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Second order nondifferentiable minimax fractional programming with square root terms

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Abstract. In this paper, we study a nondifferentiable minimax fractional programming problem under the assumptions of generalized second order $(\mathcal{F}, \alpha, \rho, d)-$convexity. A second order parametric dual is formulated. Weak, strong and converse duality theorems are established in order to relate the primal and dual problems under the afore-said assumptions.

1. Introduction

We consider the following nondifferentiable minimax fractional programming problem:

$$\text{(NFP)} \qquad \text{Minimize} \sup_{y \in Y} \frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}},$$
$$\text{subject to } h(x) \leq 0,$$

where Y is a compact subset of R^l, $f(.,.), g(.,.) : R^n \times R^l \to R$, and $h(.) : R^n \to R^m$ are twice differentiable functions. B and C are $n \times n$ positive semidefinite symmetric matrices. Throughout this paper, we assume that $g(x,y) - (x^TBx)^{1/2} > 0$ and $f(x,y) + (x^TBx)^{1/2} \geq 0$ for each (x,y) in $X \times Y$ where $X \subseteq R^n$.

Minimax mathematical programming and deriving the duality theorems for them have been of much interest in the recent past. Schmitendorf [19] presented necessary and sufficient optimality conditions for a minimax programming problem. Tanimoto [20] used these optimality conditions to construct a dual problem and discussed duality theorems, which were extended for the fractional analogue of generalized minimax problem by Yadav and Mukherjee [21]. Lai et al. [11] established necessary and sufficient optimality conditions for a nondifferentiable minimax fractional problem with generalized convexity and applied these optimality conditions to construct a parametric dual model and also discussed duality theorems. Lai and Lee [10] obtained duality theorems for two parameter-free dual models of nondifferentiable minimax

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fractional problem involving generalized convexity assumptions. Ahmad and Husain [3] derived necessary and sufficient optimality conditions for minimax fractional programming problems involving generalized convex functions.

Second order duality provides tighter bounds for the value of the objective function when approximations are used. For more datailts, the authors may consult ([16], pp. 93). One more advantage of second order duality when applicable is that if a feasible point in the primal is given and first order duality does not apply, then we can use second order duality to provide a lower bound of the value of the primal (see [9]).

Mangasarian [15] first formulated the second order dual for a nonlinear programming problem and discussed second order duality results under certain inequalities. Mond [16] reproved second order duality results assuming rather simple inequalities. Bector and Chandra [8] constructed a second order dual for a fractional programming problem and established usual duality results under the assumptions [16] by naming these as bonvex/boncave functions.

Inspired by Ahmad et al. [1, 5], Mond [17], and Zhang and Mond [22] , we formulate a second order parametric dual for a nondifferentiable minimax fractional programming problem involving square root terms of positive semidefinite quadratic forms. Second order duality results are proved by using the concept of second order generalized $(\mathcal{F}, \alpha, \rho, d)-$convexity. These results generalize a number of results [1, 4–6, 12–14] appeared in the literature.

2. Notation and preliminaries

Recall the definitions of sublinear functional and that of a unified formulation of generalized convexity, called generalized second order $(\mathcal{F}, \alpha, \rho, d)$-convexity from Ahmad and Husain [2]:

Definition 2.1. A functional $\mathcal{F} : X \times X \times R^n \mapsto R$ is said to be sublinear in its third argument, if for all $x, \bar{x} \in X$

(*i*) $\mathcal{F}(x, \bar{x}; a + b) \leq \mathcal{F}(x, \bar{x}; a) + \mathcal{F}(x, \bar{x}; b),\ \forall\, a, b \in R^n$;
(*ii*) $\mathcal{F}(x, \bar{x}; \beta a) = \beta\mathcal{F}(x, \bar{x}; a),\ \forall\, \beta \in R,\ \beta \geq 0,$ and $\forall\, a \in R^n$.

From (*ii*), it is clear that $\mathcal{F}(x, \bar{x}; 0) = 0$.

Let $\mathcal{F}$ be a sublinear functional; let $f : X \mapsto R$ be twice differentiable at $\bar{x} \in X$; and let $\rho \in R$.

Definition 2.2. A twice differentiable function f over X is said to be second order $(\mathcal{F}, \alpha, \rho, d)$-pseudoconvex at $\bar{x} \in X$, if there exist functions $\alpha : X \times X \mapsto R_+ \setminus \{0\}$, $d(.,.) : X \times X \mapsto R$ and a real number ρ such that for all $x \in X$ and for all $p \in R^n$,

$$\mathcal{F}(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f(\bar{x}) + \nabla^2 f(\bar{x})p\}) \geq -\rho d^2(x, \bar{x}) \Rightarrow f(x) \geq f(\bar{x}) - \frac{1}{2}p^T\nabla^2 f(\bar{x})p.$$

Definition 2.3. A twice differentiable function f over X is said to be second order strictly $(\mathcal{F}, \alpha, \rho, d)$-pseudoconvex at $\bar{x} \in X$, if there exist functions $\alpha : X \times X \mapsto R_+ \setminus \{0\}$, $d(.,.) : X \times X \mapsto R$ and a real number ρ such that for all $x \in X$ and for all $p \in R^n$,

$$\mathcal{F}(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f(\bar{x}) + \nabla^2 f(\bar{x})p\}) \geq -\rho d^2(x, \bar{x}) \Rightarrow f(x) > f(\bar{x}) - \frac{1}{2}p^T\nabla^2 f(\bar{x})p.$$

Definition 2.4. A twice differentiable function f over X is said to be second order $(\mathcal{F}, \alpha, \rho, d)$-quasiconvex at $\bar{x} \in X$, if there exist functions $\alpha : X \times X \mapsto R_+ \setminus \{0\}$, $d(.,.) : X \times X \mapsto R$ and a real number ρ such that for all $x \in X$ and for all $p \in R^n$,

$$f(x) \leq f(\bar{x}) - \frac{1}{2}p^T\nabla^2 f(\bar{x})p \Rightarrow \mathcal{F}(x, \bar{x}; \alpha(x, \bar{x})\{\nabla f(\bar{x}) + \nabla^2 f(\bar{x})p\}) \leq -\rho d^2(x, \bar{x}).$$

Let $S = \{x \in X : h(x) \leq 0\}$ denote the set of all feasible solutions of (NFP). Any point $x \in S$ is called the feasible point of (NFP). For each $(x, y) \in R^n \times R^l$, we define

$$\phi(x, y) = \frac{f(x, y) + (x^T Bx)^{1/2}}{g(x, y) - (x^T Cx)^{1/2}},$$

such that for each $(x, y) \in S \times Y$,

$f(x, y) + (x^T Bx)^{1/2} \geq 0$ and $g(x, y) - (x^T Cx)^{1/2} > 0$.

For each $x \in S$, we define

$$J(x) = \{j \in J : h_j(x) = 0\},$$

where

$$J = \{1, 2, \cdots, m\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x,y)+(x^T Bx)^{1/2}}{g(x,y)-(x^T Cx)^{1/2}} = \sup_{z \in Y} \frac{f(x,z)+(x^T Bx)^{1/2}}{g(x,z)-(x^T Cx)^{1/2}} \right\}.$$

$$K(x) = \Big\{ (s, t, \tilde{y}) \in N \times R^s_+ \times R^{ms} : 1 \leq s \leq n+1,\ t = (t_1, t_2, \cdots, t_s) \in R^s_+$$
$$\text{with } \sum_{i=1}^{s} t_i = 1, \tilde{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_s) \text{ with } \bar{y}_i \in Y(x) (i = 1, 2, \cdots, s) \Big\}.$$

Since f and g are continuously differentiable and Y is compact in R^l, it follows that for each $x^* \in S, Y(x^*) \neq \emptyset$, and for any $\bar{y}_i \in Y(x^*)$, we have a positive constant

$$k_\circ = \phi(x^*, \bar{y}_i) = \frac{f(x^*, \bar{y}_i) + (x^{*T} Bx^*)^{1/2}}{g(x^*, \bar{y}_i) - (x^{*T} Cx^*)^{1/2}}.$$

Generalized Schwartz Inequality

Let A be a positive-semidefinite matrix of order n. Then, for all, $x, w \in R^n$,

$x^T Aw \leq (x^T Ax)^{\frac{1}{2}} (w^T Aw)^{\frac{1}{2}}$.

Equality holds if for some $\lambda \geq 0$,

$Ax = \lambda Aw$.

Evidently, if $(w^T Aw)^{\frac{1}{2}} \leq 1$, we have

$x^T Aw \leq (x^T Ax)^{\frac{1}{2}}$.

If the functions f, g and h in problem (NFP) are continuously differentiable with respect to $x \in R^n$, then Lai et al. [11] derived the following first order necessary conditions for optimality of (NFP).

Theorem 2.1 (Necessary Conditions). If x^* is a solution of (NFP) satisfying $x^{*T} Bx^* > 0$, $x^{*T} Cx^* > 0$, and $\nabla h_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s, t^*, \bar{y}) \in K(x^*), k_\circ \in R_+, w, v \in R^n$, and $\mu^* \in R^m_+$ such that

$$\sum_{i=1}^{s} t_i^* \Big\{ \nabla f(x^*, \bar{y}_i) + Bw - k_\circ (\nabla g(x^*, \bar{y}_i) - Cv) \Big\} + \nabla \sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$$

$f(x^*, \bar{y}_i) + (x^{*T}Bx^*)^{\frac{1}{2}} - k_\circ \left(g(x^*, \bar{y}_i) - (x^{*T}Cx^*)^{\frac{1}{2}}\right) = 0,\ i = 1, 2, \cdots, s,$

$\sum_{j=1}^{m} \mu_j^* h_j(x^*) = 0,$

$t_i^* \geq 0\ (i = 1, 2, \cdots, s), \sum_{i=1}^{s} t_i^* = 1,$

$w^T Bw \leq 1,\ v^T Cv \leq 1,$

$(x^{*T}Bx^*)^{1/2} = x^{*T}Bw,$.

$(x^{*T}Cx^*)^{1/2} = x^{*T}Cv.$

3. Parametric nondifferentiable fractional duality

We formulate the following second order parametric dual associated to (NFP):

(NMD) $\max_{(s,t,\bar{y}) \in K(z)} \sup_{(z,\mu,k,v,w,p) \in H(s,t,\bar{y})} k,$

where $H(s, t, \bar{y})$ denotes the set of all $(z, \mu, k, v, w, p) \in R^n \times R^m_+ \times R_+ \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i[\nabla f(z, \bar{y}_i) + \nabla^2 f(z, \bar{y}_i)p + Bw - k\{\nabla g(z, \bar{y}_i) + \nabla^2 g(z, \bar{y}_i)p - Cv\}] - \sum_{j=1}^{m} [\nabla u_j^T h_j(z) - \nabla^2 u_j^T h_j(z)p] = 0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i[f(z, \bar{y}_i) - \frac{1}{2} p^T\ \nabla^2\ f(z, \bar{y}_i)p + z^T Bw - k\{g(z, \bar{y}_i) - \frac{1}{2} p^T\ \nabla^2\ g(z, \bar{y}_i)p - z^T Cv\}] \geq 0, \tag{3.2}$$

$$\sum_{j=1}^{m} \mu_j h_j(z) - \tfrac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j h_j(z) p \geq 0, \tag{3.3}$$

$$w^T Bw \leq 1, \tag{3.4}$$

$$v^T Cv \leq 1, \tag{3.5}$$

$$(s, t, \bar{y}) \in K(z). \tag{3.6}$$

If, for a triplet $(s, t, \bar{y}) \in K(z)$, the set $H(s, t, \bar{y}) = \emptyset$, then we define the supremum over it to be $-\infty$.

Let Z be the set of all feasible solutions of (NMD).

Theorem 3.1 (*Weak duality*). Let $x \in S$ and $(z, \mu, k, v, w, s, t, \bar{y}, p) \in Z$. If for each $(z, \mu, k, v, w, s, t, \bar{y}, p) \in Z$,

(*i*) $\sum_{j=1}^{m} \mu_j h_j(.)$ is second order $(\mathcal{F}, \alpha, \rho, d)$-quasiconvex at z,

(*ii*) $\sum_{i=1}^{s} t_i[f(., y_i) + (.)^T Bw - k\{g(., y_i) - (.)^T Cv\}]$ is second order $(\mathcal{F}, \alpha, \sigma, d)$-pseudoconvex at z; and

(*iii*) $\rho + \sigma \geq 0$,

then

$$\sup_{y\in Y}\frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}\geq k.$$

Proof. Suppose contrary to the result that

$$\sup_{y\in Y}\frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}< k.$$

Thus, we have

$$f(x,\bar{y}_i)+(x^TBx)^{1/2}-k(g(x,\bar{y}_i)-(x^TCx)^{1/2})<0,\ \text{for all } \bar{y}_i\in Y(x),\ i=1,2,\ldots,s.$$

It follows from $t_i\geq 0,\ i=1,2,\ldots,s$, that

$$t_i\left[f(x,\bar{y}_i)+(x^TBx)^{1/2}-k(g(x,\bar{y}_i)-(x^TCx)^{1/2})\right]\leq 0,$$

with at least one strict inequality, since $t=(t_1,t_2,\ldots,t_s)\neq 0$. Taking summation over i, we have

$$\sum_{i=1}^{s}t_i\left[f(x,\bar{y}_i)+(x^TBx)^{1/2}-k(g(x,\bar{y}_i)-(x^TCx)^{1/2})\right]\leqq 0,$$

which together with (3.2) and Schwartz inequality gives

$$\sum_{i=1}^{s}t_i(f(x,\bar{y}_i)+x^TBw-k(g(x,\bar{y}_i)-x^TCv))<0\leq\sum_{i=1}^{s}t_i[f(z,\bar{y}_i)-\tfrac{1}{2}p^T\,\nabla^2 f(z,\bar{y}_i)p+z^TBw$$
$$-k\{g(z,\bar{y}_i)-\frac{1}{2}p^T\,\nabla^2 g(z,\bar{y}_i)p-z^TCv\}]. \quad (3.7)$$

Using hypothesis (ii), we get

$$\mathcal{F}(x,z;\alpha(x,z)\sum_{i=1}^{s}t_i[\nabla f(z,\bar{y}_i)+\nabla^2 f(z,\bar{y}_i)p+Bw-k\{\nabla g(z,\bar{y}_i)+\nabla^2 g(z,\bar{y}_i)p-Cv\}])<-\sigma d^2(x,z). \quad (3.8)$$

Since $x\in S$ and $(z,\mu,k,v,w,s,t,\bar{y},p)\in Z$, we have

$$\sum_{j=1}^{m}\mu^Th_j(x)\leq 0\leq\sum_{j=1}^{m}(\mu^Th_j(z)-\frac{1}{2}p^T\nabla^2\mu^Th_j(z)p).$$

The second order $(\mathcal{F},\alpha,\rho,d)$-quasiconvexity of $\mu^Th(.)$ at z gives

$$\mathcal{F}(x,z;\alpha(x,z)\sum_{j=1}^{m}\{\nabla\mu^Th_j(z)+\nabla^2\mu^Th_j(z)p\})\leq-\rho d^2(x,z). \quad (3.9)$$

Adding (3.8) and (3.9), we get

$$\mathcal{F}(x,z;\alpha(x,z)\{\sum_{i=1}^{s}t_i[\nabla f(z,\bar{y}_i)+\nabla^2 f(z,\bar{y}_i)p+Bw-k\{\nabla g(z,\bar{y}_i)+\nabla^2 g(z,\bar{y}_i)p-Cv\}]-\sum_{j=1}^{m}[\nabla u_j^th_j(z)$$
$$-\nabla^2u_j^th_j(z)p]\})<-(\rho+\sigma)d^2(x,z)$$

Since $\alpha(x,z)>0$ and $\rho+\sigma\geq 0$, we have

$$\mathcal{F}(x,z;\sum_{i=1}^{s}t_i[\nabla f(z,\bar{y}_i)+\nabla^2 f(z,\bar{y}_i)p+Bw-k\{\nabla g(z,\bar{y}_i)+\nabla^2 g(z,\bar{y}_i)p-Cv\}]-\sum_{j=1}^{m}[\nabla u_j^Th_j(z)-\nabla^2u_j^Th_j(z)p])<0,$$

which contradicts (3.1), since $\mathcal{F}(x,z;0)=0$.

Theorem 3.2 (Strong duality). Assume that x^* is an optimal solution of (NFP) and $\nabla h_j(x^*), j \in J(x^*)$ is linearly independent. Then there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{p} = 0) \in H(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*, \bar{p} = 0)$ is a feasible solution of (NFD). Further, if the hypotheses of weak duality theorem (Theorem 3.1) are satisfied for all feasible solutions $(z, \mu, k, v, w, s, t, \bar{y}, \bar{p})$ of (NFD), then $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*, \bar{p} = 0)$ is an optimal solution of (NFD) and the two objectives have the same optimal values.

Proof. If x^* be an optimal solution of (NFP) and $\nabla h_j(x^*), j \in J(x^*)$ is linearly independent, then by Theorem 2.1, there exist $(\bar{s}, \bar{t}, \bar{y}^*) \in K(x^*)$ and $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{p} = 0) \in H(\bar{s}, \bar{t}, \bar{y}^*)$ such that $(x^*, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*, \bar{p} = 0)$ is feasible for (NFD) and problems (NFP) and (NFD) have the same objective values and

$$\bar{k} = \frac{f(x^*, \bar{y}_i^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}_i^*) - (x^{*T} C x^*)^{1/2}}.$$

The optimality of this feasible solution for (NFD) thus follows from Theorem 3.1.

Theorem 3.3 (Strict converse duality). Assume that x^* and $(\bar{z}, \bar{\mu}, \bar{k}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*, \bar{p} = 0)$ are the optimal solutions of (NFP) and (NFD), respectively and $\nabla h_j(x^*), j \in J(x^*)$ is linearly independent. Suppose that $\sum_{i=1}^{s} t_i\{f(., \bar{y}_i) + (.)^T Bw - \bar{k}\left(g(., \bar{y}_i) - (.)^T Cv\right)\}$ is second order srtictly $(\mathcal{F}, \alpha, \sigma, d)$-pseudoconvex at z and $\sum_{j=1}^{m} \mu_j h_j(.)$ is second order $(\mathcal{F}, \alpha, \rho, d)$-quasiconvex at z .Then $x^* = \bar{z}$, that is, $\bar{z}$ is an optimal point in (NFP) and

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + (\bar{z}^T B \bar{z})^{1/2}}{g(\bar{z}, \bar{y}^*) - (\bar{z}^T C \bar{z})^{1/2}} = \bar{k}.$$

Proof. We shall assume that $x^* \neq \bar{z}$ and reach a contradiction. From the strong duality theorem (Theorem 3.2), it follows that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}^*) - (x^{*T} C x^*)^{1/2}} = \bar{k}. \tag{3.10}$$

Now, we proceed similar as in the proof of Theorem 3.1, replacing x by x^* and $(z, \mu, k, v, w, s, t, \bar{y}, p = 0)$ by $(\bar{z}, \bar{\mu}, \bar{u}, \bar{v}, \bar{w}, \bar{s}, \bar{t}, \bar{y}^*, \bar{p} = 0)$, so that we we arrive at the strict inequality

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}^*) - (x^{*T} C x^*)^{1/2}} > k.$$

But this contradicts the fact

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + (x^{*T} B x^*)^{1/2}}{g(x^*, \bar{y}^*) - (x^{*T} C x^*)^{1/2}} = k = \bar{k},$$

and we conclude that $x^* = \bar{z}$. Hence, the proof of the theorem is complete.

4. Special cases

(*i*) If B = 0 and C = 0, then (NFP) and (NMD) reduce the problems discussed by Husain et al. [7].
(*ii*) Let B = 0 , C = 0 and p = 0. Then (NFP) and (NMD) reduce the problems discussed in [13, 14].
(*iii*) If $Y = \phi$, then (NFP) and (NMD) reduce to the following primal and dual problems:

(FP) Minimize $\frac{f(x,y)+(x^TBx)^{1/2}}{g(x,y)-(x^TCx)^{1/2}}$,

subject to $h(x) \leq 0$.

(MD) Maximize $K(z,u,v,w,z,p) = k$
!

subject to

$$[\nabla f(z) + \nabla^2 f(z)p + Bw] - k[\nabla g(z) + \nabla^2 g(z)p - Cv] - \nabla u^T h(z) - \nabla^2 u^T h(z)p = 0,$$

$$[f(z) - \tfrac{1}{2}p^T \nabla^2 f(z)p + z^T Bw] - k[g(z) - \tfrac{1}{2}p^T \nabla^2 g(z)p - z^T Cv] \leq 0,$$

$$u^T h(z) - \tfrac{1}{2}p^T \nabla^2 u^T h(z)p \geq 0,$$

$$w^T Bw \leq 1,$$

$$v^T Cv \leq 1,$$

$$u \geq 0, k \geq 0.$$

The above problems considered in [5] by changing maximize(minimize) by minimize (maximize).

5. Conclusion

It is well known [18] that second order duality is useful computationally as it provides a tighter lower bound. So it is very important to extend the validity of existing results to second order. As a follow up, the notion of second order $(\mathcal{F}, \alpha, \rho, d)$-convexity and its generalizations is adopted, which include many other generalized convexity concepts in mathematical programming as special cases. This concept is appropriate to discuss the weak, strong and strict converse duality theorems for a second order dual (NMD) of a nondifferentiable minimax fractional programming problem (NFP).

The question arise as to whether the second order fractional duality results developed in this paper hold for the following complex nondifferentiable minimax fractional problem:

Minimize $\sup\limits_{\nu \in W} \frac{Re[f(\xi,\nu)+(z^TBz)^{1/2}]}{Re[g(\xi,\nu)-(z^TCz)^{1/2}]}$,

subject to $-h(\xi) \in S, \xi \in C^{2n}$,

where $\xi = (z, \bar{z}), \nu = (\omega, \omega)$ for $z \in C^n, \omega \in C^l, f(.,.) : C^{2n} \times C^{2l} \to C$ and $g(.,.) : C^{2n} \times C^{2l} \to C$ are analytic with respect to ω, W ia a specified compact subset in C^{2l}, S is a polyhedral cone in C^m and $h : C^{2n} \to C^m$ is analytic. Also $B, C \in C^{n\times n}$ are positive semidefinite Hermitian matrices.

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