that $r_{nb}(TS) \le r_{nb}(ST)$. By a similar argument, we get $r_{nb}(ST) \le r_{nb}(TS)$ and this proves (*i*).

(*ii*) Fix a bounded set $B \subseteq X$. Suppose $\nu > r_{bb}(ST)$. There is a zero neighborhood U_3 such that $\frac{T}{\nu}(U_3) \subseteq W$. Since $S(B)$ is bounded, there is $n_1 \in \mathbb{N}$ with $\frac{(ST)^{n-1}}{\nu^{n-1}}(S(B)) \subseteq U_3$ for all $n > n_1$. Therefore,

$$\frac{(TS)^n}{\nu^n}(B) = \frac{T(ST)^{n-1}S}{\nu\nu^{n-1}}(B) \subseteq \frac{T}{\nu}(U_3) \subseteq W.$$

So, $r_{bb}(TS) \le r_{bb}(ST)$. Similarly, $r_{bb}(ST) \le r_{bb}(TS)$ and this proves (*ii*).

(*iii*) Assume $\nu > r_c(ST)$ and $\varepsilon > 0$ is given. Find a zero neighborhood U_4 with $\frac{T}{\nu}(U_4) \subseteq \varepsilon W$. There are a zero neighborhood U_5 and an n_2 such that $\frac{(ST)^{n-1}}{\nu^{n-1}}(U_5) \subseteq U_4$ for all $n > n_2$. Choose a zero neighborhood U_6 with $S(U_6) \subseteq U_5$ and so,

$$\frac{(TS)^n}{\nu^n}(U_6) = \frac{T(ST)^{n-1}S}{\nu\nu^{n-1}}(U_6) \subseteq \frac{T(ST)^{n-1}}{\nu\nu^{n-1}}(U_5) \subseteq \frac{T}{\nu}(U_4) \subseteq \varepsilon W.$$

This shows that $r_c(TS) \le r_c(ST)$. A similar argument shows that $r_c(ST) \le r_c(TS)$. □

3. Completeness

First, we consider some lemmas which are proved by Troitsky in [9].

Lemma 3.1. *Suppose that a sequence (T_n) of bb-bounded operators converges uniformly on bounded sets to a linear operator T. Then T is also bb-bounded.*

Lemma 3.2. *Suppose that a sequence (T_n) of continuous operators converges equicontinuously to a linear operator T. Then T is also continuous.*

It is easy to see that the conclusions of Lemma 3.1 and Lemma 3.2 are valid if we consider nets instead of sequences. Throughout this section, X is assumed to be a locally convex topological vector space.

Theorem 3.3. *If $B_n(X)$ is complete, then so is X.*

Proof. Let (x_α) be a Cauchy net in X. Choose $f \in X^*$ with $f \neq 0$. There exists some zero neighborhood U such that $|f(U)| < \frac{1}{2}$. Define $T_\alpha : X \to X$ by letting $T_\alpha(x) = f(x)x_\alpha$. It is not difficult to see that each T_α is *nb*-bounded. Also, (T_α) is a Cauchy net in $B_n(X)$. For, if W is an arbitrary zero neighborhood in X, then there is an α_0 such that $(x_\alpha - x_\beta) \in W$ for every $\alpha \geq \alpha_0$ and for every $\beta \geq \alpha_0$. For any $x \in U$, we have $(T_\alpha - T_\beta)(x) = f(x)(x_\alpha - x_\beta) \in W$, so that $(T_\alpha - T_\beta)(U) \subseteq W$. So, there are an *nb*-bounded operator T and a zero neighborhood $U_1 \subseteq X$ such that $(T_\alpha - T)(U_1) \subseteq W$ for sufficiently large α. Choose $e \in X$ with $f(e) = 1$. There exists $\gamma_e > 0$ such that $e \in \gamma_e U_1$, so that $(T_\alpha - T)(\frac{e}{\gamma_e}) \in W$. This means that $x_\alpha \to T(e)$. □

Note that the converse of Theorem 3.3 is not true, in general. See Example 2.22 in [9].

Theorem 3.4. *$B_b(X)$ is complete if and only if so is X.*

Proof. Suppose $B_b(X)$ is complete and (x_α) is a Cauchy net in X. There is $f \in X^*$ such that $f \neq 0$. Define the net (T_α) on X by setting $T_\alpha(x) = f(x)x_\alpha$. Fix a bounded set $B \subseteq X$. Since $f(B)$ is bounded in the scalar field, there exists $M > 0$ with $|f(B)| \leq M$. Let W be an arbitrary zero neighborhood in X. It is easy to see that each T_α is *bb*-bounded. Also, there is an α_0 such that $x_\alpha - x_\beta \in \frac{1}{M}W$ for all $\alpha \geq \alpha_0$ and for all $\beta \geq \alpha_0$. For each $x \in B$, $(T_\alpha - T_\beta)(x) = f(x)(x_\alpha - x_\beta) \in W$, so that $(T_\alpha - T_\beta)(B) \subseteq W$. This shows that (T_α) is a Cauchy net in $B_b(X)$ and so it converges. So, there is a *bb*-bounded operator T such that $(T_\alpha - T)$ converges to zero uniformly on bounded sets. Choose $e \in X$ with $f(e) = 1$. Thus, $\lim x_\alpha = \lim T_\alpha(e) = T(e)$, so that (x_α) converges.

For the converse assume that X is complete and (T_α) is a Cauchy net in $B_b(X)$. Since every singleton is bounded, for any $x \in X$, $(T_\alpha(x))$ is Cauchy net in X and therefore it converges. Put $T(x) = \lim T_\alpha(x)$. On the other hand, there exists an α_1 such that for each $\alpha \geq \alpha_1$ and for each $\beta \geq \alpha_1$, we have $(T_\alpha - T_\beta)(B) \subseteq W$. Therefore for each $x \in B$, $(T_\alpha - T_\beta)(x) \in W$ and it follows that $(T_\alpha - T)(x) \in W$. Thus, $(T_\alpha - T)(B) \subseteq W$. Now, by Lemma 3.1, T is also a *bb*-bounded operator. □

Theorem 3.5. *$B_c(X)$ is complete if and only if so is X.*

Proof. Suppose $B_c(X)$ is complete and (x_α) is a Cauchy net in X. There exists $f \in X^*$ with $f \neq 0$. Define $T_\alpha : X \to X$ by $T_\alpha(x) = f(x)x_\alpha$. Let $W \subseteq X$ be an arbitrary zero neighborhood and $\varepsilon > 0$ be given. There is a zero neighborhood $U \subseteq X$ such that $|f(U)| < \frac{1}{2}$. For each α, there is $\gamma_\alpha > 0$ with $x_\alpha \in \gamma_\alpha W$, so that $T_\alpha(U) \subseteq \gamma_\alpha W$ and hence each T_α is continuous. Also, (T_α) is a Cauchy net in $B_c(X)$. For, there is an α_0 such that $(x_\alpha - x_\beta) \in \varepsilon W$ for each $\alpha \geq \alpha_0$ and for each $\beta \geq \alpha_0$. For every $x \in U$, $(T_\alpha - T_\beta)(x) = f(x)(x_\alpha - x_\beta) \in \varepsilon W$, so that $(T_\alpha - T_\beta)(U) \subseteq \varepsilon W$. This implies that there are a continuous linear operator T and a zero neighborhood $U_1 \subseteq X$ such that $(T_\alpha - T)(U_1) \subseteq \varepsilon W$ for sufficiently large α. Choose $e \in X$ with $f(e) = 1$. Then, there is $\gamma_e > 0$ with $e \in \gamma_e U_1$. Corresponding to $\varepsilon = \frac{1}{\gamma_e}$ in the above argument, we get $(T_\alpha - T)(e) \in W$, so that $x_\alpha \to T(e)$ and it follows that X is complete.

For the converse, assume X is complete and (T_α) is a Cauchy net in $B_c(X)$. There are a zero neighborhood $U_2 \subseteq X$ and an α_1 with $(T_\alpha - T_\beta)(U_2) \subseteq \varepsilon W$ for every $\alpha \geq \alpha_1$ and for every $\beta \geq \alpha_1$. Fix $x \in X$. There is a positive scalar γ_x such that $x \in \gamma_x U_2$. Thus, for $\varepsilon = \frac{1}{\gamma_x}$, we have $(T_\alpha - T_\beta)(x) \in W$ and so $(T_\alpha(x))$ is a Cauchy net in X, so that it converges. This guarantees the existence of a linear operator T with $T(x) = \lim T_\alpha(x)$. Since this convergence is in $B_c(X)$, by Lemma 3.2, T is also continuous and this shows that $B_c(X)$ is complete. □

4. An application of the results

We know that when X is a Banach space, a bounded operator T on X is invertible with inverse $\sum_{n=0}^{\infty} T^n$ if $r(I-T) < 1$, where I denotes the identity operator on X. In the following, we prove a similar result for different types of bounded linear operators on a complete locally convex topological vector space. In what follows, X is assumed to be a complete locally convex topological vector space and as usual, I denotes the identity operator on X.

Theorem 4.1.

(i) *Suppose T is a bb-bounded operator with $r_{bb}(T) < 1$. Then $(I-T)$ is invertible in $B_b(X)$ with inverse $\sum_{n=0}^{\infty} T^n$;*

(ii) *Suppose T is a continuous operator with $r_c(T) < 1$. Then $(I-T)$ is invertible in $B_c(X)$ with inverse $\sum_{n=0}^{\infty} T^n$.*

Proof. (i) Let W be an arbitrary zero neighborhood. There is positive scalar ν such that $r_{bb}(T) < \nu < 1$. Without loss of generality, we may assume that $\nu = \frac{1}{\alpha}$ for some positive scalar $\alpha > 1$. Fix a bounded set $B \subseteq X$. There is $n_0 \in \mathbb{N}$ with $\frac{T^n}{\nu^n}(B) \subseteq \frac{\alpha-1}{\alpha} W$ for each $n > n_0$. So, $T^n(B) \subseteq \nu^n \frac{\alpha-1}{\alpha} W = \frac{(\alpha-1)}{\alpha^{n+1}} W$. Put $S_n = \sum_{k=0}^{n} T^k$. Thus, for all $m > n > n_0$ and for any $x \in B$,

$$(S_m - S_n)(x) = \sum_{k=0}^{m} T^k(x) - \sum_{k=0}^{n} T^k(x) = \sum_{k=n+1}^{m} T^k(x) \in \sum_{k=n+1}^{m} \frac{(\alpha-1)}{\alpha^{k+1}} W.$$

Therefore, $(S_m - S_n)(B) \subseteq \sum_{k=n+1}^{m} \frac{(\alpha-1)}{\alpha^{k+1}} W$. Since W is convex

$$\sum_{k=n+1}^{m} \frac{\alpha-1}{\alpha^{k+1}} W \subseteq (\sum_{k=n+1}^{m} \frac{\alpha-1}{\alpha^{k+1}}) W \subseteq (\sum_{k=0}^{\infty} \frac{\alpha-1}{\alpha^{k+1}}) W \subseteq W.$$

It follows that (S_n) is a Cauchy sequence in $B_b(X)$. By Theorem 2.4, $B_b(X)$ is complete and so (S_n) converges to some $S \in B_b(X)$. This means that the series $\sum_{n=0}^{\infty} T^n$ exists in $B_b(X)$ with sum S. Now, for each $x \in B$, $(S_n(I-T)-I)(x) = ((\sum_{k=0}^{n} T^k)(I-T)-I)(x) = T^{n+1}(x) \in \nu^{n+1} W \subseteq W$, for sufficiently large $n \in \mathbb{N}$. Therefore, $(S_n(I-T)-I)(B) \subseteq W$. By a similar argument, $((I-T)S_n - I)(B) \subseteq W$. It follows that $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$.
(ii) There is positive scalar ν such that $r_c(T) < \nu < 1$. Without loss of generality, we may assume that $\nu = \frac{1}{\alpha}$ for some positive scalar $\alpha > 1$. There is a zero neighborhood $U \subseteq X$ such that for a given $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ with $\frac{T^n}{\nu^n}(U) \subseteq \frac{\alpha-1}{\alpha} \varepsilon W$ for each $n > n_1$. So, $T^n(U) \subseteq \nu^n \frac{\alpha-1}{\alpha} \varepsilon W = \frac{(\alpha-1)}{\alpha^{n+1}} \varepsilon W$. Put $S_n = \sum_{k=0}^{n} T^k$. So, for all $m > n > n_1$ and for any $x \in U$,

$$(S_m - S_n)(x) = \sum_{k=0}^{m} T^k(x) - \sum_{k=0}^{n} T^k(x) = \sum_{k=n+1}^{m} T^k(x) \in \sum_{k=n+1}^{m} \frac{(\alpha-1)}{\alpha^{k+1}} \varepsilon W.$$

Thus, $(S_m - S_n)(U) \subseteq \sum_{k=n+1}^{m} \frac{(\alpha-1)}{\alpha^{k+1}} \varepsilon W$. Since W is convex

$$\sum_{k=n+1}^{m} \frac{\alpha-1}{\alpha^{k+1}} \varepsilon W \subseteq (\sum_{k=n+1}^{m} \frac{\alpha-1}{\alpha^{k+1}}) \varepsilon W \subseteq (\sum_{k=0}^{\infty} \frac{\alpha-1}{\alpha^{k+1}}) \varepsilon W \subseteq \varepsilon W.$$

It follows that (S_n) is a Cauchy sequence in $B_c(X)$. By Theorem 2.5, $B_c(X)$ is complete and so (S_n) converges to some $S \in B_c(X)$. This means that the series $\sum_{n=0}^{\infty} T^n$ exists in $B_c(X)$ with sum S. Now, for any $x \in U$, $(S_n(I-T)-I)(x) = ((\sum_{k=0}^{n} T^k)(I-T)-I)(x) = T^{n+1}(x) \in \nu^{n+1} \varepsilon W \subseteq \varepsilon W$, for sufficiently large $n \in \mathbb{N}$. Thus, $(S_n(I-T)-I)(U) \subseteq \varepsilon W$. Similarly, $((I-T)S_n - I)(U) \subseteq \varepsilon W$. This shows that $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$. □

We recall that when X is not locally bounded, $B_n(X)$ is not a unital algebra. So, for an nb-bounded linear operator, to examine a similar result, we can use of the concept of quasi-invertibility. Recall that for an algebra A, an element $a \in A$ is said to be quasi-invertible if there exists an element $b \in A$ such that the quasi-products $x \circ y = x + y - xy$ and $y \circ x = y + x - yx$ are equal to zero. The quasi-inverse of a quasi-invertible element x is denoted by x°. For more details about quasi-invertible elements see [6, Chapter 2, Section 2.1]. Also, if for a topological vector space X, $B_n(X)$ is complete, by a similar argument as in Theorem 4.1, we have the following.

Theorem 4.2. *Suppose T is an nb-bounded operator with $r_{nb}(T) < 1$, then the series $\sum_{n=1}^{\infty} T^n$ converges in $B_n(X)$. Also, T is quasi-invertible in $B_n(X)$ and we have $T^{\circ} = -\sum_{n=1}^{\infty} T^n$.*

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