Filomat 26:6 (2012), 1291–1295
DOI 10.2298/FIL1206291D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Note on selection principles of Kočinac

(Dedicated to Professor Ljubiša Kočinac on the occasion of his 65th birthday)

Dragan Djurčić[a], Mališa R. Žižović[b], Aleksandar Petojević[c]

[a]*University of Kragujevac, Technical Faculty in Čačak, Svetog Save 65, 32000 Čačak, Serbia*

[b]*Singidunum University Belgrade, Faculty of Business, Danijelova 32, 11010 Belgrade, Serbia*

[c]*University of Novi Sad, Faculty of Education, Podgorička 4, 25000 Sombor, Serbia*

Abstract. This paper investigates α_i^d, $i \in \{2,3,4\}$, selection principles (which are modification of known selection principles of Kočinac) on a double sequence of double sequences of real numbers which converge to a point $a \in \mathbb{R}$ in Pringsheim's sense. A stronger result than one given in [6] will be proved for the α_2^d selection principle. Also, two more propositions will be proved for the S_1^d and S_1^{φ} selection principles, which are also improvements of results given in [6].

1. Introduction

Well-known Kočinac's selection principles α_i, $i \in \{1,2,3,4\}$, are subject of the research in many papers that appeared in the literature in recent years (see, e.g., [2–8, 12]). These selection principles are based on the famous Arhangel'skiĭ's properties introduced and studied in 1972 (see, e.g., [1] and [7]). Nowadays these selection principles constitute an important subtheory of the theory of diagonalization processes and are directly related to the theory of infinite topological games, optimization theory, Karamata theory and Ramsey theory (see, e.g., [3–6]).

On the other hand, in the last few years the study of the Pringsheim convergence of double sequences [10] and various its applications has been done by a number of mathematicians. In particular, in [6] the selection principle α_2^d (as a modification of Kočinac's α_2 selection principle) was introduced and investigated for a double sequence of double sequences of real numbers which converge to a point $a \in \mathbb{R}$ in Pringsheim's sense. In this paper we continue this investigation and present several results which refine and extend the results in [6].

Let a double sequence of real numbers $\mathbf{x} = (x_{m,n})_{m,n \in \mathbb{N}}$ and a point $a \in \mathbb{R}$ be given. Then:

2010 *Mathematics Subject Classification*. Primary 41A65
Keywords. Selection principles, double sequences, convergence in Pringsheim's sense
Received: 27 May 2012; Revised: 29 September 2012; Accepted: 13 October 2012
Communicated by Vladimir Rakočević
This research is supported by Ministry of Education, Science and Technological Development of the Republic of Serbia
Email addresses: dragandj@tfc.kg.ac.rs (Dragan Djurčić), zizovic@gmail.com (Mališa R. Žižović), apetoje@ptt.rs (Aleksandar Petojević)

1° $\mathbf{x} \in c_2^{a,1}$ (that is, $\mathbf{x}$ converges to the point a in Pringsheim's sense) if for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $n \geq N_0$ and all $m \geq N$ (see, e.g., [9, 10]), that is, if

$$\lim_{\min\{m,n\}\to+\infty} x_{m,n} = a.$$

2° $\mathbf{x} \in c_2^{a,2}$ if for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $n \geq N_0$ or for all $m \geq N_0$, that is, if

$$\lim_{\max\{m,n\}\to+\infty} x_{m,n} = a.$$

3° $\mathbf{x} \in c_2^{a,3}$ if for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $m \in \mathbb{N}$ and for all $n \in \mathbb{N}$ for which $m + n \geq N_0$, that is, if

$$\lim_{m+n\to+\infty} x_{m,n} = a.$$

Then

$$c_2^{a,3} \subsetneq c_2^{a,2} \subsetneq c_2^{a,1}$$

and all three classes are important and interesting subjects in the theory of double sequences.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a non-empty set X. Then we define the following selection principles for X:

1* X satisfies $\alpha_2^d(\mathcal{A},\mathcal{B})$ if for every double sequence $(A_{j,k})$ of elements from $\mathcal{A}$ there exists an element B from $\mathcal{B}$ such that $B \cap A_{j,k}$ is infinite for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}$.

2* X satisfies $S_1^d(\mathcal{A},\mathcal{B})$ if for every double sequence $(A_{j,k})$ of elements from $\mathcal{A}$ there exists an element $B = (b_{j,k})$ from $\mathcal{B}$ such that $b_{j,k} \in A_{j,k}$ for all $j \in \mathbb{N}$ and for all $k \in \mathbb{N}$ (see, e.g., [6] and [11]).

3* X satisfies $S_1^\varphi(\mathcal{A},\mathcal{B})$ if $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is a bijection and for every sequence (A_t) of elements in $\mathcal{A}$ there exists an element $B = (b_{j,k})$ from $\mathcal{B}$ such that $b_{j,k} \in A_t$ for $t = \varphi(j,k)$ (see [6]).

Remark 1.1. 1. Analogously to the definition of selection principle $\alpha_2^d(\mathcal{A},\mathcal{B})$ corresponding to the selection principle $\alpha_2(\mathcal{A},\mathcal{B})$ in [7], it is possible to define selection principles $\alpha_i^d(\mathcal{A},\mathcal{B})$, $i \in \{1,3,4\}$ corresponding to the selection principles $\alpha_i(\mathcal{A},\mathcal{B})$, $i \in \{1,3,4\}$ (see, e.g., [7, 8]). Observe that we have

$$\alpha_1^d(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_2^d(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_3^d(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4^d(\mathcal{A},\mathcal{B}).$$

2. Selection principles $S_1^d(\mathcal{A},\mathcal{B})$ and $S_1^\varphi(\mathcal{A},\mathcal{B})$ are based on famous Rothberger's $S_1(\mathcal{A},\mathcal{B})$ selection principle (see, e.g., [11]) and they are two-dimensional versions of the Rothberger-type selection principle S_1.

In [6], among others, the following proposition was proved.

Proposition 1.2. ([6]) *Let $a \in \mathbb{R}$ be a given point. Then the following selection principles hold:*

(a) $\alpha_2^d(c_2^{a,1}, c_2^{a,1})$;

(b) $S_1^d(c_2^{a,1}, c_2^{a,1})$;

(c) $S_1^\varphi(c_2^{a,1}, c_2^{a,1})$.

In the next section we generalize and improve these results.

2. Main results

In the following propositions results from Proposition 1.2 are improved by replacing the second coordinate $c_2^{a,1}$ with its important and proper subclass $c_2^{a,3}$.

Proposition 2.1. *Let $a \in \mathbb{R}$ be a given point. Then the selection principle $\alpha_2^d(c_2^{a,1}, c_2^{a,3})$ holds.*

Proof. Let a double sequence of double sequences $(x_{m,n,j,k})$ be such that for every $j_0 \in \mathbb{N}$ and every $k_0 \in \mathbb{N}$, $(x_{m,n,j_0,k_0}) \in c_2^{a,1}$. Create the double sequence $\mathbf{y} = (y_{j,k})$ in the following way:

If $j \in \mathbb{N}$, $k \in \mathbb{N}$ and $j \neq k$, let $y_{i,k} = a$, and if $j = k$, then use the following procedure:

Firstly, take a sequence (p_r) of prime numbers in the ascending order. Then, sort the initial double sequence of double sequences into a sequence of double sequences $(x_{m,n,r})$ in any standard way. For $r \in \mathbb{N}$ and $r \geq 2$ let us define $y_{j,j} = x_{p_r^s,p_r^s,r}$ if $j = p_r^s$ for some $s \in \mathbb{N}$. If $|x_{p_r^s,p_r^s,r} - a| > \frac{1}{2^r}$, then already defined $y_{j,j}$ replace with $y_{j,j} = a$. Let $j_r = \min\{j \in \mathbb{N} \mid y_{j,j} = x_{p_r^s,p_r^s,r}\}$. If $j_{r+1} < j_r$, then already given $y_{j,j}$, $j \in \{j_{r+1}, j_{r+1}+1, \ldots, j_r - 1\}$, redefine so that $y_{j,j} = a$. If p_r does not exist and $s \in \mathbb{N}$ with $j = p_r^s$ does not exist, then take $y_{j,j} = a$. According to the construction of $\mathbf{y}$ we have: (1) $\mathbf{y} = (y_{j,k}) \in c_2^{a,3}$, and (2) $\mathbf{y}$ has infinitely many common elements at the same positions with every double sequence (x_{m,n,j_0,k_0}). This completes the proof. □

Proposition 2.2. *Let $a \in \mathbb{R}$ be a given point. Then the selection principle $S_1^d(c_2^{a,1}, c_2^{a,3})$ holds.*

Proof. Let $(x_{m,n,j,k})$ be a double sequence of double sequences in $c_2^{a,1}$. Let $\mathbf{y} = (y_{j,k})$ be the double sequence defined in the following way:

Let $(j,k) \in \mathbb{N} \times \mathbb{N}$. Then an $x_{m^*,m^*,j,k}$ can be chosen from the double sequence $(x_{m,n,j,k})$ so that

$$|x_{m^*,m^*,j,k} - a| \leq \left(\frac{1}{2}\right)^{j+k-1}.$$

For $y_{j,k}$ take $x_{m^*,m^*,j,k}$ and in that way create the double sequence $\mathbf{y} = (y_{j,k})$. Since for every $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that $(\frac{1}{2})^p \leq \varepsilon$ for every $p \in \mathbb{N}$, $p \geq p_0$, it follows that for every j and every k from $\mathbb{N}$, such that $j+k-1 \geq p_0$, it holds $|y_{j,k} - a| \leq \varepsilon$. So, $\mathbf{y} \in c_2^{a,3}$ and for all $(j,k) \in \mathbb{N} \times \mathbb{N}$ the element $y_{j,k}$ belongs to the double sequence $(x_{m,n,j,k})$. □

Proposition 2.3. *Let $a \in \mathbb{R}$ be given. Then the selection principle $S_1^{\varphi}(c_2^{a,1}, c_2^{a,3})$ holds.*

Proof. Let $(x_{m,n,t})$ be a sequence of double sequences such that $(x_{m,n,t_0}) \in c_2^{a,1}$ holds for all $t_0 \in \mathbb{N}$. Let $\mathbf{y} = (y_{j,k})$ be the double sequence formed in the following way:

Let $(j,k) \in \mathbb{N} \times \mathbb{N}$. Let $y_{j,k} = x_{m^*,m^*,\varphi(j,k)}$, where $m^* \in \mathbb{N}$ is such that $|x_{m^*,m^*,\varphi(j,k)} - a| \leq (\frac{1}{2})^{j+k-1}$. According to the construction of the double sequence $\mathbf{y} = (y_{j,k})$, it follows that $y_{j,k}$ belongs to the double sequence $(x_{m,n,\varphi(j,k)})$ for every (fixed) $(j,k) \in \mathbb{N} \times \mathbb{N}$. Also, for every $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that $(\frac{1}{2})^p \leq \varepsilon$ for all $p \in \mathbb{N}$, $p \geq p_0$. So, $|y_{j,k} - a| \leq \varepsilon$ holds for all $(j,k) \in \mathbb{N} \times \mathbb{N}$ such that $j + k - 1 \geq p_0$, and it follows that $\mathbf{y} \in c_2^{a,3}$. □

Remark 2.4. 1. For the bijection $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (from the previous considerations) it is said that it generates an *arrangement on* $\mathbb{N} \times \mathbb{N}$ in the following way:

For (m_1, n_1) and (m_2, n_2) from $\mathbb{N} \times \mathbb{N}$, it holds that $(m_1, n_1) \leq_{\varphi} (m_2, n_2)$ if $\varphi(m_1, n_1) \leq \varphi(m_2, n_2)$.

2. In Propositions 2.1, 2.2 and 2.3 the class $c_2^{a,1}$ at the first coordinate can be replaced with the class of real double sequences which have the Pringsheim limit point $\lim_{k\to+\infty} x_{m(k),n(k)} = a$ (see [6]), so that those propositions remain true.
3. Parts of the proof of Proposition 2.1 present a more precise and complete proof of the analogous proposition from [6] (where the second coordinate is the class $c_2^{a,1}$).
4. For Propositions 2.1, 2.2 and 2.3, game-theoretic and Ramsey-theoretic results, analogous to the results in [3] for the S_1 selection principle, can be obtained.
5. Obviously, from the previous it follows that selection principles $\alpha_2^d(c_2^{a,1}, c_2^{a,2})$, $S_1^d(c_2^{a,1}, c_2^{a,2})$, $S_1^{\varphi}(c_2^{a,1}, c_2^{a,2})$ and $\alpha_i(c_2^{a,1}, c_2^{a,j})$ hold for all $i \in \{2, 3, 4\}$ and for all $j \in \{2, 3\}$.
6. Let $a = 0$ and let $c_{2,+}^{0,1}$ be the set of all double sequences from $c_2^{0,1}$ with positive elements. Then for $\mathbf{y} = (y_{j,k})$, which is constructed in the proofs of Propositions 2.1, 2.2 and 2.3, it holds $\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} y_{j,k} \leq 1$.

Let $\mathbf{x} = (x_{m,n})$ be a double sequence of real numbers. Then:

4° $\mathbf{x} \in c_2^{\infty,1}$ (i.e., $\mathbf{x}$ converges to $+\infty$ in Pringsheim's sense) if for every $M > 0$ there exists $N_0 = N_0(M) \in \mathbb{N}$ such that $x_{m,n} \geq M$ for all $n \geq N_0$ and for all $m \geq N_0$ (see [6]). That is, if $\lim_{\min\{m,n\}\to+\infty} x_{m,n} = +\infty$.

5° $\mathbf{x} \in c_2^{\infty,2}$ if for every $M > 0$ there exists $N_0 = N_0(M) \in \mathbb{N}$ such that $x_{m,n} \geq M$ for all $n \geq N_0$ or for all $m \geq N_0$. That is, if $\lim_{\max\{m,n\}\to+\infty} x_{m,n} = +\infty$.

6° $\mathbf{x} \in c_2^{\infty,3}$ if for every $M > 0$ there exists $N_0 = N_0(M) \in \mathbb{N}$ such that $x_{m,n} \geq M$ for all $m \in \mathbb{N}$ and for all $n \in \mathbb{N}$ for which $m + n \geq N_0$. That is, if $\lim_{m+n\to+\infty} x_{m,n} = +\infty$.

Then

$$c_2^{\infty,3} \subsetneq c_2^{\infty,2} \subsetneq c_2^{\infty,1}.$$

Analogously as in Propositions 2.1, 2.2 and 2.3 it can be proved that selection principles $\alpha_2^d(c_2^{\infty,1}, c_2^{\infty,3})$, $S_1^d(c_2^{\infty,1}, c_2^{\infty,3})$ and $S^{\varphi}(c_2^{\infty,1}, c_2^{\infty,3})$ are true, and remarks analogous to Remark 2.4 (2., 4. and 5.) hold. Notice that to show that $\alpha_2^d(c_2^{\infty,1}, c_2^{\infty,3})$ holds, one should take $y_{i,j} = i + j$ in the proof (for $i \in \mathbb{N}$, $j \in \mathbb{N}$, $i \neq j$).

Let $\mathbf{x} = (x_{m,n})$ be a double sequence of positive real numbers. Then a double sequence $\mathbf{y} = (y_{m,n})$ of positive real numbers is said to be 1-*strongly asymptotic equivalent* (or 1-asymptotic equal) to $\mathbf{x}$, denoted $\mathbf{x} \overset{1}{\sim} \mathbf{y}$ or $\mathbf{y} \in [\mathbf{x}]_1$, if for every $\varepsilon > 1$ there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{\varepsilon} \leq \frac{x_{m,n}}{y_{m,n}} \leq \varepsilon$$

for all $n \geq N_0$ and all $m \geq N_0$.

Relations $\overset{2}{\sim}$ and $\overset{3}{\sim}$ are introduced analogously and they are (as the relation $\overset{1}{\sim}$) equivalence relations on the set of all double sequences of positive real numbers. For a real positive double sequence $\mathbf{x}$ it holds

$$[\mathbf{x}]_3 \subsetneq [\mathbf{x}]_2 \subsetneq [\mathbf{x}]_1.$$

Also, for $c_2^{a,j}$, where $a > 0$ and $j \in \{1, 2, 3\}$, it holds $c_2^{a,j} = [\mathbf{x}]_j$ whenever $\mathbf{x} \in c_2^{a,j}$.

From results in [6], Proposition 2.1 and Remark 2.4(5.) it follows that for $a > 0$ and for all $\mathbf{x} \in c_2^{a,j}$, $j \in \{1, 2, 3\}$, selection principles $\alpha_i^d([\mathbf{x}]_j, [\mathbf{x}]_j)$ and $\alpha_i([\mathbf{x}]_j, [\mathbf{x}]_j)$ hold for $i \in \{2, 3, 4\}$.

Proposition 2.5. *Let* $\mathbf{x} = (x_{m,n})$ *be a double sequence of positive real numbers. Then the selection principle* $\alpha_2^d([\mathbf{x}]_1, [\mathbf{x}]_3)$ *holds.*

Proof. Let $(z_{m,n,j,k})$ be a double sequence of double sequences of positive real numbers, such that for every $j_0 \in \mathbb{N}$ and every $k_0 \in \mathbb{N}$ it holds $(z_{m,n,j_0,k_0}) \in [\mathbf{x}]_1$. First pick an increasing sequence (p_r) of prime numbers so that $p_1 = 2$, and then arrange $(z_{m,n,j,k})$ into a sequence of double sequences $(z_{m,n,r})$ in any standard way. Define the double sequence $\mathbf{y} = (y_{j,k})$ in the following way:

First, take $\mathbf{y}_0 = \mathbf{x}$. Then for $r \in \mathbb{N}$ create the double sequence $\mathbf{y}_r$ by replacing elements at positions (p_r^s, p_r^s), $s \in \mathbb{N}$, in the double sequence $\mathbf{y}_{r-1}$ by elements $z_{p_r^s, p_r^s, r}$, $s \in \mathbb{N}$, if

$$\frac{2^r - 1}{2^r} \leq \frac{z_{p_r^s, p_r^s, r}}{x_{p_r^s, p_r^s}} \leq \frac{2^r}{2^r - 1}.$$

Let the double sequence $\mathbf{y}$ be obtained by the above procedure as $r \to +\infty$. Then, according to the construction, $\mathbf{y}$ has infinitely many common elements at the same positions with double sequences $(z_{m,n,r})$, i.e. $\mathbf{y} \in [\mathbf{x}]_3$. $\square$

Remark 2.6.
1. From the previous consideration it follows that the selection principles $\alpha_i^d([\mathbf{x}]_j, [\mathbf{x}]_3)$ hold for $i \in \{2, 3, 4\}$ and $j \in \{1, 2, 3\}$ and selection principles $\alpha_i([\mathbf{x}]_j, [\mathbf{x}]_3)$, for a given double sequence $\mathbf{x} = (x_{m,n})$ of positive real numbers.
2. From Proposition 2.5 it follows that for a given $\mathbf{x} \in c_2^{\infty,1}$ the selection principle $\alpha_2^d([\mathbf{x}]_1, [\mathbf{x}]_3)$ holds.

References

[1] A.V. Arhangel'skiĭ, The frequency spectrum of a topological space and the classification of spaces, Soviet Math. Doklady 13 (1972) 1185–1189.

[2] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, On selection principles and games in divergent processes, in: Lj.D.R. Kočinac (ed.), Selection Principles and Covering Properties in Topology, Quaderni di Matematica, Caserta, Vol. 18 (2006) 133–155.

[3] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Some properties of rapidly varying sequences, J. Math. Anal. Appl. 327 (2007) 1297–1306.

[4] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, A few remarks on divergent sequences: Rates of divergence, J. Math. Anal. Appl. 360 (2009) 588–598.

[5] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Summability of sequences and selection properties, Abst. Appl. Analysis, Vol. 2011, Article ID 213816, 8 pages, doi:10.1155/2011/213816.

[6] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Double sequences and selections, Abst. Appl. Analysis, Vol. 2012, Article ID 497594, 6 pages, doi: 10.1155/2012/497594.

[7] Lj.D.R. Kočinac, Selection principles related to α_i-properties, Taiwanese J. Math. 12 (2008) 561–571.

[8] Lj.D.R. Kočinac, On the α_i-selection principles and games, Contemp. Math. 533 (2011) 107–124.

[9] R.F. Patterson, E. Savaş, A category theorem for double sequences, Appl. Math. Letters 124 (2011) 1680–1684.

[10] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (1900) 289–321.

[11] F. Rothberger, Eine Vershärfung der Eigenschaft C, Fund. Math. 30 (1938) 50–55.

[12] B. Tsaban, On the Kočinac α_i properties, Topology Appl. 155 (2007) 141–145.