# A note on spaces between normal and $\kappa$-normal spaces

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**Abstract.** Several generalized notions of normality exist in the literature. In this paper, some variants of normality which lies between normal and $\kappa$-normal (or mildly normal) spaces are considered. Interrelation among these variants of normality is established and examples are provided in their support. Some new decompositions of normality is obtained in terms of seminormal spaces and $\beta$-normal spaces. A weaker version of seminormal spaces is introduced which provides a factorization of normality in terms of $\pi$-normality.

## 1. Introduction and preliminaries

Normality is an important topological property and many generalized notion of normality exists in the literature. $\kappa$-normal spaces was introduced by Shchepin in [9] and the same notion was independently introduced and studied in [8] by Singal and Singal. In this paper those variants of normality which lies in between normal and $\kappa$-normal spaces are considered for investigation.

Throughout the present paper no separation axiom is assumed. Let $X$ be a topological space and let $A \subset X$, then the closure of a set $A$ will be denoted by $\overline{A}$ or $clA$ and the interior by $intA$ or $A^o$. A set $U \subset X$ is said to be **regularly open** [6] if $U = int\overline{U}$. The complement of a regularly open set is called **regularly closed**. It is observed that an intersection of two regularly closed sets need not be regularly closed. A finite union of regular open sets is called $\pi$-open set and a finite intersection of regular closed sets is called $\pi$-closed set. It is obvious that the complement of a $\pi$-open set is $\pi$-closed and the complement of a $\pi$-closed set is $\pi$-open, the finite union (intersection) of $\pi$-closed sets is $\pi$-closed, but the infinite union (intersection) of $\pi$-closed sets need not be $\pi$-closed (see [5]). A point $x \in X$ is called a $\delta$-**limit point** [11] of $A$ if every regularly open neighbourhood of $x$ intersects $A$. Let $cl_\delta A$ denotes the set of all $\delta$-limit point of $A$. The set $A$ is called $\delta$-**closed** if $A = cl_\delta A$. The complement of a $\delta$-closed set will be referred to as a $\delta$-**open** set. The family of $\delta$-open sets forms a topology on $X$. The topology formed by the set of $\delta$-open sets is the semiregularization topology whose basis is the family of regularly open sets. Thus

Regularly closed $\implies$ $\pi$-closed $\implies$ $\delta$-closed $\implies$ closed.

Let $Y$ be a subspace of $X$. A subset $A$ of $X$ is concentrated on $Y$ [1] if $A$ is contained in the closure of $A \cap Y$ in $X$. We say that $X$ is normal on $Y$ if every two disjoint closed subsets of $X$ concentrated on $Y$ can be separated by disjoint open neighbourhoods in $X$ [1].

**Definition 1.1.** A topological space *X* is said to be

(i) *almost normal* [7] if every pair of disjoint closed sets one of which is regularly closed are contained in disjoint open sets.

(ii) *mildly normal* [8] ( or *κ-normal* [9]) if every pair of disjoint regularly closed sets are contained in disjoint open sets.

(iii) *quasi-normal* [5] if any two disjoint π-closed subsets *A* and *B* of *X* there exist two open disjoint subsets *U* and *V* of *X* such that $A \subset U$ and $B \subset V$.

(iv) *π-normal* [5] if for any two disjoint closed subsets *A* and *B* of *X* one of which is π-closed, there exist two open disjoint subsets *U* and *V* of *X* such that $A \subset U$ and $B \subset V$.

(v) *β-normal* [2] if for any two disjoint closed subsets *A* and *B* of *X* there exist open subsets *U* and *V* of *X* such that $A \cap U$ is dense in *A*, $B \cap V$ is dense in *B*, and $\overline{U} \cap \overline{V} = \phi$.

(vi) *Δ-normal* [3] if every pair of disjoint closed sets one of which is δ-closed are contained in disjoint open sets.

(vii) *weakly Δ-normal* [3] if every pair of disjoint δ-closed sets are contained in disjoint open sets.

(viii) *weakly functionally Δ-normal* (wf Δ-normal) [3] if for every pair of disjoint δ-closed sets *A* and *B* there exists a continuous function $f : X \rightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.

(ix) *densely normal* [1] if there exists a dense subspace *Y* of *X* such that *X* is normal on *Y*.

## 2. Interrelations

It is easy to see that every normal space is densely normal and every densely normal space is *κ*-normal. On the other hand, the converse is not true, as was shown in [4]. Zaitsev in [13] introduced the notion of quasinormality in the class of regular spaces. It is observed in [5], that a π-normal space is quasi normal but the converse need not be true. Shchepin in [9], illustrated an example of a mildly normal space which is not quasi-normal. Since every regularly closed set is π-closed, the following results are obvious.

**Theorem 2.1.** *Every Δ-normal space is π-normal.*

**Example 2.2.** A π-normal space which is not Δ-normal.

Let *X* be the Modified Fort space [10] in which *X* is the union of any infinite set *N* and two distinct one point sets *p* and *q*. Topologize *X* by calling any subset of *N* open and calling any set containing *p* or *q* is open if and only if it contains all but a finite number of points in *N*. This space is π-normal but not weakly Δ-normal, hence not Δ-normal, since disjoint δ-closed sets *p* and *q* can not be separated by disjoint open sets.

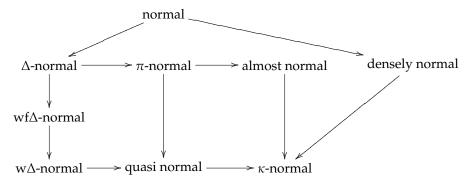**Theorem 2.3.** *Every weakly Δ-normal space is quasi normal.*

**Example 2.4.** A quasi normal space which is not weakly Δ-normal.

The space defined in Example 2.2 is a quasi normal space which is not weakly Δ-normal.

**Theorem 2.5.** ([5]) *Every π-normal space is almost normal.*

**Theorem 2.6.** ([5]) *Every quasi normal space is κ-normal.*

The following implications immediately follows from the definitions and the above results but none of these implication is reversible.

normal
Δ-normal → π-normal → almost normal     densely normal
wfΔ-normal
wΔ-normal → quasi normal → κ-normal

Seminormal spaces was introduced by Vigilino [12] which was utilized by Singal and Arya [7] to obtain a decomposition of normality in terms of almost normality. A space is said to be *seminormal* if for every closed set $F$ and each open set $U$ containing $F$, there exists a regular open set $V$ such that $F \subset V \subset U$.

**Definition 2.7.** A space is said to be *weakly seminormal* if for every closed set $F$ and each open set $U$ containing $F$, there exists a π-open set V such that $F \subset V \subset U$.

Every seminormal space is weakly seminormal but the converse need not be true.

**Example 2.8.** A weakly seminormal space which is not seminormal.

Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, X, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{a, d, e\}, \{a\}, \{d\}, \{a, d\}\}$. Here $\{b, c\}$ is a closed set contained in the open set $\{a, b, c, d\}$. But since there is no regularly open set contained in $\{a, b, c, d\}$ containing $\{b, c\}$, the space is not seminormal. However, being union of regularly open sets $\{a, b\}$ and $\{c, d\}$, the set $\{a, b, c, d\}$ is itself π-open. Thus the space is weakly seminormal.

The following theorems provide few decompositions of normality in terms of variants of normality which lies in between normal and κ-normal spaces.

**Theorem 2.9.** *In an weakly seminormal space the following statements are equivalent:*
*(a) X is normal.*
*(b) X is Δ-normal.*
*(c) X is weakly functionally Δ-normal.*
*(d) X is weakly Δ-normal.*
*(e) X is π-normal.*
*(g) X is quasi normal.*

*Proof.* The implications (a) $\implies$ (b) $\implies$ (e) $\implies$ (g) and (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (g) are obvious.

To prove (g) $\implies$ (a), let $X$ be a weakly seminormal quasi normal space. Let $A$ and $B$ be two disjoint closed sets in $X$. Then $X - B$ is an open set containing the closed set $A$. Since $X$ is weakly semi normal there exists a π-open set $U$ such that $A \subset U \subset X - B$. Thus $A$ is a closed set which is disjoint from the π-closed set $X - U$. So $X - A$ is an open set containing the π-closed set $X - U$. Again by weak seminormality of $X$, there exists a π-open set $W$ such that $X - U \subset W \subset X - A$. Thus $X - W$ and $X - U$ are disjoint π-closed sets containing $A$ and $B$ respectively. By quasi normality of $X$, there exist disjoint open sets $P$ and $Q$ such that $X - W \subset P$ and $X - U \subset Q$. Thus $P$ and $Q$ are two disjoint open sets containing $A$ and $B$ respectively. □

**Theorem 2.10.** *In a seminormal space the following statements are equivalent:*
*(a) X is normal.*
*(b) X is Δ-normal.*
*(c) X is weakly functionally Δ-normal.*
*(d) X is weakly Δ-normal.*
*(e) X is π-normal.*

*(f) X is almost normal.*
*(g) X is quasi normal.*
*(h) X is densely normal.*
*(i) X is κ-normal.*

*Proof.* The implications (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (g) $\implies$ (i) are observed in [3], [5] and Theorem 2.3. In [1], it is shown that (a) $\implies$ (h) $\implies$ (i) and implications (a) $\implies$ (b) $\implies$ (e) $\implies$ (f) $\implies$ (i) follows from [3], [5] and Theorem 2.1. The prove of the implication (i) $\implies$ (a) is similar to the prove of Theorem 2.9. $\square$

A topological space $X$ is said to be $\Delta$-*regular* [3] if for every closed set $F$ and each open set $U$ containing $F$, there exists a $\delta$-open set $V$ such that $F \subset V \subset U$.

It is obvious that every semiregular space is $\Delta$-regular and every weakly seminormal space is $\Delta$-regular.

**Example 2.11.** A $\Delta$-regular space which is not semiregular.

Let $X = \{a, b, c\}$ and let $\tau = \{\{a\}, \{a, b\}, X, \phi\}$. This space is $\Delta$-regular but not semiregular.

The following factorization of normality is established in [3].

**Theorem 2.12.** ([3]) *In a $\Delta$-regular space the following statements are equivalent:*
*(a) X is normal.*
*(b) X is $\Delta$-normal.*
*(c) X is weakly functionally $\Delta$-normal.*
*(d) X is weakly $\Delta$-normal.*

In [2], the authors proved that a space is normal if and only if it is $\beta$-normal and $\kappa$-normal. Thus the following theorem is immediate.

**Theorem 2.13.** *In a $\beta$-normal space the following statements are equivalent:*
*(a) X is normal.*
*(b) X is $\Delta$-normal.*
*(c) X is weakly functionally $\Delta$-normal.*
*(d) X is weakly $\Delta$-normal.*
*(e) X is $\pi$-normal.*
*(f) X is almost normal.*
*(g) X is densely normal.*
*(h) X is quasi normal.*
*(i) X is $\kappa$-normal.*

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