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# Screenableness in countable products

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**Abstract.** In this note, it is shown that the product $\prod_{n\in\omega} X_n$ is screenable if $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces. And a group of equivalent conditions of screenableness of products is obtained.

## 1. Introduction

The notion of $C$-scattered spaces was introduced and investigated by Telgarsky [14]. Furthermore, utilizing it to products, he proved that if $X$ is a paracompact $C$-scattered space, then the product $X \times Y$ is paracompact for each paracompact space $Y$. In these connections, $C$-scattered spaces have been widely used in study of topological spaces characterized by coverings to their countable products. As an excellent result, Friedler et al. [5], Hohti and Pelant [7] showed that if $\{X_n : n \in \omega\}$ is a countable collection of $C$-scattered paracompact spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact.

As a generalization of $C$-scattered spaces, Čech-scattered spaces introduced by Hohti and Ziqiu [8] play the same fundamental role in the study of paracompactness of countable products. In 2005, Aoki and Tanaka [1] extended the Hohti and Ziqiu's results by proving that if $Y$ is a perfect paracompact space, and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact.

Bing [3] defined a space to be *screenable* if every open cover has a $\sigma$-disjoint refinement. And Greever [6] showed that this property played an important role in study of the equivalency between countably paracompact spaces and paracompact spaces. However, Balogh [2] constructed a normal, screenable, nonparacompact space in ZFC. And Peiyong [10] demonstrated that there is a first countable regular screenable space $X$ such that $X^n$ is screenable for each $n \in \omega$, but $X^\omega$ is not screenable. In view of the above, it is natural to pose the following two questions:

**Question 1.1.** *Is the product $\prod_{n\in\omega} X_n$ screenable if $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces?*

**Question 1.2.** *If a product space $S=\prod_{n\in\omega} X_n$ is countably paracompact and for each $n \in \omega$, space $X_n$ is screenable, is $S$ screenable?*

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The aim of this note is to give an affirmative answer to question 1.1. In addition, a great advance has been got on with the results of inverse limit and Tychonoff products of topological spaces characterized by coverings, see [11,15]. Particularly, the paper [15] discussed the preserved property of the inverse limits of normal screenable spaces. Dropping the condition of normality, we prove that the screenable spaces is invariable under inverse limit operations. Using it, a group of equivalent conditions of screenableness of countable products is obtained. Moreover, the answer to Question 1.2 is positive.

Throughout this paper, each space is assumed to be a Tychonoff space in Section 3. Each space has at least two points without any separation axiom in Section 4, and all maps are continuous. Let $\omega$ be the set of natural numbers.

## 2. Preliminaries

In the rest of this section, we state some notation and basic facts. Undefined terminology can be found in Engelking [4].

Recall that a space $X$ is *scattered* if every nonempty closed subset $A$ has an isolated point $a$. And a space $X$ is said to be *C-scattered* (*Čech-scattered*) if every nonempty closed subset $A$ of $X$, there exists a point $a \in A$ which has a *compact* (*Čech-complete*) neighborhood in $A$. Evidently, all of the scattered spaces, locally compact spaces and $C$-scattered spaces are Čech-scattered.

Let $X$ be a space. For a subset $S$ of $X$, $|S|$ ($\overline{S}$) denotes its cardinality (closure). Assume that $S$ is closed. Put

$S^*=\{x \in S : x$ has no Čech-complete neighborhood in $S\}$.

Let $S^0=S$, $S^{(\alpha+1)}=(S^{(\alpha)})^*$, and $S^{(\alpha)}=\cap_{\beta<\alpha}S^{(\beta)}$ for a limit ordinal $\alpha$. Note that each $S^{(\alpha)}$ is closed in $X$. Furthermore, a space $X$ is Čech-scattered if and only if $X^{(\alpha)}=\emptyset$ for some ordinal $\alpha$. Obviously, a Čhech-scattered space is hereditary for its closed (open) subspace. A closed subset $S$ of $X$ is called *topped* if $S \cap S^{(\alpha)}$ is nonempty Čech-complete and $S \cap S^{(\alpha+1)}=\emptyset$ for some ordinal $\alpha$. For each $x \in X$, there is a unique ordinal $\alpha$ such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. Let $rank(x)=\alpha$. Then, there is an open neighborhood base $\mathcal{V}$ of $x$ in $X$ such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped in $X$ and $\alpha(\overline{V})=rank(x)$. A collection $\mathcal{V}$ of subsets of $X$ is a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ and $\cup\mathcal{V}=\cup\mathcal{U}$.

To complete our proof, the following lemmas will be needed.

**Lemma 2.1.** ([1]) *The product $X \times Y$ is Čhech-scattered if $X$ and $Y$ are Čech-scattered spaces.*

**Lemma 2.2.** ([4]) *A Tychonoff space $X$ is Čech-complete if and only if there exists a countable family $\{\mathcal{A}_i\}_{i\in\omega}$ of open covers of the space $X$ with the property that any family $\mathcal{F}$ of closed subsets of $X$, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for $i \in \omega$, has nonempty intersection.*

Note that the intersection $\cap\mathcal{F}$ is countable compact in Lemma 2.2. Therefore, if $X$ is screenable, then $\cap\mathcal{F}$ is compact.

**Lemma 2.3.** ([9]) *A space $X$ is $\lambda$-paracompact if and only if for every directed open cover $\mathcal{U}$ of $X$ with cardinality $\leq \lambda$, there is a locally finite open cover $\mathcal{V}$ of $X$ such that $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$. A space $X$ is countably paracompact if and only if $\lambda=\omega$.*

## 3. Countable products of screenable spaces

The following lemmas play important roles in the study of our main result. They will be stated briefly.

**Lemma 3.1.** *Let $X$ be a Čech-scattered screenable space, and $\alpha=inf\{\beta : \beta$ is an ordinal number and $X^{(\beta)}=\emptyset\}$. Then there exists a $\sigma$-disjoint open cover $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ of $X$ such that for each $V \in \mathcal{V}$,*

*(a) if $\alpha$ is a successor ordinal, then $\overline{V}^{(\alpha)}$ is Čech-complete,*

*(b) if $\alpha$ is a limit ordinal, then $\overline{V}^{(\beta)}=\emptyset$ for some $\beta < \alpha$.*

**Lemma 3.2.** *Let $X$ be a locally Čech-complete screenable space, and $\alpha$=inf{$\beta$ : $\beta$ is an ordinal number and $X^{(\beta)}=\emptyset$}. For every open cover $\mathcal{U}$ of $X$, there exists a $\sigma$-disjoint open cover $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ of $X$ such that for each $V \in \mathcal{V}$,*

*(a) $\overline{V} \subset U$ for some $U \in \mathcal{U}$,*

*(b) $\overline{V}^{(\beta)}$ is Čech-complete for some $\beta < \alpha$.*

**Lemma 3.3.** *If $X$ is a Čech-scattered screenable space, then for every open cover $\mathcal{U}$ of $X$, there exists a $\sigma$-disjoint open cover $\mathcal{V}=\cup_{n\in\omega}\mathcal{V}_n$ of $X$ such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped and is contained in some element of $\mathcal{U}$.*

*Proof.* This proof is a modification of [Lemma 3.3, 12]. Then, by Lemma 3.1 and Lemma 3.2, it is true. □

Now, we state our main theorem.

**Theorem 3.4.** *Let $X$ be a Čech-scattered space with $Top(X)=\{a\}$. Then the product $X^\omega$ is screenable if $X$ is screenable.*

*Proof.* Let $\mathcal{G}$ be an arbitrary open covering of $X^\omega$. We can assume that $\mathcal{G}$ is closed under finite unions. We are going to find a $\sigma$-disjoint open refinement of $\mathcal{G}$.

Let $\mathcal{B}$ be a base of $X^\omega$, consisting of all sets of the form $D=\prod_{i\in\omega} D_i$ and for each $i \in \omega$, $\overline{D_i}$ is topped, i.e, $Top(\overline{D_i})$ is Čech-complete. Then, there is a sequence $\{\mathcal{W}_{i,m}(D) : m \in \omega\}$ of open covers of $Top(\overline{D_i})$, such that if $\mathcal{F}$ is a collection of nonempty closed subset of $Top(\overline{D_i})$ with the finite intersection property such that for each $m \in \omega$, there are $F_m \in \mathcal{F}$ and $W_m \in \mathcal{W}_{i,m}(D)$ with $F_m \subset W_m$, then the intersection $\cap\mathcal{F}$ is nonempty. Let $n(D)=\inf\{i : D_j=X, \text{ for } j \geq i\}$. And then, define $C$ as follows:

(*) $(D, \mathcal{W}_{i,m}(D)) \in C$, $m \in \omega$, if $D=\prod_{i\in\omega} D_i \in \mathcal{B}$ and $\mathcal{W}_{i,m}(D)$ is an open cover of $Top(\overline{D_i})$, satisfying the conditions described above.

Let $(D, \mathcal{W}_{i,m}(D)) \in C$ for each $m \in \omega$. In case of that $i < n(D)$, let $m$=1. Then for each $W \in \mathcal{W}_{i,1}(D)$, there is an open subset $W'$ of $\overline{D_i}$ such that $W=W' \cap Top(\overline{D_i})$. Moreover, $\{W' : W \in \mathcal{W}_{i,m}(D)\} \cup \{\overline{D_i} - Top(\overline{D_i})\}$ covers $\overline{D_i}$ and hence, it follows from Lemma 3.3 that there is an open covering $\mathcal{H}_i(D)=\cup_{j<\omega}\mathcal{H}_{i,j}(D)$ of $D_i$ such that

(i) for each $j \in \omega$, $\mathcal{H}_{i,j}(D)$ is disjoint,

(ii) for each $A \in \mathcal{H}_{i,j}(D)$, $j \in \omega$, $\overline{A}$ is topped and contained in some member of $\{W' : W \in \mathcal{W}_{i,m}(D)\} \cup \{\overline{D_i} - Top(\overline{D_i})\}$.

In case of that $i=n(D)$, we can also take a $\sigma$-disjoint open covering $\mathcal{H}_{n(D)}(D)$ of $D_i$ such that for each $A \in \mathcal{R}_i(D)$, $\overline{A}$ is topped. And there is a proper member $A_0 \in \mathcal{H}_{n(D)}(D)$ with $a \in A_0$ and for each $A^* \in \mathcal{H}_{n(D)}(D) - \{A_0\}$, $a \notin A^*$.

Let $\mathcal{R}(D)_j=\prod_{i\leq n(D)} \mathcal{H}_{i.j}(D)$. Then $\mathcal{R}(D)=\cup_{j\in\omega}\mathcal{R}_j(D)$ is a $\sigma$-disjoint open covering of $\prod_{i\leq n(D)} D_i$ and $\mathcal{R}(D) \subset \mathcal{B}$. Fix an $R=\prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$ with $Top(\overline{R}) \cap Top(\prod_{i\leq n(D)} \overline{D_i}) \neq \emptyset$. Then, $Top(\overline{R_i}) \cap Top(\overline{D_i}) \neq \emptyset$ for each $i \leq n(R)$. And then, we deduce that

$$Top(\overline{R_i}) \cap Top(\overline{D_i})=\overline{R_i} \cap Top(\overline{D_i}) = Top(\overline{R_i}).$$

Hence, by (ii), $Top(\overline{R_i}) \subset W$ for some $W \in \mathcal{W}_{i,1}(D)$. Put $P(R)=R \times X \times \cdots =\prod_{i\in\omega} P(R)_i$. Then $P(R) \in \mathcal{B}$ and $Top(\overline{P(R)})=Top(\overline{R}) \times \{a\} \times \cdots$. Namely, $Top(\overline{P(R)})$ is Čech-complete. Next, we define $R$ satisfying (**) as follows:

(**) if there are some basic open subsets $E_1$, $E_2$ and $E_3$ in $X^\omega$ and some $G \in \mathcal{G}$ such that $Top(\overline{P(R)}) \subset E_1 \subset \overline{E_1} \subset E_2 \subset \overline{E_2} \subset E_3 \subset \overline{E_3} \subset G$.

Assume that $R$ satisfies the condition (**). Let $k(R)=\inf\{n(E_1) : E_1, E_2$ and $E_3$ are some basic open subsets in $X^\omega$ with $n(E_1)=n(E_2)=n(E_3)$ such that $Top(\overline{P(R)}) \subset E_1 \subset \overline{E_1} \subset E_2 \subset \overline{E_2} \subset E_2 \subset E_3 \subset \overline{E_3} \subset G$ for some $G \in \mathcal{G}\}$. Then, we can take some basic open subsets $E_1(R)=\prod_{i\in\omega} E_1(R)_i$, $E_2(R)=\prod_{i\in\omega} E_2(R)_i$ and $E_3(R)=\prod_{i\in\omega} E_3(R)_i$ in $X^\omega$ and some $G(R) \in \mathcal{G}$ such that

(1) (a) $Top(\overline{P(R)}) \subset E_1(R) \subset \overline{E_1(R)} \subset E_2(R) \subset \overline{E_2(R)} \subset E_3(R) \subset \overline{E_3(R)} \subset G(R)$.

(b) $k(R)=n(E_1(R))$.

Let $r(R)=\max\{n(D)+1, k(R)\}$. Define $Z(R)=\prod_{i<r(R)}(P(R)_i \cap E_3(R)_i) \times X \times \cdots =\prod_{i\in\omega} Z(R)_i$. By the definition of $Z(R)$, we assume that

(2) (a) for $i \in \omega$ with $k(R) \leq i < r(R)$, let $Z(R)_i=P(R)_i$,

(b) for $i \in \omega$ with $i < k(R)$ and $i < n(D)$, let $Z(R)_i = P(R)_i \cap E_3(R)_i$,

(c) for $i \in \omega$ with $n(D) \leq i < k(R)$, let $Z(R)_i = \{a\}$,

(d) in case of that $r(R) = n(D) + 1$, let $Z(R)_i = X$ for each $i \geq n(D) + 1$; in case of that $r(R) = k(R) > n(D) + 1$, let $Z(H)_i = X$ for $i \geq k(R)$.

Then, $Z(R)$ is a basic open subset of $X^\omega$ such that $Top(\overline{P(R)}) \subset Z(R)$, and contained in some member of $\mathcal{G}$ and for each $i \in \omega$, $\overline{Z(R)_i}$ is topped.

Put $\mathcal{A}(R) = \mathcal{P}(\{0, 1, \cdots, r(R) - 1\})$. Fix an $A \in \mathcal{A}(R)$. We define $D_A(R) = \prod_{i\in\omega} D_{A,i}(R)$ as follows:

(3) (a) if $i \in A$ with $i < n(D)$, let $D_{A,i}(R) = P(R)_i - \overline{P(R)_i \cap E_2(R)_i}$,

(b) if $i \in A$ with $r(R) = k(R) > i \geq n(D)$, let $D_{A,i}(R) = X - \{a\}$,

(c) if $i < r(R)$ with $i \notin A$, let $D_{A,i}(R) = P(R)_i \cap E_1(R)_i$,

(d) for each $i$ with $i \geq r(R)$, let $D_{A,i}(R) = X$.

Clearly, for each $A, B \in \mathcal{A}(R)$ with $A \neq B$, $D_A(R) \cap D_B(R) = \emptyset$. And if $i$ satisfies (3) (c) or (d), then $\overline{D_{A,i}(R)}$ is topped. And if $i \in A$ with $k(R) \leq i < r(R)$, then $D_{A,i}(R) = \emptyset$. Now, we consider the other cases:

(i) if $i \in A$ with $i < \max\{n(D) + 1, k(R)\}$;

(ii) if $i \in A$ with $r(R) = k(R) > i \geq n(D)$;

(iii) if $i = r(R)$.

If $i$ satisfies the conditions (i) or (ii), then $\overline{D_{A,i}(R)}$ does not need to be topped and hence, there is an open covering $\mathcal{B}(\overline{D_{A,i}(R)})$ of $\overline{D_{A,i}(R)}$ such that for each $B \in \mathcal{B}(\overline{D_{A,i}(R)})$, $\overline{B}$ is topped. Then, there is a $\sigma$-disjoint, open refinement $\mathcal{D}_{A,i}(R) = \cup_{n\in\omega}\mathcal{D}_{A,i,n}(R)$ of $\mathcal{B}(\overline{D_{A,i}(R)})$, covering $D_{A,i}(R)$ and for each $D^*_i \in \mathcal{D}_{A,i}(R)$, $\overline{D^*_i}$ is topped. If $i$ satisfies (iii), there is a proper $\sigma$-disjoint, open collection $\mathcal{D}_{A,r(R)}(R) = \cup_{n\in\omega}\mathcal{D}_{A,r(R),n}(R)$, covering $X$ and for each $D^*_i \in \mathcal{D}_{A,r(R)}(R)$, $\overline{D^*_i}$ is topped. Next, we define $\mathcal{D}^*_{A,n}(R)$, $n \in \omega$, as follows:

(4) $D^* = \prod_{i\in\omega} D^*_i \in \mathcal{D}^*_{A,n}(R)$ if for each $i \in \omega$,

(a) if $i \in A$ with $k(R) \leq i < n(D)$, let $D^*_i = \emptyset$,

(b) if $i$ satisfies one of the conditions (i), (ii) and (iii), let $D^*_i \in \mathcal{D}_{A,i,n}(R)$,

(c) if $i \notin A$ with $i < r(R)$, let $D^*_i = D_{A,i,n}(R)$,

(d) let $D^*_i = X$ for each $i > r(R)$.

Put $\mathcal{D}_{A,n}(R) = \{D^* \in \mathcal{D}^*_{A,n}(R) : D^* \neq \emptyset\}$. Then, we infer that

(5) $\mathcal{D}_{A,n}(R)$, $n \in \omega$, is disjoint in $X^\omega$.

Indeed, let $D^1 = \prod_{i\in\omega} D^1_i$, $D^2 = \prod_{i\in\omega} D^2_i \in \mathcal{D}_{A,n}(R)$ with $D^1 \neq D^2 (\neq \emptyset)$. Then $D^1_i \neq D^2_i$ for some $i \in \omega$. By 4 (b), $D^1_i \cap D^2_i$ is disjoint. Hence the proof of (5) is true.

Moreover, let $\mathcal{D}_n(R) = \cup\{\mathcal{D}_{A,n}(R) : A \in \mathcal{A}(R)\}$. Hence, by (5) and the definition of $Z(R)$, $\mathcal{D}(R) = \cup_{n\in\omega}\mathcal{D}_n(R)$ satisfies the following:

(6) $\mathcal{D}(R)$ is $\sigma$-disjoint in $X^\omega$ and $R = Z(R) \cup (\cup\mathcal{D}(R))$.

Fix an $A \in \mathcal{A}(R)$. Take a $D^* = \prod_{i\in\omega} D^*_i \in \mathcal{D}_{A,n}(R)$ for each $n \in \omega$. Clearly, the length of $D^*$ is $r(R)$. Then $n(D^*) > n(D)$ and for each $i \in \omega$, $\alpha(\overline{D^*_i}) \leq \alpha(\overline{D_i})$.

Let $i \leq n(D)$. If $\alpha(\overline{D^*_i}) = \alpha(\overline{D_i})$, then $Top(\overline{D^*_i}) \subset Top(\overline{R_i})$ since $Top(\overline{R_i}) \cap Top(\overline{D_i}) \neq \emptyset$. And let $\mathcal{W}_{i,m}(D^*) = \{W \cap \overline{D^*} : W \in \mathcal{W}_{i,m+1}(D)\}$, $m \in \omega$. Then $(D^*, \mathcal{W}_{i,m}(D^*)) \in C$.

Since $A \neq \emptyset$, there is an $i \in A$ such that if $i < k(R)$, $Top(\overline{D_i}) \cap \overline{D^*_i} = \emptyset$. In other words, $\alpha(\overline{D^*_i}) < \alpha(\overline{D_i})$. And if $i \geq k(R)$, $D^*_i = \emptyset$. Hence, there is an $i < k(R)$ such that $\alpha(\overline{D^*_i}) < \alpha(\overline{D_i})$ if $k(R) < n(D)$.

For each $j \in \omega$, we put

$\mathcal{Z}_j(D) = \{Z(R) : R \in \mathcal{R}_j(D)\}$,

$\mathcal{D}_j(D) = \cup\{\mathcal{D}(R) : R \in \mathcal{R}_j(D)\}$.

When $R$ does not satisfy (**) or $Top(\overline{R}) \cap Top(\prod_{i\leq n(D)} \overline{D_i}) = \emptyset$, let $\mathcal{Z}_j(D) = \{\emptyset\}$, $\mathcal{D}_j(D) = \{D^*\}$ for each $j \in \omega$, where $D^* = R \times X \times \cdots$. We can also take a proper sequence $\{\mathcal{W}_{i,m}(D^*) : m \in \omega\}$ such that $(D^*, \mathcal{W}_{i,m}(D^*)) \in C$, $m \in \omega$, as before.

The following statements are straightforward from the above.

(7) (a) $\mathcal{Z}(D) = \cup_{j\in\omega}\mathcal{Z}_j(D)$ is a $\sigma$-disjoint collection of basic open subsets of $X^\omega$ such that every member of $\mathcal{Z}(D)$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{D}(D) = \cup_{j\in\omega}\mathcal{D}_j(D)$ is a $\sigma$-disjoint collection of basic open subsets of $X^\omega$,

(c) $D=\cup\mathcal{Z}(D)\cup(\cup\mathcal{D}(D))$,

for each $D^*=\prod_{i\in\omega}D_i^*\in\mathcal{D}(R)$, $R=\prod_{i\leq n(D)}R_i\in\mathcal{R}(D)$,

(d) $n(D^*)>n(D)$ and for each $i\in\omega$, $\alpha(\overline{D_i^*})\leq\alpha(\overline{D_i})$,

(e) $(D^*,\mathcal{W}_{i,m}(D^*))\in C$ such that for each $i\leq n(D)$, if $\alpha(\overline{D_i^*})=\alpha(\overline{D_i})$, then $Top(\overline{D_i^*})\subset Top(\overline{R_i})$ and for each $m\in\omega$, $\mathcal{W}_{i,m}(D^*)=\{W\cap\overline{D^*}:W\in\mathcal{W}_{i,m+1}(D)\}$,

(f) if $R$ satisfies (∗∗) with $k(R)<n(D)$, then there is an $i<k(R)$ such that $\alpha(\overline{D_i^*})<\alpha(\overline{D_i})$.

Now, proceeding by induction on $n\in\omega$, we define two families $\mathcal{Z}_n$ and $\mathcal{D}_n$ as follows. Let $\mathcal{Z}_0=\{\emptyset\}$, $\mathcal{D}_0=\{D(0)\}$, where $D(0)=X^\omega$. Put $\mathcal{W}_{i,m}=\{\{a\}\}$ for each $i,m\in\omega$. Now assume that $n=m$. Both of the families $\mathcal{Z}_n$ and $\mathcal{D}_n$ of basic open subsets of $X^\omega$, satisfy the following:

(8) (a) $\mathcal{Z}_n=\cup\{\mathcal{Z}(D):D\in\mathcal{D}_{n-1}\}$ is a $\sigma$-disjoint collection of basic open subsets of $X^\omega$ such that every member of $\mathcal{Z}_n$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{D}_n=\cup\{\mathcal{D}(D):D\in\mathcal{D}_{n-1}\}$ is a $\sigma$-disjoint collection of basic open subsets of $X^\omega$,

for each $D=\prod_{i\in\omega}D_i\in\mathcal{D}_{n-1}$, $D^*=\prod_{i\in\omega}D_i^*\in\mathcal{D}(R)$, $R=\prod_{i\leq n(D)}R_i\in\mathcal{R}(D)$,

(c) $(D,\mathcal{W}_{i,m}(D))\in C$

(d) $D=\cup\mathcal{Z}(D)\cup(\cup\mathcal{D}(D))$,

(e) $n(D^*)>n(D)$,

(f) for each $i\in\omega$, $\alpha(\overline{D_i^*})\leq\alpha(\overline{D_i})$,

(g) $(D^*,\mathcal{W}_{i,m}(D^*))\in C$ such that for each $i\leq n(D)$, if $\alpha(\overline{D_i^*})=\alpha(\overline{D_i})$, then $Top(\overline{D_i^*})\subset Top(\overline{R_i})$ and for each $m\in\omega$, $\mathcal{W}_{i,m}(D^*)=\{W\cap\overline{D^*}:W\in\mathcal{W}_{i,m+1}(D)\}$,

(h) if $R$ satisfies (∗∗) with $k(R)<n(D)$, then there is an $i<k(R)$ such that $\alpha(\overline{D_i^*})<\alpha(\overline{D_i})$.

By the above constructions, we infer that the families $\mathcal{Z}_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the consequents of (8) (a) ∼ (h). Put $\mathcal{Z}=\cup_{n\in\omega}\mathcal{Z}_n$. Our proof will be complete if we show the following claim.

**Claim 3.5.** *$\mathcal{Z}$ is a $\sigma$-disjoint open refinement of $\mathcal{G}$.*

By (8) (a), (b) and the induction, $\mathcal{Z}$ is a $\sigma$-disjoint collection of open sets in $X^\omega$. It suffices to show that $\mathcal{Z}$ covers $X^\omega$. To show this, assume the contrary. Let $x=(x_k)\in X^\omega-\cup\mathcal{Z}$. By (7) and (8) repeatedly, there are $\{A(m):m\geq 1\}\subset\mathcal{A}(R(m))$, $\{R(m):m\geq 1\}\subset\mathcal{R}(D(m-1))$, $\{D(m):m\in\omega\}\subset\mathcal{D}(R(m))\subset\mathcal{B}$, where $D(m)$ and $R(m)$ are denoted by $\prod_{i\in\omega}D(m)_i$ and $\prod_{i\leq D(m-1)}R(m)_i$ respectively, satisfying that: for each $m\geq 1$,

(9) (a) $x=(x_k)\in D(m)$, $n(D(m))>n(D(m-1))$ and for each $i\in\omega$, $\alpha(\overline{D(m)_i})\leq\alpha(\overline{D(m-1)_i})$,

(b) for each $i\leq n(D(m))$, if $\alpha(\overline{D(m+1)_i})=\alpha(\overline{D(m)_i})$, then $Top(\overline{D(m+1)_i})\subset Top(\overline{R(m)_i})$ and for each $j\in\omega$, $\mathcal{W}_{i,j}(D(m+1))=\{W\cap\overline{D(m+1)_i}:W\in\mathcal{W}_{i,j+1}(D(m))\}$,

(c) if each $R(m)$ satisfies (∗∗) with $k(R(m))<n(D(m-1))$, then there is an $i<k(R(m))$ such that $\alpha(\overline{D(m)_i})<\alpha(\overline{D(m-1)_i})$.

Fix an $i\in\omega$. By (9) (a), $n(D(m))>n(D(m-1))$ for each $m\geq 1$. Then there is an $s_i\in\omega$ such that $i<n(D(s_i))$. Let $s_i^*=\inf\{m\in\omega:i<n(D(m))\}$. And then, $n(D(m))>i$ for each $m\geq s_i^*$. In addition, by (9) (a), $\alpha(\overline{D(m)_i})\leq\alpha(\overline{D(m-1)_i})$ for each $m\geq 1$. Then, there is a $t_i\in\omega$ such that $\alpha(\overline{D(t)_i})=\alpha(\overline{D(t_i)_i})$ for each $t\geq t_i$. Let $m_i^*=\max\{s_i^*,t_i\}$. Hence, $i<n(D(m_i^*))$ and $\alpha(\overline{D(m)_i})=\alpha(\overline{D(m_i^*)_i})$ for $m\geq m_i^*$. Moreover, by (9) (b), $Top(\overline{D(m+1)_i})\subset Top(\overline{R(m)_i})$ for $m\geq m_i^*$. Then there is a sequence $\{W(m):m\geq m_i^*\}$ of open subsets of $X$ such that for each $m\geq m_i^*$, $W(m)\in\mathcal{W}_{i,m-m_i^*+1}(D(m_i^*))$ and $Top(\overline{R(m)_i})\subset W_m$.

Let $K_i=\cap_{m\geq m_i^*}Top(\overline{D(m)_i})=\cap_{m\geq m_i^*}Top(\overline{R(m)_i})$. It follows from Lemma 2.2 that $K_i$ is nonempty and compact. And then, let $K=\prod_{i\in\omega}K_i$. Clearly, $K$ is compact. Hence, by Wallace theorem in Engelking [4], $K\subset G$ for some $G\in\mathcal{G}$. Define $p=\inf\{n(V):K\subset V\subset\overline{V}\subset G\}$, where $V=\prod_{i\in\omega}V_i$ is an open subset of $X^\omega$. Then, there exists an $m_0\in\omega$ such that $p<n(D(m_0))$. Let $m_1=\max\{m_i^*:i<p\}$. And let $m^*=\max\{m_0,m_1\}$. Then, we infer that $p<n(D(m^*))$ and for each $i<p$, $m_i^*\leq m^*$ and $Top(\overline{D(m^*)_i})\subset V_i$.

Then $R(m^*+1)\subset V$ and hence, $R(m^*+1)$ satisfies (∗∗). Then, by (9) (c), $k(R(m^*+1))\leq p<n(D(m^*))$. Thus, there is an $i<k(R(m^*+1))$ such that $\alpha(\overline{D(m^*+1)_i})<\alpha(\overline{D(m^*)_i})$, which is a contradiction.

It follows from the Claim 3.5 that $X^\omega$ is screenable. □

**Remark 3.6.** If $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered screenable spaces, we can assume that $X_n$=$X$ for each $n \in \omega$, and $X$ is topped with $Top(X)$=$\{a\}$ for some $a \in X$, see [1, 8]. Therefore, by Theorem 3.4, the product $\prod_{n\in\omega} X_n$ is screenable.

## 4. The equivalency of screenable spaces

For an inverse system $\{X_n, \pi^n_m, \omega\}$ and its limit $S$, let $\pi_n$ be the projection from $S$ into $X_n$ for each $n \in \omega$ (for the detailed definition of inverse limit, see [2.5, 4]).

**Lemma 4.1.** *Let $\{X_n, \pi^n_m, \omega\}$ be an inverse system and $S$ its inverse limit with each projection $\pi_n$ being an open and onto map. Suppose that $S$ is countably paracompact. If each space $X_n$ is screenable, then so is $S$.*

*Proof.* Let $\mathcal{G}$=$\{G_\xi : \xi \in \Xi\}$ be an open cover of $S$. For each $n \in \omega$ and $\xi \in \Xi$, let $V_{n,\xi}$=$\cup\{V : V$ is open in $X_n$ and $\pi_n^{-1}(V) \subset G_\xi\}$ and put $V_n$=$\cup\{V_{n,\xi} : \xi \in \Xi\}$. Then $\{\pi_n^{-1}(V_n) : n \in \omega\}$ is an open cover of $S$ with $\pi_n^{-1}(V_n) \subset \pi_{n+1}^{-1}(V_{n+1})$ for each $n \in \omega$. By the countably paracompactness of $S$, there is a monotone increasing collection $\{E_n : n \in \omega\}$ of open sets of $S$ such that $\overline{E_n} \subset \pi_n^{-1}(V_n)$ for each $n \in \omega$. Fix an $n \in \omega$. Define $Z_n$=$\cup\{V : V$ is open in $X_n$ and $\pi_n^{-1}(V) \subset E_n\}$. Then $\{\pi_n^{-1}(Z_n) : n \in \omega\}$ is an open cover of $S$ with $\pi_n^{-1}(Z_n) \subset E_n$. Again since $S$ is countably paracompact, there is a locally finite open covering $\{W_n : n \in \omega\}$ of $S$ such that $\overline{W_n} \subset \pi_n^{-1}(Z_n)$ for each $n \in \omega$. Take an $n \in \omega$. Clearly, $\overline{Z_n} \subset V_n$ and hence, there is an open refinement $\mathcal{O}_n$=$\cup_{i\in\omega}\mathcal{O}_{n,i}$ of $\{V_{n,\xi} : \xi \in \Xi\}$ such that for each $i \in \omega$, $\mathcal{O}_{n,i}$=$\{O_{n,i,\xi} : \xi \in \Xi\}$ is disjoint in $\overline{Z_n}$.

For each $n, i \in \omega$, put $\mathcal{H}_{n,i}$=$\{W_n \cap \pi_n^{-1}(O_{n,i,\xi}) : \xi \in \Xi\}$. Then, it is easy to check that $\mathcal{H}$=$\cup_{n\in\omega}\cup_{i\in\omega}\mathcal{H}_{n,i}$ is a $\sigma$-disjoint open refinement of $\mathcal{G}$. □

**Theorem 4.2.** *If a product space $S$=$\prod_{i\in\omega} X_i$ is countably paracompact, then the following are equivalent:*
*(1) $S$ is screenable;*
*(2) The product $\prod_{i\in\sigma} X_i$ is screenable for each $\sigma \in [\omega]^{<\omega}$;*
*(3) The product $\prod_{i<n} X_i$ is screenable for each $n \in \omega$.*

*Proof.* (1)⇒(2)⇒(3) hold trivially. Now, we infer that (3)⇒(1):

For each $n \in \omega$, let $\sigma_n$=$\{0, 1, \cdots, n\}$. Then, $\Sigma$=$\{\sigma_n : n \in \omega\}$ is directed by the relation ⊂. Define $X_{\sigma_n}$=$\prod_{i<n} X_i$ for each $n \in \omega$. For each $i, j \in \omega$ with $i \le j$, let $\pi^{\sigma_i}_{\sigma_j} : X_{\sigma_j} \to X_{\sigma_i}$ be the projection map. Then, each $\pi^{\sigma_i}_{\sigma_j}$ is open and onto. Denote $S'$=$\varprojlim\{X_{\sigma_i}, \pi^{\sigma_i}_{\sigma_j}\}$. Then, it is easy to check that $S'$ is homeomorphism with $S$. It follows from Lemma 4.1 that $S'$ is screenable, and hence so is $S$. □

By [Theorem 2, 9] and Theorem 4.2, the following result is obtained.

**Corollary 4.3.** *Let $\{X_n : n \in \omega\}$ be a countable collection of screenable spaces. Then the product $\prod_{n\in\omega} X_n$ is paracompact if and only if it is countably paracompact.*

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