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The growth of functions with derivatives in $L^p(\mathbb{R}^n)$

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Abstract. We establish bounds on the growth of $|u(x)|$ as $|x| \to \infty$ for functions u all of whose derivatives of order k are in $L^p(\mathbb{R}^n)$ and $k > n/p$.

In memory of Časlav V. Stanojević

1. Introduction

Let $L^{p,k}(\mathbb{R}^n)$, $k = 1, 2, \ldots$, denote the class of functions u on $\mathbb{R}^n$ all of whose (distributional) derivatives of order k are in $L^p(\mathbb{R}^n)$ and set

$$\|u\|_{L^{p,k}(\mathbb{R}^n)} = \left\{ \sum_{|\nu|=k} \|D^\nu u\|^p_{L^p(\mathbb{R}^n)} \right\}^{1/p} .$$

These classes arise in various applications, for some approximation theoretic examples see [6–8]. We ask how fast can such functions $u(x)$ grow as $|x| \to \infty$. Now, such functions need not be continuous unless $k > n/p$, for example see [1]. So, accordingly, we assume that this constraint is valid in the considerations below. Furthermore, since the case $p = 2$ is somewhat technically more transparent we consider it first.

The null space of $L^{2,k}(\mathbb{R}^n)$ consists of the class of polynomials of degree $\le k-1$ so it is reasonable to expect that the bound on the growth of such functions $u(x)$ should be no less than $O(|x|^{k-1})$ as $|x| \to \infty$. Indeed, in the case $n = 1$ approximating u by its Taylor polynomial of degree $k-1$ and applying Schwarz's inequality to the error term results in the bound $|u(x)| = O(|x|^{k-1/2})$ as $|x| \to \infty$.

In the case of general n we have the following.

Proposition 1.1. *If $k > n/2$ then every u in $L^{2,k}(\mathbb{R}^n)$ can be expressed as $u = v + w$ where v is a polynomial of degree no greater than $k-1$ and w is a continuous function that enjoys the following properties:*

$$|w(x)| \le C\|u\|_{L^{2,k}(\mathbb{R}^n)} \begin{cases} (1+|x|)^{k-n/2} & \text{if } n \text{ is odd} \\ (1+|x|)^{k-n/2}\Big(\log(2+|x|)\Big)^{1/2} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

$$|w(x)| = \begin{cases} o(|x|^{k-n/2}) & \text{if } n \text{ is odd} \\ o(|x|^{k-n/2}(\log|x|)^{1/2}) & \text{if } n \text{ is even} \end{cases} \quad \text{as } |x| \to \infty. \tag{2}$$

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$$\int_{|x|\geq 2}\left(\frac{|w(x)|}{|x|^{k-n/2}}\right)^2\frac{dx}{|x|^n}\leq C\|u\|^2_{L^{2,k}(\mathbb{R}^n)}\quad \textit{if } n \textit{ is odd.} \tag{3}$$

The transformations $P : u \to v = Pu$ and $Q : u \to w = Qu = u - Pu$ can be defined via linear projection operators.

The constants in (1) and (3) may depend on k and n but are otherwise independent of u.

This proposition significantly improves and extends [6, Proposition 2] and [7, item(3.3)].

Since v is a polynomial of degree $\leq k-1$ it should be clear that

$$\|w\|_{L^{2,k}(\mathbb{R}^n)} = \|u\|_{L^{2,k}(\mathbb{R}^n)}. \tag{4}$$

As we shall see, the decomposition $u = v + w$ is not unique and the projections P and Q are not uniquely defined.

Consider the following examples:

First, note that $u(x) = (1+|x|^2)^{a/2}$ is in $L^{2,k}(\mathbb{R}^n)$ whenever $a < k - n/2$, which suggests that the bound (1) is on target in the case of odd n. Next, if n is even and ν is a multi-index such that $|\nu| = k - n/2$ then $u(x) = x^\nu\Big(\log(2+|x|^2)\Big)^b$ is in $L^{2,k}(\mathbb{R}^n)$ whenever $b < 1/2$, which suggests that (1) is also on target for even n. Finally, if $n = 1$ and $r > 0$ let

$$u_r(x) = \begin{cases} x/\sqrt{r} & \text{if } 0 < x \leq r \\ \sqrt{r} & \text{if } r < x \end{cases} \qquad \text{and} = 0 \text{ if } x \leq 0.$$

Then

$$\|u_r\|_{L^{2,1}(\mathbb{R})} = 1, \quad |u_r(x)| \leq |x|^{1/2}, \quad \text{and} \quad \sup_{r>0}|u_r(x)| = |x|^{1/2} \text{ for } x > 0,$$

which implies that in the case $n = k = 1$ the bound (1) is asymptotically optimal.

Section 2 is devoted to the definition of the projection $Q : u \to w$ and the proof of Proposition 1. A statement and proof of the corresponding result in the more general case when the derivatives of order k are in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, can be found in Section 3.

2. Details

Notation In what follows differentiations, Fourier transforms, and equalities are to be interpreted in the distributional sense unless they are meaningful otherwise. The Fourier transform of a function u in $L^1(R^n)$ is defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle}u(x)dx$$

and $u^\vee$ denotes the inverse Fourier transform of u, thus $(\widehat{u})^\vee = u$.

For convenience we often use pointwise notation, e.g. $u(\xi)$, even when u is a distribution which is not necessarily defined pointwise. We expect that there will be no misunderstanding as to the precise meaning of such expressions.

Let $\phi(t)$ be a non-negative infinitely differentiable function on $\mathbb{R}_+ = (0,\infty)$ with support in the interval $1/2 \leq t \leq 1$ and normalized such that

$$\int_0^\infty \phi(t)\frac{dt}{t} = \int_{1/2}^1 \phi(t)\frac{dt}{t} = 1.$$

Then $\phi(t|\xi|)$, $0 < t < \infty$, is a partition of unity of $\mathbb{R}^n \setminus \{0\}$ as a function of ξ in the sense that $\phi(t|\xi|)$ has support in $\frac{1}{2t} \leq |\xi| \leq \frac{1}{t}$ and

$$\int_0^\infty \phi(t|\xi|)\frac{dt}{t} = 1 \quad \text{if} \quad |\xi| \neq 0.$$

The collection of function $\phi(t|\xi|)$, $0 < t < \infty$, may be thought of as a continuous analog of the well known partition found, for example, in [5, 9]. See also [3, 4].

Let

$$\chi(\xi) = \begin{cases} \int_1^\infty \phi(t|\xi|)\frac{dt}{t} & \text{if } |\xi| \neq 0 \\ 1 & \text{if } |\xi| = 0 \end{cases}$$

then $\chi(\xi)$ is non-negative, in $C^\infty(\mathbb{R}^n)$,

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1/2 \\ 0 & \text{if } |\xi| \geq 1, \end{cases}$$

and for $\epsilon > 0$

$$\chi(\epsilon\xi) = \int_\epsilon^\infty \phi(t|\xi|)\frac{dt}{t} \quad \text{when } |\xi| \neq 0.$$

Note that

$$\lim_{\epsilon\to 0} \chi(\epsilon\xi)u(\xi) = u(\xi)$$

in $\mathcal{S}$ for every u in $\mathcal{S}$ and hence in $\mathcal{S}'$ for every u in $\mathcal{S}'$. On the other hand as r goes to infinity $\chi(r\xi)u(\xi)$ does not converge in $\mathcal{S}$ and hence not in $\mathcal{S}'$ except for certain classes of distributions u. For example, the fact that if u is in $L^2(\mathbb{R}^n)$ then $\lim_{r\to\infty} \chi(r\xi)u(\xi) = 0$ in $L^2(\mathbb{R}^n)$ and hence also in $\mathcal{S}'$ will be useful in what follows.

If u is in $L^{2,k}(\mathbb{R}^n)$ then $|\xi|^k\widehat{u}(\xi)$ is in $L^2(\mathbb{R}^n)$ and the (semi)norm $\||\xi|^k\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}$ is equivalent to $\|u\|_{L^{2,k}(\mathbb{R}^n)}$. This fact, which is a consequence of Plancherel's formula, will be used often in what follows.

Definitions For u in $L^{2,k}(\mathbb{R}^n)$ define $w = Qu$ by its Fourier transform $\widehat{w}$ evaluated at a test function ψ as

$$\begin{aligned} \langle \widehat{w}, \psi\rangle = \langle \widehat{w}(\xi), \psi(\xi)\rangle &= \lim_{r\to\infty} \langle (1 - \chi(r\xi))\widehat{u}(\xi), \psi(\xi) - \psi_{m-1}(\xi)\chi(\xi)\rangle \\ &= \lim_{r\to\infty} \int_0^r \langle \phi(t|\xi|)\widehat{u}(\xi), \psi(\xi) - \psi_{m-1}(\xi)\chi(\xi)\rangle \frac{dt}{t} \end{aligned}$$

where ψ_{m-1} is the Taylor polynomial of ψ of degree $m - 1$,

$$\psi_{m-1}(\xi) = \sum_{|\nu|\leq m-1} \frac{D^\nu\psi(0)}{\nu!}\xi^\nu$$

and m is the integer which satisfies $k - n/2 < m \leq k - n/2 + 1$.

That $w = Qu$ is well defined follows from

$$|\langle w, \psi\rangle| \leq C\|u\|_{L^{2,k}(\mathbb{R}^n)} \sum_{|\nu|\leq m} \|D^\nu\psi\|_{L^\infty(\mathbb{R}^n)}$$

which in turn follows from

$$|\langle \phi(t|\xi|)\widehat{u}(\xi), \psi(\xi) - \psi_{m-1}(\xi)\chi(\xi)\rangle| \leq C \begin{cases} A\sum_{|\nu|\leq m} \|D^\nu\psi\|_{L^\infty(\mathbb{R}^n)} & \text{if } t > 1/2 \\ B\|\psi\|_{L^\infty(\mathbb{R}^n)} & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} A &= \||\xi|^m\phi(t|\xi|)\widehat{u}(\xi)\|_{L^1(\mathbb{R}^n)} \\ &\leq \||\xi|^{m-k}\phi(t|\xi|)\|_{L^2(\mathbb{R}^n)}\||\xi|^k\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \leq C\,t^{-m+k-n/2}\|u\|_{L^{2,k}(\mathbb{R}^n)} \end{aligned}$$

and

$$\begin{aligned}B &= \|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^1(\mathbb{R}^n)}\\ &\le t^k \||t\xi|^{-k}\phi(t|\xi|)\|_{L^2(\mathbb{R}^n)}\||\xi|^k\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \le C\, t^{k-n/2}\|u\|_{L^{2,k}(\mathbb{R}^n)}.\end{aligned}$$

Note that $\xi^\nu\widehat{w}(\xi) = \xi^\nu\widehat{u}(\xi)$ for multi-indexes ν such that $|\nu| = k$. It follows that $\widehat{w}(\xi) = \widehat{u}(\xi)$ for $|\xi| \neq 0$, w is in $L^{2,k}(\mathbb{R}^n)$, $\|u - w\|_{L^{2,k}(\mathbb{R}^n)} = 0$, and $\|w\|_{L^{2,k}(\mathbb{R}^n)} = \|u\|_{L^{2,k}(\mathbb{R}^n)}$. Hence

$$\widehat{v} = \widehat{u} - \widehat{w} = \widehat{Pu}$$

has support at the origin and thus

$$v = u - w = Pu$$

is a polynomial. That the degree of v is no greater than $k-1$ follows from that fact $\|v\|_{L^{2,k}(\mathbb{R}^n)} = \|u-w\|_{L^{2,k}(\mathbb{R}^n)} = 0$.

Proof of (1) To estimate the size of $|w(x)|$ write

$$w = \left(\chi\widehat{w}\right)^\vee + \left((1-\chi)\widehat{u}\right)^\vee$$

and

$$\begin{aligned}\|(1-\chi)\widehat{u}\|_{L^1(\mathbb{R}^n)} &\le \int_0^1 \|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^1(\mathbb{R}^n)}\frac{dt}{t}\\ &= \int_0^1 t^k\left\|\frac{\phi(t|\xi|)}{|t\xi|^k}|\xi|^k\widehat{u}(\xi)\right\|_{L^1(\mathbb{R}^n)}\frac{dt}{t}\\ &\le \int_0^1 t^k\left\|\frac{\phi(t|\xi|)}{|t\xi|^k}\right\|_{L^2(\mathbb{R}^n)}\||\xi|^k\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}\frac{dt}{t}\\ &= \left\{\int_0^1 t^{k-n/2}\frac{dt}{t}\right\}\left\|\frac{\phi(|\xi|)}{|\xi|^k}\right\|_{L^2(\mathbb{R}^n)}\||\xi|^k\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}\\ &\le C\|u\|_{L^{2,k}(\mathbb{R}^n)}.\end{aligned}$$

So that

$$\|\left((1-\chi)\widehat{u}\right)^\vee\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{L^{2,k}(\mathbb{R}^n)}. \tag{5}$$

To estimate the size of $|\left(\chi\widehat{w}\right)^\vee(x)|$ write

$$\begin{aligned}(2\pi)^n\left(\chi\widehat{w}\right)^\vee(x) &= \langle\widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle\\ &= \int_{1/2}^\infty \langle\phi(t|\xi|)\widehat{u}(\xi), \left(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle)\right)\chi(\xi)\rangle\frac{dt}{t}\end{aligned}$$

and

$$\begin{aligned}|\langle\widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| &\le \int_{1/2}^\infty \left|\langle\phi(t|\xi|)\widehat{u}(\xi), \left(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle)\right)\chi(\xi)\rangle\right|\frac{dt}{t}\\ &\le \int_{1/2}^\infty \|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}\left\|\left(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle)\right)\Phi(t|\xi|)\right\|_{L^2(\mathbb{R}^n)}\frac{dt}{t}\end{aligned} \tag{6}$$

where $\Phi(t)$ is a non-negative function on $\mathbb{R}_+ = (0,\infty)$ which is infinitely differentiable and satisfies

$$\Phi(t) = \begin{cases}1 & \text{if } 1/2 \le t \le 1\\ 0 & \text{if } t \le 1/4 \text{ or } t \ge 2.\end{cases} \quad \text{and} \quad p_{m-1}(s) = \sum_{j=0}^{m-1}\frac{(is)^j}{j!}.$$

Observe that

$$\begin{aligned}\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} &= t^k \left\| \frac{\phi(t|\xi|)}{|t\xi|^k} |\xi|^k \widehat{u}(\xi) \right\|_{L^2(\mathbb{R}^n)} \\ &\le t^k \left\| \frac{\phi(t|\xi|)}{|t\xi|^k} \right\|_{L^\infty(\mathbb{R}^n)} \left\| |\xi|^k \widehat{u}(\xi) \right\|_{L^2(\mathbb{R}^n)} \\ &\le C t^k \|u\|_{L^{2,k}(\mathbb{R}^n)}\end{aligned} \tag{7}$$

and

$$\left\| \left(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle) \right) \Phi(t|\xi|) \right\|_{L^2(\mathbb{R}^n)} \le C t^{-n/2} \begin{cases} \left(\frac{|x|}{t}\right)^m & \text{if } |x| \le t \\ \left(\frac{|x|}{t}\right)^{m-1} & \text{if } |x| \ge t. \end{cases} \tag{8}$$

Hence if $|x| \le 1/2$ then

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le C\|u\|_{L^{2,k}(\mathbb{R}^n)} |x|^m \int_{1/2}^\infty t^{k-n/2-m} \frac{dt}{t} \le C_1 \|u\|_{L^{2,k}(\mathbb{R}^n)}$$

since $m > k - n/2$.

When $|x| > 1/2$ we can compute as follows:

If n is odd, so that $k - n/2$ is not an integer and $k - n/2 < m < k - n/2 + 1$, using (8) we may write

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le C\|u\|_{L^{2,k}(\mathbb{R}^n)} \{ |x|^{m-1} \int_{1/2}^{|x|} t^{k-n/2+1-m} \frac{dt}{t} + |x|^m \int_{|x|}^\infty t^{k-n/2-m} \frac{dt}{t} \}$$

which simplifies to

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le C\|u\|_{L^{2,k}(\mathbb{R}^n)} |x|^{k-n/2}.$$

and which together with (5) implies (1) in the case of odd n.

On the other hand if n is even so that $k - n/2$ is an integer and $k - n/2 < m = k - n/2 + 1$ we may write

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le AB$$

where

$$A^2 = \int_{1/2}^\infty \left(t^{-k} \|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \right)^2 \frac{dt}{t}$$

and

$$B^2 = \int_{1/2}^\infty \left(t^k \left\| \left(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle) \right) \Phi(t|\xi|) \right\|_{L^2(\mathbb{R}^n)} \right)^2 \frac{dt}{t}$$

Now

$$\begin{aligned} A^2 &\le \int_0^\infty \left(t^{-k} \|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \right)^2 \frac{dt}{t} = \int_0^\infty \left\| \frac{\phi(t|\xi|)}{|t\xi|^k} |\xi|^k \widehat{u}(\xi) \right\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \\ &= \int_{\mathbb{R}^n} \left\{ \int_0^\infty \left(\frac{\phi(t|\xi|)}{|t\xi|^k} \right)^2 \frac{dt}{t} \right\} \left| |\xi|^k \widehat{u}(\xi) \right|^2 d\xi = C \| |\xi|^k \widehat{u}(\xi) \|_{L^2(\mathbb{R}^n)}^2 \le C \|u\|_{L^{2,k}(\mathbb{R}^n)}^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} B^2 &\le \int_{1/2}^{|x|} \left(C t^{k-n/2+1-m} |x|^{m-1} \right)^2 \frac{dt}{t} + \int_{|x|}^\infty \left(C t^{k-n/2-m} |x|^m \right)^2 \frac{dt}{t} \\ &= C_1 |x|^{2(k-n/2)} \log 2|x| + C_2 |x|^{2(k-n/2)} \end{aligned}$$

since $m-1 = k-n/2$. Hence when n is even we get

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le C\|u\|_{L^{2,k}(\mathbb{R}^n)}|x|^{k-n/2}(1+\log 2|x|)^{1/2}$$

which together with (5) implies (1) in the case of even n.

Proof of (2) To see (2) in the case of odd n it suffices to show that for every positive ϵ we have for sufficiently large $|x|$ the inequality

$$|w(x)| \le \epsilon|x|^{k-n/2}. \tag{10}$$

To see (10) let

$$w_r = \Big((1-\chi(r\xi))\widehat{u}(\xi)\Big)^{\vee}.$$

Then w_r is a bounded function for every positive r, $Qw_r = w_r$, and

$$\lim_{r\to\infty}\|w-w_r\|_{L^{2,k}(\mathbb{R}^n)} = 0.$$

Write

$$\begin{aligned}|w(x)| &\le |w(x)-w_r(x)| + |w_r(x)|\\ &\le C\|w-w_r\|_{L^{2,k}(\mathbb{R}^n)}|x|^{k-n/2} + |w_r(x)|\end{aligned}$$

and choose r so that $\|w-w_r\|_{L^{2,k}(\mathbb{R}^n)} < \epsilon/(2C)$. Then for x such that $|x|^{k-n/2} > 2\|w_r\|_{L^\infty(\mathbb{R}^n)}/\epsilon$ we have (10).

The same reasoning is also does the job in the case of even n, *mutatis mutandis*.

Proof of (3) To see (3), in view of (5)), it suffices to show that

$$\int_{|x|>2}\left(\frac{|\langle \widehat{w}, e^{i\langle x,\xi\rangle}\chi(\xi)\rangle|}{|x|^{k-n/2}}\right)^2\frac{dx}{|x|^n} \le C\|u\|_{L^{2,k}(\mathbb{R}^n)}. \tag{11}$$

when n is odd.

By virtue of (6) we may write

$$|\langle \widehat{w}, e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le I_1(x) + I_2(x) + I_3(x)$$

where

$$I_1 = \int_{1/2}^{2}\cdots\frac{dt}{t},\quad I_2 = \int_{2}^{|x|}\cdots\frac{dt}{t},\quad I_3 = \int_{|x|}^{\infty}\cdots\frac{dt}{t}.$$

and the integrand in each case is

$$\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}\Big\|\Big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle)\Big)\Phi(t|\xi|)\Big\|_{L^2(\mathbb{R}^n)}.$$

In view of (7) and (8)

$$I_1(x) \le C\|u\|_{L^{2,k}(\mathbb{R}^n)}|x|^{m-1}\int_{1/2}^{2} t^{k-n/2+1-m}\frac{dt}{t}$$

and hence

$$\int_{|x|>2}\left(\frac{I_1(x)}{|x|^{k-n/2}}\right)^2\frac{dx}{|x|^n} \le C\|u\|^2_{L^{2,k}(\mathbb{R}^n)}\int_{|x|>2}|x|^{2(m-1-k+n/2)}\frac{dx}{|x|^n} = C_1\|u\|^2_{L^{2,k}(\mathbb{R}^n)}$$

where the last equality follows from the fact that $k-n/2+1-m>0$.

Applying Schwarz's inequality and (8) yields, with ϵ satisfying $0 < \epsilon < k - n/2 + 1 - m$,

$$I_2(x) \le \left\{ \int_2^{|x|} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k-\epsilon}} \right)^2 \frac{dt}{t} \right\}^{1/2} \left\{ \int_2^{|x|} \left(t^{k-n/2+1-m-\epsilon} |x|^{m-1} \right)^2 \frac{dt}{t} \right\}^{1/2}$$

$$= C|x|^{k-n/2-\epsilon} \left\{ \int_2^{|x|} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k-\epsilon}} \right)^2 \frac{dt}{t} \right\}^{1/2}.$$

Thus

$$\begin{aligned}
\int_{|x|>2} \left(\frac{I_2(x)}{|x|^{k-n/2}} \right)^2 \frac{dx}{|x|^n} &\le C \int_{|x|>2} |x|^{-2\epsilon} \left\{ \int_2^{|x|} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k-\epsilon}} \right)^2 \frac{dt}{t} \right\} \frac{dx}{|x|^n} \\
&= C \int_2^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k-\epsilon}} \right)^2 \left\{ \int_{|x|>t} |x|^{-2\epsilon} \frac{dx}{|x|^n} \right\} \frac{dt}{t} \\
&= C_1 \int_2^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k}} \right)^2 \frac{dt}{t} \\
&\le C_2 \|u\|^2_{L^{2,k}(\mathbb{R}^n)}
\end{aligned}$$

where the last inequality in the above string follows by virtue of (9).

Again applying Schwarz's inequality and (8) yields, with ϵ satisfying $0 < \epsilon < m - (k - n/2)$,

$$I_3(x) \le \left\{ \int_{|x|}^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k+\epsilon}} \right)^2 \frac{dt}{t} \right\}^{1/2} \left\{ \int_{|x|}^{\infty} \left(t^{k-n/2-m+\epsilon} |x|^{m} \right)^2 \frac{dt}{t} \right\}^{1/2}$$

$$= C|x|^{k-n/2+\epsilon} \left\{ \int_{|x|}^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k+\epsilon}} \right)^2 \frac{dt}{t} \right\}^{1/2}.$$

and so

$$\begin{aligned}
\int_{|x|>2} \left(\frac{I_3(x)}{|x|^{k-n/2}} \right)^2 \frac{dx}{|x|^n} &\le C \int_{|x|>2} |x|^{2\epsilon} \left\{ \int_{|x|}^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k+\epsilon}} \right)^2 \frac{dt}{t} \right\} \frac{dx}{|x|^n} \\
&= C \int_2^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k+\epsilon}} \right)^2 \left\{ \int_{2\le|x|\le t} |x|^{2\epsilon} \frac{dx}{|x|^n} \right\} \frac{dt}{t} \\
&= C_1 \int_2^{\infty} \left(\frac{\|\phi(t|\xi|)\widehat{u}(\xi)\|_{L^2(\mathbb{R}^n)}}{t^{k}} \right)^2 \frac{dt}{t} \\
&\le C_2 \|u\|^2_{L^{2,k}(\mathbb{R}^n)}.
\end{aligned}$$

The above bounds on $\int_{|x|>2} \left(t^{-(k-n/2)} I_j(x) \right)^2 \frac{dt}{t}$, $j = 1, 2, 3$, of course imply (11)

3. The case $1 < p < \infty$

In the somewhat more general case where 2 is extended to p, $1 < p < \infty$, we have the following:

Proposition 3.1. *Suppose p satisfies* $1 < p < \infty$. *If* $k > n/p$ *then every* u *in* $L^{p,k}(\mathbb{R}^n)$ *can be expressed as* $u = v + w$ *where* v *is a polynomial of degree no greater than* $k - 1$ *and* w *is a continuous function that enjoys the following properties:*

$$|w(x)| \le C\|u\|_{L^{p,k}(\mathbb{R}^n)} W(n, p; x) \tag{12}$$

where

$$W(n,p;x) = \begin{cases} (1+|x|)^{k-n/p} & \text{if } n/p \text{ is not an integer} \\ (1+|x|)^{k-n/p}\Big(\log(2+|x|)\Big)^{1/2} & \begin{cases}\text{if } n/p \text{ is an integer} \\ \text{and } 1 < p \le 2\end{cases} \\ (1+|x|)^{k-n/p}\Big(\log(2+|x|)\Big)^{1-1/p} & \begin{cases}\text{if } n/p \text{ is an integer} \\ \text{and } 2 \le p < \infty.\end{cases} \end{cases}$$

$$|w(x)| = o\Big(W(n,p;x)\Big) \quad as \quad |x| \to \infty. \tag{13}$$

$$\int_{|x|\ge 2} \left(\frac{|w(x)|}{|x|^{k-n/p}}\right)^p \frac{dx}{|x|^n} \le C\|u\|^p_{L^{p,k}(\mathbb{R}^n)} \text{ if } n/p \text{ is not an integer.} \tag{14}$$

The transformations $P : u \to v = Pv$ and $Q : u \to w = Qu$ can be defined via the same type of linear projection operators as in Proposition 1.

For the most part the proof of Proposition 2 follows the same lines as that of Proposition 1 with Hölder's inequality in the role played by Schwarz's inequality. The fact that $\|\Big(|\xi|^k\widehat{u}(\xi)\Big)^\vee\|_{L^p(\mathbb{R}^n)}$ is equivalent to $\|u\|_{L^{p,k}(\mathbb{R}^n)}$ whenever u is in $L^{p,k}(\mathbb{R}^n)$, $1 < p < \infty$, is a consequence of the appropriate variant of the Fourier multiplier theorem of Marcinkiewicz Hörmander, [5, 10]. The definition of w and v is the same as in Section 2 with the exception that now m is the integer which satisfies $k - n/p < m \le k - n/p + 1$.

Proof of (12) **when** n/p **is not an integer** The analog of (5) is

$$\begin{aligned} \|\Big((1-\chi)\widehat{u}\Big)^\vee\|_{L^\infty(\mathbb{R}^n)} &\le \int_0^1 t^k \left\|\Big(\frac{\phi(t|\xi|)}{|t\xi|^k}\Big)^\vee\right\|_{L^q(\mathbb{R}^n)} \left\|\Big(|\xi|^k\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} \frac{dt}{t} \\ &= \left\{\int_0^1 t^{k-n/p}\frac{dt}{t}\right\} \left\|\Big(\frac{\phi(|\xi|)}{|\xi|^k}\Big)^\vee\right\|_{L^q(\mathbb{R}^n)} \left\|\Big(|\xi|^k\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} \\ &\le C\|u\|_{L^{p,k}(\mathbb{R}^n)}, \end{aligned}$$

where $1/q = 1 - 1/p$, while the analog of (6) and (7) are

$$\begin{aligned} &|\langle\widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \\ &\quad \le \int_{1/2}^\infty \left\|\Big(\phi(t|\xi|)\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} \left\|\Big(\Big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle)\Big)\Phi(t|\xi|)\Big)^\vee\right\|_{L^q(\mathbb{R}^n)} \frac{dt}{t} \end{aligned}$$

and

$$\begin{aligned} \left\|\Big(\phi(t|\xi|)\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} &= t^k \left\|\Big(\frac{\phi(t|\xi|)}{|t\xi|^k}|\xi|^k\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} \\ &\le t^k \left\|\Big(\frac{\phi(t|\xi|)}{|t\xi|^k}\Big)^\vee\right\|_{L^1(\mathbb{R}^n)} \left\|\Big(|\xi|^k\widehat{u}(\xi)\Big)^\vee\right\|_{L^p(\mathbb{R}^n)} \\ &\le Ct^k\|u\|_{L^{p,k}(\mathbb{R}^n)}. \end{aligned}$$

Next we note that as a function of s, $-\infty < s < \infty$,

$$\frac{e^{is} - p_{m-1}(s)}{s^a}$$

is the Fourier transform of a finite measure on $\mathbb{R}$ for $a = m$ and $a = m - 1$. Hence for such a and every η

$$\mu_\eta = \Big(\frac{e^{i\langle \eta, \xi\rangle} - p_{m-1}(\langle \eta, \xi\rangle)}{\langle \eta, \xi\rangle^a}\Big)^\vee$$

is a finite measure on $\mathbb{R}^n$. So using $\|\mu\|_{\mathcal{M}}$ to denote the total variation of the finite measure μ on $\mathbb{R}^n$ we may write

$$\begin{aligned}&\Big\|\Big(\Big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x, \xi\rangle)\Big)\Phi(t|\xi|)\Big)^\vee\Big\|_{L^q(\mathbb{R}^n)}\\ &\quad\le \Big\|\Big(\frac{e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x, \xi\rangle)}{\langle x, \xi\rangle^a}\Big)^\vee\Big\|_{\mathcal{M}} \Big\|\Big(\langle \eta, t\xi\rangle^a\Phi(t|\xi|)\Big)^\vee\Big\|_{L^q(\mathbb{R}^n)}\Big(\frac{|x|}{t}\Big)^a.\end{aligned}$$

Since $\|\mu_\eta\|_{\mathcal{M}}$ is independent of η for $|\eta| = 1$ choosing $\eta = x/|x|$ allows us to conclude that

$$\Big\|\Big(\frac{e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x, \xi\rangle)}{\langle x, \xi\rangle^a}\Big)^\vee\Big\|_{\mathcal{M}} = \|\Big(\widehat{\mu}_\eta(|x|\xi)\Big)^\vee\|_{\mathcal{M}} = \|\mu_\eta\|_{\mathcal{M}} = C_a$$

where C_a is a constant which depends only on a. Choosing a accordingly results in

$$\Big\|\Big(\Big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x, \xi\rangle)\Big)\Phi(t|\xi|)\Big)^\vee\Big\|_{L^q(\mathbb{R}^n)} \le Ct^{-n/p}\begin{cases}\Big(\frac{|x|}{t}\Big)^m & \text{if } |x| \le t\\ \Big(\frac{|x|}{t}\Big)^{m-1} & \text{if } |x| \ge t,\end{cases} \tag{15}$$

which is the analog of (8)

Finally, computing as in Section 2 with these inequalities it follows that

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le C\|u\|_{L^{p,k}(\mathbb{R}^n)}|x|^{k-n/p}.$$

which as in Section 2 implies (12) in the case when n/p is not an integer.

Proof of (12) **when** n/p **is an integer** The case when n/p is an integer is a bit more involved. First of all, it suffices to restrict attention to the case $|x| > 1/2$ and we do so below.

If $1 < p \le 2$ write

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle}\chi(\xi)\rangle| \le AB$$

where

$$A^2 = \int_{1/2}^{\infty}\Big(t^{-k}\Big\|\Big(\phi(t|\xi|)\widehat{u}(\xi)\Big)^\vee\Big\|_{L^p(\mathbb{R}^n)}\Big)^2\ \frac{dt}{t}$$

and

$$B^2 = \int_{1/2}^{\infty}\Big(t^{k}\Big\|\Big(\Big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x, \xi\rangle)\Big)\Phi(t|\xi|)\Big)^\vee\Big\|_{L^q(\mathbb{R}^n)}\Big)^2\ \frac{dt}{t}$$

To estimate B use (15) and compute as in Section 2 to get

$$B^2 = C(|x|^{2(k-n/p)}(1 + \log 2|x|).$$

To estimate A write

$$\begin{aligned}A &\le \Big\{\int_0^{\infty}\Big(t^{-k}\|\Big(\phi(t|\xi|)\widehat{u}(\xi)\Big)^\vee\|_{L^p(\mathbb{R}^n)}\Big)^2\ \frac{dt}{t}\Big\}^{1/2}\\ &\le \Big\|\Big\{\int_0^{\infty}\Big|\Big(\frac{\phi(t|\xi|)}{|t\xi|^k}|\xi|^k\widehat{u}(\xi)\Big)^\vee\Big|^2\frac{dt}{t}\Big\}^{1/2}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

and note that

$$g_2(U) = \Big\{ \int_0^\infty \Big| \Big(\frac{\phi(t|\xi|)}{|t\xi|^k} |\xi|^k \widehat{u}(\xi) \Big)^\vee \Big|^2 \frac{dt}{t} \Big\}^{1/2}$$

is simply a variant of the Littlewood-Paley function of $U = \big(|\xi|^k \widehat{u}(\xi)\big)^\vee$ which enjoys the property that

$$\|g_2(U)\|_{L^p(\mathbb{R}^n)} \le C_p \|U\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$, [2, 5, 10]. Hence

$$A \le C\|u\|_{L^{p,k}(\mathbb{R}^n)}$$

and together with the estimate of B this implies (12) in the case $k - n/p$ is an integer and $1 < p \le 2$.

If $2 \le p < \infty$ again write

$$|\langle \widehat{w}(\xi), e^{i\langle x,\xi\rangle} \chi(\xi) \rangle| \le AB$$

but now

$$A^p = \int_{1/2}^\infty \Big(t^{-k} \|\big(\phi(t|\xi|)\widehat{u}(\xi)\big)^\vee\|_{L^p(\mathbb{R}^n)} \Big)^p \ \frac{dt}{t}$$

and

$$B^q = \int_{1/2}^\infty \Big(t^k \Big\| \Big(\big(e^{i\langle x,\xi\rangle} - p_{m-1}(\langle x,\xi\rangle) \big) \Phi(t|\xi|) \Big)^\vee \Big\|_{L^q(\mathbb{R}^n)} \Big)^q \ \frac{dt}{t}.$$

To estimate B^q proceed as in the earlier cases:

$$B^q \le C \Big\{ \int_{1/2}^{|x|} \Big(t^{k-n/p+1-m} |x|^{m-1} \Big)^q \frac{dt}{t} + \int_{|x|}^\infty \Big(t^{k-n/p-m} |x|^m \Big)^q \frac{dt}{t} \Big\}$$

which results in

$$B \le C|x|^{k-n/p} \big(1 + \log 2|x| \big)^{1/q}.$$

To estimate A write

$$\begin{aligned} A &\le \Big\{ \int_0^\infty \Big(t^{-k} \|\big(\phi(t|\xi|)\widehat{u}(\xi)\big)^\vee\|_{L^p(\mathbb{R}^n)} \Big)^p \ \frac{dt}{t} \Big\}^{1/p} \\ &\le \Big\| \Big\{ \int_0^\infty \Big| \Big(\frac{\phi(t|\xi|)}{|t\xi|^k} |\xi|^k \widehat{u}(\xi) \Big)^\vee \Big|^p \frac{dt}{t} \Big\}^{1/p} \Big\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

and note that

$$g_p(U) = \Big\{ \int_0^\infty \Big| \Big(\frac{\phi(t|\xi|)}{|t\xi|^k} |\xi|^k \widehat{u}(\xi) \Big)^\vee \Big|^p \frac{dt}{t} \Big\}^{1/p}$$

is simply the Littlewood-Paley like function of $U = \big(|\xi|^k \widehat{u}(\xi)\big)^\vee$ which enjoys the bound

$$g_p(U) \le g_2(U)^{2/p} g_\infty(U)^{1-2/p}$$

where for each z in $\mathbb{R}^n$

$$g_\infty(U,z) = \sup_{t>0} \Big| \Big(\frac{\phi(t|\xi|)}{|t\xi|^k} \widehat{U}(\xi) \Big)^\vee (z) \Big|.$$

Since the $L^p(\mathbb{R}^n)$ norms of both $g_2(U)$ and $g_\infty(U)$ are bounded by constant multiples of the $L^p(\mathbb{R}^n)$ norm of U, $1 < p < \infty$, we may conclude that

$$\|g_p(U)\|_{L^p(\mathbb{R}^n)} \leq C_p \|U\|_{L^p(\mathbb{R}^n)}$$

for $2 \leq p < \infty$.

Hence

$$A \leq C\|u\|_{L^{p,k}(\mathbb{R}^n)}$$

and together with the estimate of B this implies (12) in the case $k - n/p$ is an integer and $2 \leq p < \infty$.

Proofs of items (13) **and** (14) follow along the lines of items (2) and (3) outlined in Section 2, *mutatis mutandis*.

Remark The transformations $P : u \to v = Pv$ and $Q : u \to w = Qu$ are projections, i. e. $P^2u = Pu$ and $Q^2u = Qu$. That Q is a projection follows directly from its definition or from (12) and the observation that $\|u - Qu\|_{L^{p,k}(\mathbb{R}^n)} = 0$.

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