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Conditional integral transforms with related topics on function space

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Abstract. In this paper we study the conditional integral transform, the conditional convolution product and the first variation of functionals on function space. For our research, we modify the class $\mathcal{S}_\alpha$ of functionals introduced in [7]. We then give the existences of the conditional integral transform, the conditional convolution product and the first variation for functionals in $\mathcal{S}_\alpha$. Finally, we give various relationships and formulas among conditional integral transforms, conditional convolution products and first variations of functionals in $\mathcal{S}_\alpha$.

1. Introduction and definitions

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x).$$

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{M}$-measurable for all $\rho \geq 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.) [9].

Let $K = K_0[0,T]$ be the space of all complex-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$ and whose real and imaginary parts are elements of $C_0[0,T]$. In many papers [3–5, 10, 13, 14], the authors studied the integral transform

$$\mathcal{F}_{\gamma,\beta}F(y) \equiv \mathcal{F}_{\gamma,\beta}(F)(y) \equiv \int_{C_0[0,T]} F(\gamma x + \beta y)dm(x), \quad y \in K, \tag{1}$$

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and the convolution product

$$(F * G)_\gamma(y) = \int_{C_0[0,T]} F\Big(\frac{y+\gamma x}{\sqrt{2}}\Big) G\Big(\frac{y-\gamma x}{\sqrt{2}}\Big) dm(x), \quad y \in K \tag{2}$$

for functionals in various classes. Also, they established various basic relationships between the integral transform and the convolution product involving the first variation. In [7], the authors studied a generalized integral transform and a convolution product by using a Gaussian process for functionals in $\mathcal{S}_\alpha$.

Now we will the useful simple formula introduced by Park and Skoug in [16].

Let $F : C_0[0,T] \to \mathbb{C}$ be a Wiener integrable functional and let $X : C_0[0,T] \to \mathbb{R}$ be a $\mathcal{B}(C_0[0,T])$-measurable functional defined by

$$X(x) = x(T). \tag{3}$$

For real number η, $E[F|X](\eta)$ denotes the conditional Wiener integral of F given X. For a more detailed study of the conditional Wiener integral, see [2, 6, 15–17].

In [16], Park and Skoug obtained a simple formula for expressing conditional Wiener integrals with a vector-valued conditioning function in terms of ordinary Wiener integral, and then used the formula to derive the Kac-Feynman integral equation for time dependent potential function on Wiener space. In particular, for given conditioning function X given by (3), they gave a useful simple formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals, namely that,

$$E[F|X](\eta) = \int_{C_0[0,T]} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) dm(x). \tag{4}$$

The integration formula (4) is called a simple formula for the conditioning function X given by (3). In [11], using the simple formula the authors studied the conditional integral transform(CIT) and the conditional convolution product(CCP) involving the first variation of functionals in a class E_0. Also see paper [12] for further work involving CITs and CCPs.

In this paper, we obtain various relationships among the CIT, the CCP and the first variation for functionals in $\mathcal{S}_\alpha$ which was introduced in [7].

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. One can show that for each $v \in L_2[0,T]$, $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$ and if $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$. Also, $\langle v, x\rangle$ has the expected linearity property. Furthermore, if $v \neq 0$, then $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [3, 4, 7, 8, 10, 11, 13].

First we give the definition of the CIT of a functional F on K.

Definition 1.1. *Let F be a functional defined on K and let X be given by* (3). *For each nonzero complex numbers γ and β, the CIT $\mathcal{F}_{\gamma,\beta}(F\|X)$ of F given X is given by the formula*

$$\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta) = \int_{C_0[0,T]} F\Big(\gamma(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta) + \beta y(\cdot)\Big) dm(x) \tag{5}$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

Next, we give the definition of the CCP of functionals F and G on K.

Definition 1.2. *Let F and G be functionals defined on K and let X be given by* (3). *For each nonzero complex number γ, the CCP $((F * G)_\gamma\|X)$ with respect to γ of F and G given X is given by the formula*

$$\begin{aligned}((F * G)_\gamma\|X)(y,\eta) = \int_{C_0[0,T]} & F\Big(\frac{1}{\sqrt{2}}y(\cdot) + \frac{\gamma}{\sqrt{2}}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big) \\ & \cdot G\Big(\frac{1}{\sqrt{2}}y(\cdot) - \frac{\gamma}{\sqrt{2}}\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big) dm(x)\end{aligned} \tag{6}$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

We finish this section by giving the definition of the first variation of a functional F on K.

Definition 1.3. *Let F be a functional defined on K. Then the first variation of F is defined by the formula*

$$\delta F(x|u) = \frac{\partial}{\partial k} F(x+ku)\Big|_{k=0}, \qquad x, u \in K, \tag{7}$$

if it exists.

2. A class $\mathcal{S}_\alpha$ of functionals

In this section we define a modified class $\mathcal{S}_\alpha$ of functionals [7].

First, we recall an integration formula which will be used several times in this paper. For each $\alpha \in \mathbb{C}$ and for $v \in L_2[0,T]$,

$$\int_{C_0[0,T]} \exp\{\alpha \langle v, x\rangle\} dm(x) = \exp\Big\{\frac{\alpha^2}{2}\|v\|_2^2\Big\}. \tag{8}$$

For each complex number α, let $\mathcal{S}_\alpha$ be the class of functionals which has the form

$$F(x) = \int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle\} df(v) \tag{9}$$

and exists for s-a.e. $x \in C_0[0,T]$, where f is in $M(L_2[0,T])$, the class of all complex valued countably additive Borel measures on $L_2[0,T]$.

Remark 2.1. *One can show that for each $\alpha = ip, p \in \mathbb{R}$, the class $\mathcal{S}_\alpha$ is a Banach algebra with the norm*

$$\|F\| = \|f\| = \int_{L_2[0,T]} |df(v)|, \ f \in M(L_2[0,T]).$$

One can show that the correspondence $f \to F$ is injective, carries convolution into pointwise multiplication and that for each complex number α, the space $\mathcal{S}_\alpha$ is a Banach algebra. In particular, if $\alpha = i$, then $\mathcal{S}_i$ is the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [1].

Definition 2.2. *Let $\mathbb{C}$ be the class of all complex numbers. For each $\alpha \in \mathbb{C}$, let*

$$E_\alpha \equiv \{(\gamma, \beta) \in \mathbb{C}\times\mathbb{C} : Re(\alpha^2\gamma^2) \le 0 \text{ and } Re(\alpha^2\beta^2) \le 0\}.$$

Note that for $F \in \mathcal{S}_\alpha$, as we will see theorems and formulas below, when evaluate the CIT, the CCP and the first variation we encounter the PWZ stochastic integral $\langle v, x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\rangle$. Let

$$b_v = \frac{1}{T}\int_0^T v(t)dt$$

for $v \in L_2[0,T]$. Then b_v is an element of $L_2[0,T]$ and

$$\langle v, x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\rangle = \langle v - b_v, x\rangle + \eta b_v.$$

Remark 2.3. *When we evaluate the CIT, the CCP and the first variation of functionals in $\mathcal{S}_\alpha$, we have to consider the existences of some integrals as follows;*

(1) First we could consider the following integral,

$$\int_{L_2[0,T]} \exp\{\alpha\langle v,x\rangle + \zeta b_v\} df(v),\ \ \zeta \in \mathbb{C}. \tag{10}$$

If we assume that

$$\int_{L_2[0,T]} \exp\Big\{|\zeta| \int_0^T |v(t)|dt\Big\}|df(v)| < \infty \tag{11}$$

for all complex number ζ, then

$$\int_{L_2[0,T]} \exp\{\alpha\langle v,x\rangle\} df(v) \qquad \textit{and} \qquad \int_{L_2[0,T]} \exp\{\alpha\gamma\eta b_v\} df(v)$$

exist. However, the integral (10) *might not exist because the product of L_1-functionals might not be in L_1.*

(2) In view of (1) *in Remark 2.3, we need a condition for f in $M(L_2[0,T])$ to show the existence of the integral in equation* (10).

i) If $v \in L_2[0,T]$ is a function of bounded variation, then for each $y \in C_0[0,T]$,

$$|\langle v,x\rangle| \le \|x\|_\infty(|v(T)| + V_0^T(v)) < \infty,$$

where $V_0^T(v)$ is the total variation of v on $[0,T]$. Hence if we assume that

$$\int_{L_2[0,T]} \exp\Big\{|\zeta|\Big[|v(T) + |V_0^T(v)| + \int_0^T |v(t)|dt\Big]\Big\}|df(v)| < \infty, \tag{12}$$

then the integral the integral (10) *always exists.*

ii) Let v be an element of $L_2[0,T]$. Then we note that

$$|\langle v,x\rangle| = \lim_{n\to\infty} |\langle v_n,x\rangle| \le \lim_{n\to\infty} \|x\|_\infty(|v_n(T)| + V_0^T(v_n)) \tag{13}$$

where $v_n(t) = \sum_{k=1}^{n}(v,\alpha_k)_2\alpha_k(t)$, $\{\alpha_k\}$ is a complete orthonormal set in $L_2[0,T]$ and $(\cdot,\cdot)_2$ is the inner product on $L_2[0,T]$. Hence if we add a condition

$$\lim_{n\to\infty}(|v_n(T)| + V_0^T(v_n)) \tag{14}$$

exists, then we can obtain the existence of the integral (10) *under the condition which is similar to the condition* (12).

iii) In fact, if ζ is a purely imaginary, then the integral (10) *always exists.*

(3) As mentioned above, we can give the condition (14) *because the expression* (13) *is independent of the choice of the complete orthonormal set $\{\alpha_k\}$ and the all expressions in equation* (13) *exists for s-a.e. $x \in C_0[0,T]$. Hence, we assume that for $f \in M(L_2[0,T])$ which satisfies the condition* (11) *above, the integral* (10) *always exists.*

3. Existence theorems

In this section, we establish the existence of the CIT, the CCP and the first variation for functionals in $\mathcal{S}_\alpha$.

In our first theorem, we obtain the formula for the CIT of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_{\alpha\beta}$.

Theorem 3.1. *Let $F \in \mathcal{S}_\alpha$ be given by equation* (9) *whose associated measure f satisfies the condition* (11) *above. Then for all $(\gamma, \beta) \in E_\alpha$, the CIT $\mathcal{F}_{\gamma,\beta}(F\|X)$ of F given X exists and is given by the formula*

$$\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta) = \int_{L_2[0,T]} \exp\Big\{\alpha\beta\langle v, y\rangle + \frac{\alpha^2\gamma^2}{2}\|v - b_v\|_2^2 + \alpha\gamma\eta b_v\Big\} df(v) \tag{15}$$

for s-a.e. $y \in C_0[0,T]$ and real number η. Furthermore, the CIT $\mathcal{F}_{\gamma,\beta}(F\|X)$, as a function of y, is an element of $\mathcal{S}_{\alpha\beta}$. In fact,

$$\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta) = \int_{L_2[0,T]} \exp\{\alpha\beta\langle v, y\rangle\} d\phi_1^\eta(v),$$

where ϕ_1^η is an element of $M(L_2[0,T])$.

Proof. Using equations (5) and (8) it follows that for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta) \\
&= \int_{C_0[0,T]} \int_{L_2[0,T]} \exp\{\alpha\gamma\langle v - b_v, x\rangle + \alpha\gamma\eta b_v + \alpha\beta\langle v, y\rangle\} df(v) dm(x) \\
&= \int_{L_2[0,T]} \exp\Big\{\alpha\beta\langle v, y\rangle + \frac{\alpha^2\gamma^2}{2}\|v - b_v\|_2^2 + \alpha\gamma\eta b_v\Big\} df(v)
\end{aligned} \tag{16}$$

and so we have established equation (15). Since $(\gamma,\beta) \in E_\alpha$, f satisfies the condition (11) above and by the hypothesis of Remark 2.3, the last expression in equation (16) exists. Now let ϕ_1^η be a set function defined by

$$\phi_1^\eta(E) = \int_E \exp\Big\{\frac{\alpha^2\gamma^2}{2}\|v - b_v\|_2^2 + \alpha\gamma\eta b_v\Big\} df(v)$$

for $E \in \mathcal{B}(L_2[0,T])$. Then ϕ_1^η is an element of $M(L_2[0,T])$ since $(\gamma,\beta) \in E_\alpha$ and so the last expression in equation (16) becomes

$$\int_{L_2[0,T]} \exp\{\alpha\beta\langle v, y\rangle\} d\phi_1^\eta(v).$$

Hence the CIT $\mathcal{F}_{\gamma,\beta}(F\|X)$ is an element of $\mathcal{S}_{\alpha\beta}$. □

Remark 3.2. *Note that for a given α, we can take complex numbers γ and β so that $Re(\alpha^2\gamma^2) \le 0$ and $Re(\alpha^2\beta^2) \le 0$. Let $\alpha\gamma = a + ib$ and $\alpha\beta = c + id$. Then $\alpha^2\gamma^2 = (a^2 - b^2) + 2iab$ and $\alpha^2\beta^2 = (c^2 - d^2) + 2icd$ and hence $Re(\alpha^2\gamma^2) \le 0$ and $Re(\alpha^2\beta^2) \le 0$ imply $a^2 - b^2 \le 0$ and $c^2 - d^2 \le 0$. Hence we can take γ so that $|a| \le |b|$ and $|c| \le |d|$. For example, if $\alpha = 1 + 2i$, then we can take $\gamma = 3 + 2i$ and $\beta = 1 + 2i$.*

In the next theorem, we obtain the formula for the CCP of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_\alpha$.

Theorem 3.3. *Let F and f be as in Theorem 3.1. Let $G \in \mathcal{S}_\alpha$ be given by*

$$G(x) = \int_{L_2[0,T]} \exp\{\alpha\langle w, x\rangle\} dg(w)$$

for s-a.e. $x \in C_0[0,T]$, where g is an element of $M(L_2[0,T])$ which satisfies the condition (11) *above. Then for all $(\gamma,\beta) \in E_\alpha$, the CCP $((F * G)_\gamma\|X)$ of F and G for given X exists and is given by the formula*

$$\begin{aligned}
&((F * G)_\gamma\|X)(y,\eta) \\
&= \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\Big\{\frac{\alpha}{\sqrt{2}}\langle v + w, y\rangle + \frac{\alpha^2\gamma^2}{4}\|v - w - b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta}{\sqrt{2}} b_{v-w}\Big\} df(v) dg(w)
\end{aligned} \tag{17}$$

for s-a.e. $y \in C_0[0,T]$ and real number η. Furthermore, the CCP $((F*G)_\gamma \| X)$, as a function of y, is an element of $\mathcal{S}_\alpha$. In fact,

$$((F*G)_\gamma \| X)(y,\eta) = \int_{L_2[0,T]} \exp\{\alpha\langle k, y\rangle\} d\phi_2^\eta(k),$$

where ϕ_2^η is an element of $M(L_2[0,T])$.

Proof. Using equations (6) and (8) it follows that for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned} &((F*G)_\gamma \| X)(y,\eta) \\ &= \int_{C_0[0,T]} \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\Big\{ \frac{\alpha\gamma}{\sqrt{2}} \langle (v - b_v - w + b_w), x\rangle \\ &\qquad + \frac{\alpha\gamma\eta}{\sqrt{2}}(b_v - b_w) + \frac{\alpha}{\sqrt{2}}\langle v + w, y\rangle \Big\} df(v) dg(w) dm(x) \\ &= \int_{L_2[0,T]} \int_{L_2[0,T]} \exp\Big\{ \frac{\alpha}{\sqrt{2}}\langle v+w, y\rangle + \frac{\alpha^2\gamma^2}{4}\|v - w - b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta}{\sqrt{2}} b_{v-w} \Big\} df(v) dg(w) \end{aligned} \tag{18}$$

and so we have established equation (17). By a similar method in the proof of Theorem 3.1, we can show that the last expression in equation (18) exists. Now, let ϕ^η be a set function defined by

$$\phi^\eta(E) = \int_E \exp\Big\{ \frac{\alpha^2\gamma^2}{4}\|v - w - b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta}{\sqrt{2}} b_{v-w} \Big\} df(v) dg(w)$$

for $E \in \mathcal{B}(L_2[0,T] \times L_2[0,T])$ and let $\rho : L_2[0,T] \times L_2[0,T] \to L_2[0,T]$ be a function defined by $\rho(v,w) = (v+w)/\sqrt{2}$. Then $\phi_2^\eta = \phi^\eta \circ \rho^{-1}$ is an element of $M(L_2[0,T])$ since $(\gamma,\beta) \in E_\alpha$ and so the last expression in equation (18) becomes

$$\int_{L_2[0,T]} \exp\{\alpha\langle k, y\rangle\} d\phi_2^\eta(k).$$

Hence the CCP $((F*G)_\gamma \| X)$ is an element of $\mathcal{S}_\alpha$. □

Let

$$\mathcal{A} = \Big\{ u \in C_0[0,T] : u(t) = \int_0^t z(s) ds \text{ for some } z \in L_2[0,T] \Big\}.$$

Note that for all $w, v \in L_2[0,T]$, we have

$$|(w,v)_2| \le \|w\|_2 \|v\|_2.$$

Furthermore, for $u \in \mathcal{A}$ and $v \in L_2[0,T]$, the PWZ integral $\langle v, u\rangle$ exists and is given by the formula

$$\langle v, u\rangle = \int_0^T v(s) z(s) ds = (v,z)_2$$

and hence $|\langle v, u\rangle| \le \|v\|_2 \|z\|_2$.

The following observation below will be very useful in the development of our theorems. For $F \in \mathcal{S}_\alpha$, we will assume that the associated measure f in $M(L_2[0,T])$ of F always satisfies the following inequality

$$\int_{L_2[0,T]} |\alpha| \|v\|_2 |df(v)| < \infty. \tag{19}$$

In our next theorem, we obtain the formula for the first variation of functionals from $\mathcal{S}_\alpha$ into $\mathcal{S}_\alpha$.

Theorem 3.4. *Let F and f be as in Theorem 3.1 and let $u \in \mathcal{A}$. Assume that*

$$\Big|\frac{\partial}{\partial k}\exp\{\alpha\langle v, x+ku\rangle\}\Big| \le L(x) \tag{20}$$

where $L(x)$ is integrable on $C_0[0,T]$. Then the first variation $\delta F(x|u)$ of F exists and is given by the formula

$$\delta F(x|u) = \int_{L_2[0,T]} \alpha\langle v, u\rangle \exp\{\alpha\langle v, x\rangle\}df(v) \tag{21}$$

for s-a.e. $x \in C_0[0,T]$. Furthermore, as a function of x, δF is an element of $\mathcal{S}_\alpha$. In fact,

$$\delta F(x|u) = \int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle\}d\phi_3(v),$$

where ϕ_3 is an element of $M(L_2[0,T])$.

Proof. Using equation (7) it follows that for s-a.e. $x \in C_0[0,T]$,

$$\begin{aligned}\delta F(x|u) &= \frac{\partial}{\partial k}F(x+ku)\Big|_{k=0}\\ &= \frac{\partial}{\partial k}\Big(\int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle + \alpha k\langle v, u\rangle\}df(v)\Big)\Big|_{k=0}\\ &= \int_{L_2[0,T]} \alpha\langle v, u\rangle \exp\{\alpha\langle v, x\rangle\}df(v).\end{aligned} \tag{22}$$

The third equality of (22) follows from condition (20) and so by a similar method in the proof of Theorem 3.1, the last expression in equation (22) exists. Thus we have established equation (21). Now let ϕ_3 be a set function defined by

$$\phi_3(E) = \int_E \alpha\langle v, u\rangle df(v)$$

for $E \in \mathcal{B}(L_2[0,T])$. Then we see that ϕ_3 is an element of $M(L_2[0,T])$ by using equation (19) and the last expression in equation (22) becomes

$$\int_{L_2[0,T]} \exp\{\alpha\langle v, x\rangle\}d\phi_3(v).$$

Hence δF is an element of $\mathcal{S}_\alpha$. □

4. Relationships involving exactly two operations

In this section we consider the relationships involving exactly two of the three operations CIT, CCP and first variation for functionals in $\mathcal{S}_\alpha$. These relationships and formulas, as well as alternative expressions for some of them are given by equations (23), (25), (26), (27) and (28) below.

In our first theorem we obtain the relationship involving the CIT and the CCP, that is to say, the CIT of the CCP is the product of their CITs.

Theorem 4.1. *Let F, G, f and g be as in Theorem 3.3. Then for all $(\gamma,\beta) \in E_\alpha$,*

$$\mathcal{F}_{\gamma,\beta}(((F*G)_\gamma\|X)(\cdot,\eta_1)\|X)(y,\eta_2) = \mathcal{F}_{\gamma,\beta}(F\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\mathcal{F}_{\gamma,\beta}(G\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}) \tag{23}$$

as elements of $\mathcal{S}_{\alpha\beta}$. Also, both sides of the expression in equation (23) are given by the formula

$$\int_{L_2[0,T]}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha\beta}{\sqrt{2}}\langle v+w, y\rangle + \frac{\alpha^2\gamma^2}{2}(\|v-b_v\|_2^2 + \|w-b_w\|_2^2) + \frac{\alpha\gamma}{\sqrt{2}}(\eta_2 b_{v+w} + \eta_1 b_{v-w})\Big\}df(v)dg(w).$$

Proof. The left hand side of equation (23) exists by Theorems 3.1 and 3.3, while the right hand side of equation (23) exists by Theorem 3.1. Hence equation (23) immediately follows from Theorem 2.1 in [11]. Furthermore, using equations (17) and (15) it follows that for s-a.e. $y \in C_0[0,T]$ and real numbers η_1, η_2 and η_3,

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}(((F*G)_\gamma\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&= \int_{C_0[0,T]}\int_{L_2[0,T]}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha\beta}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha\gamma}{\sqrt{2}}\langle v+w-b_{v+w},x\rangle\\
&\quad + \frac{\alpha^2\gamma^2}{4}\|v-w-b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta_2 b_{v+w}}{\sqrt{2}} + \frac{\alpha\gamma\eta_1 b_{v-w}}{\sqrt{2}}\Big\}df(v)dg(w)dm(x)\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha\beta}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha^2\gamma^2}{4}\|v+w-b_{v+w}\|_2^2\\
&\quad + \frac{\alpha^2\gamma^2}{4}\|v-w-b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta_2 b_{v+w}}{\sqrt{2}} + \frac{\alpha\gamma\eta_1 b_{v-w}}{\sqrt{2}}\Big\}df(v)dg(w)\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha\beta}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha^2\gamma^2}{2}(\|v-b_v\|_2^2+\|w-b_w\|_2^2) + \frac{\alpha\gamma}{\sqrt{2}}(\eta_2 b_{v+w}+\eta_1 b_{v-w})\Big\}df(v)dg(w).
\end{aligned} \tag{24}$$

The second equality in equation (24) follows immediately from equation (8) and the last equality in equation (24) follows from the parallelogram law of the norm. Hence we have the desired result. □

In our next theorem we obtain a formula relating the CIT and the first variation, that is to say, the CIT of the first variation is the first variation of the CIT.

Theorem 4.2. *Let F, f and u be as in Theorem 3.4. Assume that F satisfies the hypothesis of Theorem 3.4. Then for all $(\gamma,\beta) \in E_\alpha$,*

$$\beta\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|u)\|X)(y,\eta) = \delta\mathcal{F}_{\gamma,\beta}(F\|X)(y|u,\eta) \tag{25}$$

as elements of $\mathcal{S}_{\alpha\beta}$. Also, both sides of the expression in equation (25) *are given by the formula*

$$\int_{L_2[0,T]} \alpha\beta\langle v,u\rangle \exp\Big\{\alpha\beta\langle v,y\rangle + \frac{\alpha^2\gamma^2}{2}\|v-b_v\|_2^2 + \alpha\gamma\eta b_v\Big\}df(v).$$

Proof. Both sides of equation (25) exists by Theorems 3.4 and 3.1. Furthermore, using equations (21), (15) and (8), it follows that for s-a.e. $y \in C_0[0,T]$ and real number η,

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|u)\|X)(y,\eta)\\
&= \int_{C_0[0,T]}\int_{L_2[0,T]} \alpha\langle v,u\rangle \exp\Big\{\alpha\beta\langle v-b_v,x\rangle + \alpha\beta\langle v,y\rangle + \alpha\gamma\eta b_v\Big\}df(v)dm(x)\\
&= \int_{L_2[0,T]} \alpha\langle v,u\rangle \exp\Big\{\alpha\beta\langle v,y\rangle + \frac{\alpha^2\gamma^2}{2}\|v-b_v\|_2^2 + \alpha\gamma\eta b_v\Big\}df(v).
\end{aligned}$$

On the other hand, using equations (15), (7) and (8), it follows that for s-a.e. $y \in C_0[0,T]$ and real number η,

$$\begin{aligned}
&\delta\mathcal{F}_{\gamma,\beta}(F\|X)(y|u,\eta)\\
&= \frac{\partial}{\partial k}\Big[\int_{L_2[0,T]} \exp\Big\{\alpha\beta\langle v,y\rangle + \alpha\beta k\langle v,u\rangle + \frac{\alpha^2\gamma^2}{2}\|v-b_v\|_2^2 + \alpha\gamma\eta b_v\Big\}df(v)\Big]_{k=0}\\
&= \int_{L_2[0,T]} \alpha\beta\langle v,u\rangle \exp\Big\{\alpha\beta\langle v,y\rangle + \frac{\alpha^2\gamma^2}{2}\|v-b_v\|_2^2 + \alpha\gamma\eta b_v\Big\}df(v),
\end{aligned}$$

which completes the proof of Theorem 4.2. □

Remark 4.3. *From Theorems 3.1 thru 4.2 above, we gave some conditions for existences of all operations. In the similar method, we can give appropriate conditions for existence of all operations for the rest theorems and formulas in Sections 3 and 4. Now, to simplify the expressions, we will only state the formulas without conditions for existences.*

Now, we consider relationships involving the CIT, the CCP and the first variation.

In our first formula, we obtain a relationship for the CCP of CITs. Equation (26) follows from Theorems 3.1, 3.3 and equation (8).

(1) A formula for the CCP with respect to the first argument of the CIT : Let F, G, f and g be as in Theorem 4.1. Then $((\mathcal{F}_{\gamma,\beta}(F||X)(\cdot|\eta_1) * \mathcal{F}_{\gamma,\beta}(G||X)(\cdot|\eta_2))_\gamma)||X)(y,\eta_3)$ exists as an element of $\mathcal{S}_{\alpha\beta}$ and is given by the formula

$$\begin{aligned}&((\mathcal{F}_{\gamma,\beta}(F||X)(\cdot|\eta_1) * \mathcal{F}_{\gamma,\beta}(G||X)(\cdot|\eta_2)_\gamma)||X)(y,\eta_3)\\ &= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\Big\{\frac{\alpha\beta}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha^2\gamma^2\beta^2}{4}(\|v-w-b_{v-w}\|_2^2\\ &\qquad + \frac{\alpha^2\gamma^2}{2}(\|v-b_v\|_2^2+\|w-b_w\|_2^2) + \frac{\alpha\beta\gamma\eta_3}{\sqrt{2}}b_{v-w} + \alpha\gamma(\eta_1 b_v+\eta_2 b_w)\Big\}df(v)dg(w)\end{aligned} \tag{26}$$

for s-a.e. $y \in C_0[0,T]$ and real numbers η_1, η_2 and η_3.

In our next formula, we obtain a relationship for CCP with respect to the first argument of the first variations. Equation (27) immediately follows from equations (21), (17) and (8).

(2) A formula for the CCP with respect to the first argument of functionals : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then $((\delta F(\cdot|u) * \delta G(\cdot|u)_\gamma)||X)(y,\eta)$ exists as an element of $\mathcal{S}_\alpha$ and is given by the formula

$$\begin{aligned}&((\delta F(\cdot|u) * \delta G(\cdot|u)_\gamma)||X)(y,\eta)\\ &= \int_{L_2[0,T]}\int_{L_2[0,T]} \alpha^2\langle v,u\rangle\langle w,u\rangle \exp\Big\{\frac{\alpha}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha^2\gamma^2}{4}(\|v-w-b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta}{\sqrt{2}}b_{v-w}\Big\}df(v)dg(w)\end{aligned} \tag{27}$$

for s-a.e. $y \in C_0[0,T]$ and real number η.

In our next formula, we obtain a relationship for the first variation of the CCP. Equation (28) immediately follows from equations (21), (17) and (8).

(3) A formula for the first variation of the CCP : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then $\delta((F * G)_\gamma||X)(\cdot,\eta)(y|u)$ exists as an element of $\mathcal{S}_\alpha$ and is given by the formula

$$\begin{aligned}&\delta((F * G)_\gamma||X)(\cdot,\eta)(y|u)\\ &= \int_{L_2[0,T]}\int_{L_2[0,T]} \frac{\alpha\gamma}{\sqrt{2}}\langle v+w,u\rangle \exp\Big\{\frac{\alpha}{\sqrt{2}}\langle v+w,y\rangle + \frac{\alpha^2\gamma^2}{4}\|v-w-b_{v-w}\|_2^2 + \frac{\alpha\gamma\eta}{\sqrt{2}}b_{v-w}\Big\}df(v)dg(w)\end{aligned} \tag{28}$$

for s-a.e. $y \in C_0[0,T]$ and real number η.

5. Relationships involving all three concepts

In this section we establish various relationships involving the CIT, the CCP and the first variation where each operation is used exactly once. It turns out that there are five distinct formulas, and these are given by equation (30) through (35) below. Equations (23) and (25) are very useful relationships for developments of this section.

Note that for $F, G \in \mathcal{S}_\alpha$ whose associated measures f and g satisfy inequality (19), we have that

$$\delta(FG)(x|u) = \delta F(x|u)G(x) + F(x)\delta G(x|u) \tag{29}$$

for s-a.e. $x \in C_0[0,T]$ and $u \in \mathcal{A}$.

In our first formula below, we obtain a relationship for taking the first variation of equation (23) and using equation (29).

(1) A formula for the first variation of CIT of the CCP : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then

$$\begin{aligned}&\delta[\mathcal{F}_{\gamma,\beta}(((F*G)_\gamma\|X))(\cdot,\eta_1)\|X)(\cdot,\eta_2)](y|u)\\&=\delta\mathcal{F}_{\gamma,\beta}(F\|X)(\cdot,\frac{\eta_2+\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{u}{\sqrt{2}})\mathcal{F}_{\gamma,\beta}(G\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\\&\quad+\mathcal{F}_{\gamma,\beta}(F\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\delta\mathcal{F}_{\gamma,\beta}(G\|X)(\cdot,\frac{\eta_2-\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{u}{\sqrt{2}})\end{aligned}\tag{30}$$

as elements of $\mathcal{S}_{\alpha\beta}$.

In our second formula, we obtain a relationship using equation (25) with $\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|u)\|X)$ replaced with $\mathcal{F}_{\gamma,\beta}(\delta(F*G)_\gamma(\cdot|u)\|X)$.

(2) Formulas for the CIT with respect to the first argument of the first variation of CCP : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then

$$\beta\mathcal{F}_{\gamma,\beta}(\delta((F*G)_\gamma\|X))(\cdot,\eta_1)(\cdot|u)\|X)(y,\eta_2)=\delta[\mathcal{F}_{\gamma,\beta}((((F*G)_\gamma\|X))(\cdot,\eta_1)\|X)(\cdot,\eta_2)](y|u)\tag{31}$$

as elements of $\mathcal{S}_{\alpha\beta}$. Furthermore using equation (30), we get another relationship

$$\begin{aligned}&\beta\mathcal{F}_{\gamma,\beta}(\delta((F*G)_\gamma\|X))(\cdot,\eta_1)(\cdot|u)\|X)(y,\eta_2)\\&=\delta\mathcal{F}_{\gamma,\beta}(F\|X)(\cdot,\frac{\eta_2+\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{u}{\sqrt{2}})\mathcal{F}_{\gamma,\beta}(G\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\\&\quad+\mathcal{F}_{\gamma,\beta}(F\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\delta\mathcal{F}_{\gamma,\beta}(G\|X)(\cdot,\frac{\eta_2-\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|\frac{u}{\sqrt{2}})\end{aligned}\tag{32}$$

as elements of $\mathcal{S}_{\alpha\beta}$.

In our third formula, we obtain a relationship using equation (23) with F and G replaced with δF and δG, respectively and using equation (25).

(3) Formulas for the CIT of the CCP with respect to the first argument of the first variation : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then

$$\begin{aligned}&\beta^2\mathcal{F}_{\gamma,\beta}(((\delta F(\cdot|u)*\delta G(\cdot|u))_\gamma\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\&=\beta^2\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|u)\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2+\eta_1}{\sqrt{2}})\mathcal{F}_{\gamma,\beta}(\delta G(\cdot|u)\|X)(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}})\\&=\delta\mathcal{F}_{\gamma,\beta}(F\|X)(\cdot,\frac{\eta_2+\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|u)\delta\mathcal{F}_{\gamma,\beta}(G\|X)(\cdot,\frac{\eta_2-\eta_1}{\sqrt{2}})(\frac{y}{\sqrt{2}}|u)\end{aligned}\tag{33}$$

as elements of $\mathcal{S}_{\alpha\beta}$.

In our next formula, we obtain a relationship using equation (25) with respect to each arguments.

(4) A formula for the CCP of the CIT of the first variation : Let F, G, f and g be as in Theorem 4.1 and let u be as in Theorem 3.4. Then

$$\begin{aligned}&\beta^2(\mathcal{F}_{\gamma,\beta}((\delta F(\cdot|u)\|X)(\cdot,\eta_1)*\mathcal{F}_{\gamma,\beta}(\delta G(\cdot|u)\|X)(\cdot,\eta_2)_\gamma)\|X)(y,\eta_3)\\&=((\delta\mathcal{F}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)(\cdot|u)*\delta\mathcal{F}_{\gamma,\beta}(G\|X)(\cdot,\eta_2)_\gamma)(\cdot|u)\|X)(\cdot,\eta_2)\|X)(y,\eta_3)\end{aligned}\tag{34}$$

as elements of $\mathcal{S}_{\alpha\beta}$.

In our last formula in this section, we obtain a relationship using the linearity of CIT, equation (29) and equation (25) with replaced F with FG.

(5) Formulas for the first variation of the CIT of the product functional : Let F, G, f and g be as in Theorem 4.1. Then

$$\begin{aligned} &\beta[\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|u)G(\cdot)\|X)(y,\eta) + \mathcal{F}_{\gamma,\beta}(F(\cdot)\delta G(\cdot|u)\|X)(y,\eta)] \\ &= \beta\mathcal{F}_{\gamma,\beta}(\delta(FG)(\cdot|u)\|X)(y,\eta) \\ &= \delta\mathcal{F}_{\gamma,\beta}(FG\|X)(\cdot,\eta)(y|u) \end{aligned} \tag{35}$$

as elements of $\mathcal{S}_{\alpha\beta}$.

6. Further results

See the four remarks below for some additional information and related results.

Remark 6.1. *We obtain some observations for the CIT, the CCP and the first variation as below:*

(1) Let $r(t) = \frac{1}{\sqrt{T}}$ *for all* $t \in [0,T]$ *and let* v *be an element of* $L_2[0,T]$ *such that* $(r,v)_2 = 0$. *Then we have* $b_v = \frac{1}{T}\int_0^T v(t)dt = \frac{1}{\sqrt{T}}\int_0^T r(t)v(t)dt = 0$ *and so*

$$\langle v, x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\rangle = \langle v, x\rangle.$$

Hence the CIT reduces the integral transform $\mathcal{F}_{\gamma,\beta}$ *defined by* (1), *that is to say,*

$$\mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta) = \mathcal{F}_{\gamma,\beta}(F)(y)$$

for s-a.e. $y \in C_0[0,T]$ *and real number* η.

(2) The convolution product $(F * G)_\gamma$ *defined by* (2) *is commutative. While the CCP is not commutative because*

$$((F * G)_\gamma\|X)(y,\eta) = ((G * F)_\gamma\|X)(y,-\eta)$$

for s-a.e. $y \in C_0[0,T]$ *and real number* η. *However the usual additive distribution held for the CCP as see above theorems and formulas in Sections 3 and 4.*

(3) Note that for $F \in \mathcal{S}_\alpha$, *the first variation* $\delta F(x|u)$ *acts like a directional derivative in the direction of* u.

To get an interesting relationship, let

$$A_\alpha \equiv \{(\gamma,\beta) \in E_\alpha : \gamma^2 + \beta^2 = 1\}.$$

Remark 6.2. *Let* F *and* f *be as in Theorem 3.1, and let* (γ_1,β_1) *and* (γ_2,β_2) *be elements of* A_α *with* $(\gamma_1,\beta_1) \in A_{\alpha\beta_2}$ *and* $(\gamma_2,\beta_2) \in A_{\alpha\beta_1}$. *Then*

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}(F\|X)(\cdot,\eta_2)\|X)(y,\eta_1) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2) \tag{36}$$

as elements of $\mathcal{S}_{\alpha\beta_1\beta_2}$ *if and only if* $\gamma_1\eta_1 = \gamma_2\eta_2$. *Also, both sides of the expression in equation* (36) *are given by the formula*

$$\int_{L_2[0,T]} \exp\Big\{\alpha\beta_1\beta_2\langle v,y\rangle + \frac{\alpha^2}{2}(\gamma_1^2 + \beta_1^2\gamma_2^2)\|v - b_v\|_2^2 + \alpha b_v(\gamma_1 + \beta_2 + \gamma_2\eta_2)\Big\}df(v).$$

In [4, 7], the authors obtained a formula

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma_0,\beta_0}(F)(y)$$

for some complex numbers γ_0 *and* β_0 *where* $\mathcal{F}_{\gamma,\beta}$ *is the integral transform defined by equation* (1). *Unfortunately we can see that there is no pair* $(\gamma_0,\beta_0) \in E_\alpha$ *(or* A_α*) such that*

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}(F\|X)(\cdot,\eta_2)\|X)(y,\eta_1) = \mathcal{F}_{\gamma_0,\beta_0}(F\|X)(y,\eta_3).$$

Remark 6.3. *Let $F \in \mathcal{S}_\alpha$ be given by equation* (9) *and let $v \in L_2[0,T]$ be a function of bounded variation. Let $\{(\gamma_n, \beta_n)\}$ be a sequence in E_α with $\gamma_n \to \gamma \neq 0$ and $\beta_n \to \beta \neq 0$ as $n \to \infty$ for some $(\gamma, \beta) \in E_\alpha$. Suppose that $|\alpha| \leq M$ for some positive real number M. Then using the dominated convergence theorem,*

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{F}_{\gamma_n,\beta_n}(F\|X)(y,\eta)\\
&= \lim_{n\to\infty} \int_{L_2[0,T]} \exp\Big\{\alpha\beta_n\langle v, y\rangle + \frac{\alpha^2\gamma_n^2}{2}\|v - b_v\|_2^2 + \alpha\gamma_n\eta b_v\Big\} df(v)\\
&= \int_{L_2[0,T]} \exp\Big\{\alpha\beta\langle v, y\rangle + \frac{\alpha^2\gamma^2}{2}\|v - b_v\|_2^2 + \alpha\gamma\eta b_v\Big\} df(v)\\
&= \mathcal{F}_{\gamma,\beta}(F\|X)(y,\eta).
\end{aligned}$$

We finish this paper by stating the inverse CIT.

Remark 6.4. *In many paper [3, 4, 7, 10, 14], the author gave the existence of the inverse integral transform for the integral transform $\mathcal{F}_{\gamma,\beta}$ defined by* (1). *That is to say, they obtained the following equation*

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}F)(y) = F(y) = \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}F)(y).$$

Also, they established several relationships involving the inverse integral transform. Unlike the integral transform, the inverse CIT does not exist because

$$\begin{aligned}
&\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}(F\|X)(\cdot,\eta_1)\|X)(y,\eta_2)\\
&= \int_{L_2[0,T]} \exp\Big\{\alpha\langle v, y\rangle + \alpha\gamma b_v(\eta_1 + i\eta_2)\Big\} df(v)\\
&\neq F(y).
\end{aligned}$$

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