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On strong double matrix summability via ideals

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Abstract. In this paper, we define some new double sequence spaces by combining the notion of ideal, Orlicz function and nonnegative four dimensional matrix. We make certain investigations on the classes of sequences arising out of this new summability method. In addition, we shall establish inclusion theorems between these spaces and other sequence spaces.

1. Introduction and background

Spaces of strongly summable sequences were studied by Kuttner [10], Maddox [11], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [12] as an extension of the definition of strongly Cesàro summable sequences. Connor [1] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. Also recently Savas and Patterson [19] extended a few results known in the literature for ordinary (single) sequences to multiply sequences of real and complex numbers. In [14] the notion of convergence for double sequences was presented by A. Pringsheim. Also, in [7] and [16] the four dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ was studied extensively by Hamilton and Robison. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise.

On the other hand, ideals were used in [8] to generalize the notion of statistical convergence ([5, 6, 17, 18]). More recent applications of ideals can be seen from ([2, 3, 20]) where more references can be found. In [21], the notion of strong A^I- summability with respect to an Orlicz function for single sequence was defined and studied.

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $I \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $A, B \in I$ implies $A \cup B \in I$; (ii) $A \in I, B \subset A$ implies $B \in I$, while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. If I is a proper ideal in Y (i.e,$Y \notin I, Y \neq \phi$) then the family of sets $F(I) = \{M \subset Y : \text{there exists } A \in I : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal I. Throughout I will stand for a proper non-trivial admissible ideal of $\mathbb{N}$ and e will denote a sequence all of whose elements are 1. Also let s'' denote the set of all double complex or real valued sequences and as usual,

$$l_\infty^2 = \left\{ x = (x_{k,l}) \in s'' : \|x\| = \sup_{k,l} \left| x_{k,l} \right| < \infty \right\}.$$

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In this paper we extend some fundamental theorems of summability theory results from ordinary (single) sequences spaces to multiply sequence spaces. This will be accomplished by presenting the following sequence spaces:

$$\left\{x \in s'' : \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} F(\left|x_{k,l}\right|) \geq \delta\right\} \in I \text{ for any } \delta > 0\right\}$$

and

$$\left\{x \in s'' : \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} F(\left|x_{k,l} - L\right|) \geq \delta\right\} \in I \text{ for any } \delta > 0 \text{ for some } L\right\},$$

where F is an Orlicz function, and A is a nonnegative four dimensional matrix. Other implications and variations will also be presented.

Recall [9] that an Orlicz function is a function $F : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing and convex with $F(0) = 0, F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If the convexity of an Orlicz function F is replaced by

$$F(x+y) \leq F(x) + F(y)$$

then it is called a modulus function (see, [12, 15]).

An Orlicz function F is said to satisfy the Δ_2-*condition* for all real values of u if there exists a constant $M > 0$ such that

$$F(2u) \leq MF(u) \quad (u \geq 0).$$

It can be readily observed that F satisfies the Δ_2-condition if and only if

$$F(tu) \leq MtF(u)$$

for all values of u and $t > 1$ [13].

Before continuing with this paper we recall some notations and basic definitions used in this paper. By the convergence in a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \epsilon$ whenever $k,l > N$ [14]. We shall describe such an x more briefly as " *P-convergent*".

Definition 1.1. Let $A = (a_{m,n,k,l})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn-th term to Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Such transformation is said to be non-negative if $a_{m,n,k,l}$ is nonnegative for all m, n, k, and l.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz in [22] and [23], respectively. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

Definition 1.2. The four dimensional matrix A is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

In addition to this definition, Robison and Hamilton also presented the following Silverman-Toeplitz type multidimensional characterization of regularity in [7] and [16]:

Theorem 1.3. *The four dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
RH_2: $P\text{-}lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: $P\text{-}lim_{m,n} \sum_{k=1}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each l;*
RH_4: $P\text{-}lim_{m,n} \sum_{l=1}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each k;*
RH_5: $\sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right|$ *is P-convergent; and*
RH_6: *there exist positive numbers A and B such that*
$\sum_{k,l>B} \left|a_{m,n,k,l}\right| < A$.

Definition 1.4. Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix . A sequence $x = (x_{k,l})$ is said to be A^I- double statistically convergent to L if for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l \in K_2(x-Le,\varepsilon)} a_{m,n,k,l} \geq \delta\right\} \in I$$

where $K_2(x - Le, \epsilon) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \left|x_{k,l} - L\right| \geq \epsilon\right\}$. In this case we write $x_{k,l} \xrightarrow{A^I-st} L$. We denote the class of all A^I- double statistically convergent sequences by $S_A^2(I)$.

2. Main results

We introduce the following definitions.

Definition 2.1. Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix and let I be an admissible ideal of $\mathbb{N}$. We define

$$\begin{aligned} W_0^I(A)_2 &= \left\{x \in s'' : \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| \geq \delta\right\} \in I \text{ for any } \delta > 0\right\}, \\ W^I(A)_2 &= \left\{x \in s'' : for \text{ some } L,\, x - Le \in W_0^I(A)_2\right\}. \end{aligned}$$

If $x \in W^I(A)_2$, we say that x is strongly A^I- double summable to L. We now introduce the following definitions by using ideals as well as Orlicz function.

Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix of real entries, F be an Orlicz function and let I be an admissible ideal of $\mathbb{N}$. We define

$$\begin{aligned} W_0^I(A,F)_2 &= \left\{x \in s'' : \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F(\left|x_{k,l}\right|) \geq \delta\right\} \in I \text{ for any } \delta > 0\right\}, \\ W^I(A,F)_2 &= \left\{x \in s'' : \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \geq \delta\right\} \in I, \text{ for some } L\right\}. \end{aligned}$$

If $x \in W^I(A,F)_2$, we say that x is strongly A^I- double summable to L with respect to an Orlicz function F. Let us consider a few special cases of the above sets:

(1) If $F(x) = x$ for all $x \in [0,\infty)$, then we have $W_0^I(A)_2$ and $W^I(A)_2$ respectively.

(2)If we take $A = (C, 1, 1)$,i.e., the double Cesàro matrix, then the above classes of sequences reduce to the following sequence spaces

$$W_0^I(F)_2 = \left\{ x \in s'' : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=0,0}^{n,m} a_{m,n,k,l} F(|x_k|) \geq \delta, \text{ for any } \delta > 0 \right\} \in I \right\}$$

and

$$W^I(F)_2 = \left\{ x \in s'' : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{nm} \sum_{k,l=0,0}^{n,m} a_{m,n,k,l} F(|x_k - L|) \geq \delta \right\} \in I, \text{ for some } L \right\}.$$

(3) Let us consider the following notations and definitions. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$

$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and let $\bar{h}_{r,s} = h_r h_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(i,j) : k_{r-1} < i \leq k_r \ \& \ l_{s-1} < j \leq l_s\}$. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}}, & \text{if } (k,l) \in I_{r,s}; \\ 0 & \text{otherwise.} \end{cases}$$

We write

$$W_0^I(\theta, F)_2 = \left\{ x \in s'' : \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{h}_{r,s}} \sum_{k,l \in I_{rs}} F(|x_{k,l}|) \geq \delta \right\} \in I, \text{ for any } \delta > 0 \right\}$$

and

$$W^I(\theta, F)_2 = \left\{ x \in s'' : \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{h}_{r,s}} \sum_{k,l \in I_{rs}} F(|x_{k,l} - L|) \geq \delta \right\} \in I \text{ for some } L \right\}.$$

(4) As a final illustration let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in L_j = [j - \lambda_j + 1, j] \\ 0, & \text{otherwise} \end{cases}$$

where we shall denote $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$ and $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$. Then our definition reduces to the following

$$W_0^I(\bar{\lambda}, F)_2 = \left\{ x \in s'' : \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} F(|x_k|) \geq \delta \right\} \in I, \text{ for any } \delta > 0 \right\}$$

$$W^I(\bar{\lambda}, F)_2 = \left\{ x \in s'' : \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} F(|x_k - L|) \geq \delta \right\} \in I \text{ for some } L \right\}.$$

It is easy to see that $W_0^I(A)_2 \subset W_0^I(A, F)_2$ and $W^I(A)_2 \subset W^I(A, F)_2$ for an Orlicz function F which satisfies the Δ_2-condition.

We now prove the following result.

Lemma 2.2. *If $A = (a_{m,n,k,l})$ is a non-negative RH-regular four dimensional matrix and F is an Orlicz function which satisfies the Δ_2-condition then*

$$W_0^I(A,F)_2 \cap l_\infty^2$$

is an ideal in l_∞^2.

Proof. Let $x = (x_{k,l}) \in W_0^I(A,F)_2$ and let $y \in l_\infty^2$. We show that $xy \in W_0^I(A,F)_2 \cap l_\infty$. Since $y \in l_\infty^2$ there exists a $M_0 > 1$ such that $\|y\| < M_0$. Then $|x_{k,l}y_{k,l}| < M_0 |x_{k,l}|$ for all $(k,l) \in \mathbb{N}\times\mathbb{N}$. Since F is non-decreasing and satisfies the Δ_2-condition, we have

$$F(|x_{k,l}y_{k,l}|) < F(M_0 |x_{k,l}|) \leq MM_0F\left(|x_{k,l}|\right),\ (M > 0).$$

Since $x \in W_0^I(A,F)_2$, so

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(|x_{k,l}|\right) \geq \delta\right\} \in I,\ \text{for any } \delta > 0.$$

Hence for $\delta > 0$,

$$\begin{aligned} &\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(|x_{k,l}y_{k,l}|\right) \geq \delta\right\} \\ \subset\ &\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}MM_0F\left(|x_{k,l}|\right) \geq \delta\right\} \\ =\ &\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(|x_{k,l}|\right) \geq \frac{\delta}{MM_0}\right\}. \end{aligned}$$

Since the set on the right hand side belongs to I so it follows that $xy \in W_0^I(A,F)_2 \cap l_\infty^2$. □

Lemma 2.3. *Let J be an ideal in l_∞^2 and let $x \in l_\infty^2$. Then x is in the closure of J in l_∞^2 if and only if $\chi_{K_2(x,\epsilon)} \in J$ for any $\epsilon > 0$, where χ_A is the characteristic function of A and $K_2(x,\epsilon) = \{(k,l) \in \mathbb{N}\times\mathbb{N} : |x_{k,l}| \geq \varepsilon\}$.*

The proof of Lemma 2.2 is straightforward. So we omit it.

Lemma 2.4. *If $A = (a_{m,n,k,l})$ is a non-negative RH-regular four dimensional matrix then $W_0^I(A)_2 \cap l_\infty^2$ is a closed ideal of l_∞^2.*

Proof. Let $x = (x_{k,l}), y = (y_{k,l})$ and $x, y \in W_0^I(A)_2 \cap l_\infty^2$. It is clear that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l} + y_{k,l}\right| \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| + \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|y_{k,l}\right|$$

and so for any $\delta > 0$,

$$\begin{aligned} &\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l} + y_{k,l}\right| \geq \delta\right\} \\ &\subset \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| \geq \frac{\delta}{2}\right\} \\ &\cup \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|y_{k,l}\right| \geq \frac{\delta}{2}\right\}. \end{aligned}$$

Since $x, y \in W_0^I(A)_2$, the sets on the right hand side belong to I and so is their union. Therefore

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}+y_{k,l}\right| \geq \delta\right\} \in I$$

which shows that $x + y \in W_0^I(A)_2 \cap l_\infty^2$.

Now let $x \in W_0^I(A)_2 \cap l_\infty^2$ and $y \in l_\infty^2$. Then there is $K > 0$ such that $\left|y_{k,l}\right| \leq K$ for all $(k,l) \in \mathbb{N}\times\mathbb{N}$. Now $\left|x_{k,l}y_{k,l}\right| \leq K\left|x_{k,l}\right|$ and we have

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}y_{k,l}\right| \geq \delta\right\} \subset \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| \geq \frac{\delta}{K}\right\}$$

for any $\delta > 0$. This easily implies that $xy \in W_0^I(A)_2 \cap l_\infty^2$.

Finally let $(x^{m,n}) \subset W_0^I(A)_2 \cap l_\infty^2$ and let $x^{m,n} \to x$ in l_∞^2. We have to show that $x \in W_0^I(A)_2 \cap l_\infty^2$. Let $\delta > 0$ be given. we choose $(p,q) \in \mathbb{N}\times\mathbb{N}$ such that $\|x^{p,q} - x\|_\infty < \frac{\delta}{2}$.
Now

$$\begin{aligned}\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}^{p,q} - x_{k,l}\right| + \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}^{p,q}\right| \\ &\leq \frac{\delta}{2} + \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}^{p,q}\right|\end{aligned}$$

as $A = (a_{m,n,k,l})$ is regular. Evidently then

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}\right| \geq \delta\right\} \subset \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left|x_{k,l}^{p,q}\right| \geq \frac{\delta}{2}\right\}$$

and it follows that $x \in W_0^I(A)_2 \cap l_\infty^2$. □

We now have

Theorem 2.5. *Let x be a double bounded sequence, F be an Orlicz function which satisfies* Δ_2-*condition and A be a non-negative regular matrix summability method. Then*

$$W^I(A,F)_2 \cap l_\infty^2 = W^I(A)_2 \cap l_\infty^2.$$

Proof. We will only show that $W_0^I(A,F)_2 \cap l_\infty^2 = W_0^I(A)_2 \cap l_\infty^2$. Clearly $W_0^I(A)_2 \cap l_\infty^2 \subset W_0^I(A,F)_2 \cap l_\infty^2$ as F satisfies Δ_2-condition. Write that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(\chi_{K(x-Le,\epsilon)}(k,l)\right) = F(1)\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\chi_{K(x-Le,\epsilon)}(k,l)$$

for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. Let $x \in W_0^I(A,F)_2 \cap l_\infty^2$ and let $\epsilon > 0$ be given. Take the sequence $y \in l_\infty^2$ by

$$y_{k,l} = \{ \begin{array}{ll} \frac{1}{x_{k,l}} & \text{if } \left|x_{k,l}\right| \geq \epsilon \\ 0 & \text{otherwise.} \end{array}$$

It is easy to see that $xy = \chi_{K_2(x,\epsilon)}$ which again belongs to $W_0^I(A,F)_2 \cap l_\infty^2$ (by Lemma 2.1). Then for $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(\chi_{K(x,\epsilon)}(k,l)\right) \geq \delta\right\} \in I.$$

But then

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\chi_{K_2(x,\epsilon)}(k,l) \geq \delta\right\}$$

$$= \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F\left(\chi_{K_2(x,\epsilon)}(k,l)\right) \geq \delta F(1)\right\} \in I.$$

This shows that $\chi_{K(x,\epsilon)} \in W_0^I(A)_2 \cap l_\infty^2$ for any $\epsilon > 0$ and then it follows from Lemmas 2.2 and 2.3 that $x \in W_0^I(A)_2 \cap l_\infty^2$. □

Theorem 2.6. *Let A be a non-negative RH-regular matrix summability method. Then*
(i) $W^I(A,F)_2 \subset S_A^2(I)$
(ii)$S_A^2(I) \cap l_\infty^2 \subset W^I(A,F)_2$ *if F satisfies the* Δ_2-*condition.*

Proof. (i) Let $x = (x_{k,l}) \in W^I(A,F)_2$. Then there exists a $L \in C$ such that for any $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \geq \delta\right\} \in I.$$

Now for a fixed $\epsilon > 0$,

$$\begin{aligned} &\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \\ = &\sum_{k,l,\left|x_{k,l}-L\right|\geq\epsilon} a_{m,n,k,l}F(|x_k - L|) + \sum_{k,l,\left|x_{k,l}-L\right|<\epsilon} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \\ \geq &\sum_{k,l,\left|x_{k,l}-L\right|\geq\epsilon} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \geq F(\varepsilon)\sum_{k,l,\left|x_{k,l}-L\right|>\epsilon} a_{m,n,k,l}. \end{aligned}$$

Therefore

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l,\left|x_{k,l}-L\right|\geq\epsilon} a_{m,n,k,l} \geq \delta\right\} \subset \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}F(\left|x_{k,l} - L\right|) \geq \delta F(\varepsilon)\right\}.$$

Since the set on the right hand side belongs to I so it follows that $x \in S_A^2(I)$.

(ii) If $x \in S_A^2(I) \cap l_\infty^2$ then from the definition $\chi_{K(x-Le,\epsilon)} \in W_0^I(A)_2 \cap l_\infty^2$ for every $\varepsilon > 0$ where as usual $K(x - Le, \epsilon) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \left|x_{k,l} - L\right| \geq \epsilon\right\}$ for some $L \in C$.
From Lemmas 2.2 and 2.3, $x \in W^I(A)_2 \cap l_\infty^2$. From Theorem 2.4 it now follows that $x \in W^I(A,F)_2$. □

Remark 2.7. Theorem 2.5 presents an improved version of Theorem 3.9 [19] and in a more general form.

Remark 2.8. It is easy to see that $W_0^I(A,F)_2 \cap l_\infty^2 = S_A^2(I) \cap l_\infty^2$.

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