Filomat 27:1 (2013), 143-148
DOI 10.2298/FIL1301143C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Majorization for certain subclasses of analytic functions involving the generalized Noor integral operator

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Abstract. In the present investigation, we study the majorization properties for certain classes of multivalent analytic functions defined by using the generalized Noor integral operator. Moreover, we point out some new or known consequences of our main result.

1. Introduction

Let $\mathcal{A}_m(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=m}^{\infty} a_{p+k} z^{p+k} \quad (p, m \in \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. In particular $\mathcal{A}_1(p) \equiv \mathcal{A}(p)$ and $\mathcal{A}_1(1) \equiv \mathcal{A}$. Let $\mathcal{S}_p^*(\gamma)$ and $\mathcal{K}_p(\gamma)$ be the subclasses of $\mathcal{A}_m(p)$ consisting of all analytic functions which are, respectively, $p-$valently starlike and $p-$valently convex of order $\gamma (0 \leq \gamma < p)$. Also, we note that $\mathcal{S}_1^*(\gamma) \equiv \mathcal{S}^*(\gamma)$ and $\mathcal{K}_1(\gamma) \equiv \mathcal{K}(\gamma)$ are, respectively, the usual classes of starlike and convex functions of order $\gamma\, (0 \leq \gamma < 1)$ in $\mathcal{U}$. In special cases, $\mathcal{S}_1^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}_1 \equiv \mathcal{K}$ are, respectively, the familiar classes of starlike and convex functions in $\mathcal{U}$. Suppose that $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in $\mathcal{U}$ with $|w(z)| \leq |z|$ for all $z \in \mathcal{U}$, such that $g(z) = f(w(z))$, denoted $g \prec f$ or $g(z) \prec f(z)$. In case $f(z)$ is univalent in $\mathcal{U}$ we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathcal{U}) \subset f(\mathcal{U})$.

For functions $f_t \in \mathcal{A}_m(p)$ given by

$$f_t(z) = z^p + \sum_{k=m}^{\infty} a_{p+k,t} z^{p+k} \quad (t = 1, 2;\ p, m \in \mathbb{N}), \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50.
Keywords. (Analytic functions, Starlike functions, Hadamard product, Subordination, Majorization.)
Received: 3 August 2011; Accepted: 15 October 2011
Communicated by Prof. Dr. H. M. Srivastava
Research supported by Ataturk University Rectorship under BAP Project (The Scientific and Research Project of Ataturk University) Project no: 2010/28.
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we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=m}^{\infty} a_{p+k,1}a_{p+k,2}z^{p+k} = (f_2 * f_1)(z).$$

Let $f(z)$ and $g(z)$ be two analytic functions in $\mathcal{U}$. Then we say that the function $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$ (see [8]), and write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}), \tag{3}$$

if there exists a function $\varphi(z)$ analytic in $\mathcal{U}$, such that

$$\left|\varphi(z)\right| \leq 1 \text{ and } f(z) = \varphi(z)g(z) \quad (z \in \mathcal{U}). \tag{4}$$

It may be noted that the notion of majorization (3) is closely related to the concept of quasi-subordination between analytic functions in $\mathcal{U}$.

For real or complex numbers a, b, c not belonging to the set $\{0, -1, -2, ...\}$, the hypergeometric series is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ... \tag{5}$$

We note that the series in (5) converges absolutely for all $z \in \mathcal{U}$ so that it represents an analytic function in $\mathcal{U}$.

The authors (see [4]) introduced a function $(z^p{}_2F_1(a, b; c; z))^{(-1)}$ given by

$$(z^p{}_2F_1(a, b; c; z)) * (z^p{}_2F_1(a, b; c; z))^{(-1)} = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p), \tag{6}$$

which leads us to the following family of linear operators:

$$I^{\lambda}_{p,m}(a, b; c)f(z) = (z^p{}_2F_1(a, b; c; z))^{(-1)} * f(z) \tag{7}$$

where $f(z) \in \mathcal{A}_m(p)$, $a, b, c \in \mathbb{R}\backslash\mathbb{Z}^-_0 = \{0, -1, -2, ...\}$, $\lambda > -p$, $z \in \mathcal{U}$. It is evident that $I^1_{1,1}(n+1, c; c) = I_n$ is the Noor integral operator. The operator $I^{\lambda}_{p,1}(a, 1; c) = I^{\lambda}_p(a; c)$ was defined recently by Cho et al. [3], $I^1_{p,1}(n+p, c; c) = I_{n,p}$ was introduced by Liu and Noor [7] (see also [10]) and $I^{\lambda}_{p,1}(a, \lambda+p; c) = I_p(a; c)$ was investigated by Saitoh [11]. By some easy calculations we obtain

$$I^{\lambda}_{p,m}(a, b; c)f(z) = z^p + \sum_{k=m}^{\infty} \frac{(c)_k(\lambda+p)_k}{(a)_k(b)_k} a_{p+k}z^{p+k}, \tag{8}$$

where $(\kappa)_n$ denote the Pochhammer symbol defined by

$$(\kappa)_0 = 1 \text{ and } (\kappa)_n = \kappa(\kappa+1)...(\kappa+n-1), \quad n \in \mathbb{N}.$$

It is readily verified from the definition (7) that

$$I^0_{p,m}(a, p; a)f(z) = f(z) \text{ and } I^1_{p,m}(a, p; a)f(z) = \frac{zf'(z)}{p}, \tag{9}$$

$$z\left(I^{\lambda}_{p,m}(a, p+\lambda; a)f(z)\right)' = (\lambda+p)\, I^{\lambda+1}_{p,m}(a, b; c)f(z) - \lambda I^{\lambda}_{p,m}(a, b; c)f(z), \tag{10}$$

$$z\left(I^{\lambda}_{p,m}(a+1, b; c)f(z)\right)' = aI^{\lambda}_{p,m}(a, b; c)f(z) - (a-p)I^{\lambda}_{p,m}(a+1, b; c)f(z) \tag{11}$$

By using the linear operator $I^{\lambda}_{p,m}(a, b; c)f(z)$, we now define some subclasses of $\mathcal{A}_m(p)$ as follows:

Definition 1.1. *A function $f \in \mathcal{A}_m(p)$ is said to be in the class $\mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma;A,B)$, if and only if*

$$\frac{1}{p-\gamma}\left(\frac{z\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j+1)}}{\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j)}}-\gamma+j\right)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathcal{U}),\tag{12}$$

where $\lambda > -p$, $-1 \le B < A \le 1$, $0 \le \gamma < p$, $j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and the real numbers a,b,c are not belonging to the set $\{0,-1,-2,...\}$.

We note that $\mathcal{S}_{p,1}^{\lambda,0}(a,\lambda+p,a;\gamma;1,-1) \equiv \mathcal{S}_{p,1}^{0,0}(a,p,a;\gamma;1,-1) = \mathcal{S}_p^*(\gamma)$ and $\mathcal{S}_{p,1}^{1,0}(a,p,a;\gamma;1,-1) \equiv \mathcal{K}_p(\gamma)$ are the classes of p-valently starlike and p-valently convex functions of order γ in $\mathcal{U}$, respectively. In particular, we denote $\mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma;1,-1)$ by $\mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma)$. Furthermore, $\mathcal{S}_{1,1}^{1,0}(k+p,c,c;\gamma;A,B)$ is the class introduced and studied by Cho [2]; $\mathcal{S}_{p,1}^{1,0}(k+p,c,c;\gamma;A,B)$ is the class introduced and studied by Patel and Cho [10]; $\mathcal{S}_{p,1}^{\lambda,0}(\lambda+p,b,c;\gamma;A,B)$ is the class introduced and studied by Srivastava and Patel [12].

In [12], authors obtained their subordinate relations, inclusion relations, the integral preserving properties in connection with the operator $I_{p,m}^{\lambda}(a,b;c)f(z)$ the sufficient conditions for a function to be in the class $\mathcal{S}_{p,m}^{\lambda,0}(a,b,c;\gamma;A,B)$.

A mojorization problem for the class $\mathcal{S}^*$ have been investigated by MacGregor [8], and Altıntaş et al. [1] generalized this result for $p-$valent starlike functions of complex order. Recently, Goyal and Goswami [5] and Goswami et al. [6] extented these results for the fractional derivative operator and a multiplier transformation, respectively. In the present paper we investigate a majorization problem for the class $\mathcal{S}_{p,n}^{\lambda,j}(a,b,c;\gamma;A,B)$, and we give some special ceses of our main result obtained for approciate choices of the parameters $a,b,c,\gamma,A,B,j,\lambda,n$ and p.

2. Majorization problem for the class $\mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma;A,B)$

We begin by proving the following main result.

Theorem 2.1. *Let the function $f \in \mathcal{A}_m(p)$, and suppose that $g(z) \in \mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma;A,B)$. If $\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j)}$ is majorized by $\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)}$ in $\mathcal{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\left(I_{p,m}^{\lambda+1}(a,b;c)f(z)\right)^{(j)}\right| \le \left|\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}\right| \quad \text{for } |z| \le r_1,\tag{13}$$

where $r_1 = r_1(p,\gamma,\lambda,A,B)$ is the smallest positive root of the equation

$$\left|(\lambda+\gamma)B-(\gamma-p)A\right|r^3-(\lambda+p+2|B|)r^2-\left(\left|(\lambda+\gamma)B-(\gamma-p)A\right|+2\right)r+\lambda+p=0,\tag{14}$$

where $\lambda > -p$, $-1 \le B < A \le 1$, $0 \le \gamma < p$.

Proof. Since $g(z) \in \mathcal{S}_{p,m}^{\lambda,j}(a,b,c;\gamma;A,B)$, we find from (12) that

$$\frac{1}{p-\gamma}\left(\frac{z\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j+1)}}{\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)}}-\gamma+j\right)=\frac{1+Aw(z)}{1+Bw(z)},\tag{15}$$

where $w(z)$ is analytic in $\mathcal{U}$, with $w(0)=0$ and $|w(z)|<1$ for all $z\in\mathcal{U}$. From (15), we get

$$\frac{z\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j+1)}}{\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)}}=\frac{p-j+\left[\gamma(B-A)+Ap-Bj\right]w(z)}{1+Bw(z)}.\tag{16}$$

Now, making use of the relation

$$z\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j+1)} = (\lambda+p)\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)} - (\lambda+j)\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)} \tag{17}$$

from (16) we get

$$\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)} = \frac{(\lambda+p)(1+Bw(z))}{(\lambda+p)+[\lambda B+\gamma(B-A)+Ap]\,w(z)}\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}.$$

The above relation implies that

$$\left|\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)}\right| \leq \frac{(\lambda+p)(1+|B|\,|z|)}{(\lambda+p)-\left|(\lambda+\gamma)B-(\gamma-p)A\right||z|}\left|\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}\right|. \tag{18}$$

Since $\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j)}$ is majorized by $\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)}$ in $\mathcal{U}$, there exists an analytic function $\varphi(z)$ such that

$$\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j)} = \varphi(z)\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)} \quad (z\in\mathcal{U}) \tag{19}$$

and $\left|\varphi(z)\right| \leq 1$. Thus we have

$$z\left(I_{p,m}^{\lambda}(a,b;c)f(z)\right)^{(j+1)} = z\varphi'(z)\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)} + z\varphi(z)\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j+1)}. \tag{20}$$

Using (17), in the above equation, we get

$$\left(I_{p,m}^{\lambda+1}(a,b;c)f(z)\right)^{(j)} = \frac{z}{\lambda+p}\varphi'(z)\left(I_{p,m}^{\lambda}(a,b;c)g(z)\right)^{(j)} + \varphi(z)\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}. \tag{21}$$

Thus, by noting that the Schwarz function $\varphi(z)$ satisfies the inequality (see, e.g. Nehari [9])

$$\left|\varphi'(z)\right| \leq \frac{1-\left|\varphi(z)\right|^2}{1-|z|^2} \quad (z\in\mathcal{U}), \tag{22}$$

and using (18) and (22) in (21), we get

$$\begin{aligned}&\left|\left(I_{p,m}^{\lambda+1}(a,b;c)f(z)\right)^{(j)}\right| \\ &\leq \left(\left|\varphi(z)\right| + \frac{1-\left|\varphi(z)\right|^2}{1-|z|^2}\,\frac{|z|(1+|B|\,|z|)}{(\lambda+p)-\left|(\lambda+\gamma)B-(\gamma-p)A\right||z|}\right)\left|\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}\right|,\end{aligned} \tag{23}$$

which upon setting

$$|z| = r \text{ and } \left|\varphi(z)\right| = \rho \quad (0\leq\rho\leq1)$$

leads us to the inequality

$$\left|\left(I_{p,m}^{\lambda+1}(a,b;c)f(z)\right)^{(j)}\right| \leq \frac{\Theta(\rho)}{(1-r^2)\left[(\lambda+p)-\left|(\lambda+\gamma)B-(\gamma-p)A\right|r\right]}\left|\left(I_{p,m}^{\lambda+1}(a,b;c)g(z)\right)^{(j)}\right|, \tag{24}$$

where

$$\Theta(\rho) = -r(1+|B|r)\rho^2 + \left(1-r^2\right)\left[(\lambda+p)-\left|(\lambda+\gamma)B-(\gamma-p)A\right|r\right]\rho + r(1+|B|r)$$

takes its maximum value at $\rho = 1$ with $r = r_1(p, \gamma, \lambda, A, B)$ the smallest positive root of the equation (14). Furthermore, if $0 \le \sigma \le r_1(p, \gamma, \lambda, A, B)$, then the function

$$\Phi(\rho) = -\sigma(1 + |B|\sigma)\rho^2 + \left(1 - \sigma^2\right)\left[(\lambda + p) - \left|(\lambda + \gamma)B - (\gamma - p)A\right|\sigma\right]\rho + \sigma(1 + |B|\sigma)$$

increases in the interval $0 \le \rho \le 1$, so that $\Phi(\rho)$ does not exceed

$$\Phi(1) = \left(1 - \sigma^2\right)\left[(\lambda + p) - \left|(\lambda + \gamma)B - (\gamma - p)A\right|\sigma\right].$$

Therefore, from this fact, (24) gives the inequality (13). □

Setting $A = 1$ and $B = -1$ in Theorem 2.1, equation (14) becomes

$$\left|2\gamma + \lambda - p\right|r^3 - (\lambda + p + 2)r^2 - \left(\left|2\gamma + \lambda - p\right| + 2\right)r + \lambda + p = 0. \tag{25}$$

We see that $r = -1$ is one of the roots of this equation, and the other two roots are given by

$$\left|2\gamma + \lambda - p\right|r^2 - \left(\left|2\gamma + \lambda - p\right| + \lambda + p + 2\right)r + \lambda + p = 0,$$

so we can easily find the smallest positive root of (25). Hence, we have the following result:

Corollary 2.2. *Let the function $f(z) \in \mathcal{A}_m(p)$ and suppose that $g(z) \in \mathcal{S}_{p,m}^{\lambda,j}(a, b, c; \gamma; 1, -1)$. If $\left(I_{p,m}^{\lambda}(a, b; c)f(z)\right)^{(j)}$ is majorized by $\left(I_{p,m}^{\lambda}(a, b; c)g(z)\right)^{(j)}$ in $\mathcal{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\left(I_{p,m}^{\lambda+1}(a, b; c)f(z)\right)^{(j)}\right| \le \left|\left(I_{p,m}^{\lambda+1}(a, b; c)g(z)\right)^{(j)}\right| \quad \text{for } |z| \le r_1,$$

where

$$r_1 = \frac{\delta - \sqrt{\delta^2 - 4\left|2\gamma + \lambda - p\right|(\lambda + p)}}{2\left|2\gamma + \lambda - p\right|}$$

with $\delta = \left|2\gamma + \lambda - p\right| + \lambda + p + 2$, $\lambda > -p$, $0 \le \gamma < p$.

As a special case of Corollary 2.2, when $b = 1$ and $m = 1$ we obtain the following result for the operator Cho-Kwon-Srivastava $I_p^{\lambda}(a; c)f(z)$:

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}(p)$ and suppose that $g(z) \in \mathcal{S}_{p,1}^{\lambda,j}(a, 1, c; \gamma; 1, -1)$. If $\left(I_p^{\lambda}(a; c)f(z)\right)^{(j)}$ is majorized by $\left(I_p^{\lambda}(a; c)g(z)\right)^{(j)}$ in $\mathcal{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\left(I_p^{\lambda+1}(a; c)f(z)\right)^{(j)}\right| \le \left|\left(I_p^{\lambda+1}(a; c)g(z)\right)^{(j)}\right| \quad \text{for } |z| \le r_1,$$

where

$$r_1 = \frac{\delta - \sqrt{\delta^2 - 4\left|2\gamma + \lambda - p\right|(\lambda + p)}}{2\left|2\gamma + \lambda - p\right|}$$

with $\delta = \left|2\gamma + \lambda - p\right| + \lambda + p + 2$, $\lambda > -p$, $0 \le \gamma < p$.

For $\lambda = \gamma = 0$, $a = c$ and $j = p = 1$ Corollary 2.3 reduces to the following result:

Corollary 2.4. *[8] Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in $\mathcal{U}$ and suppose that $g(z) \in \mathcal{S}^*$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \quad \text{for } |z| \le 2 - \sqrt{3}.$$

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