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Remarks on neighborhood star-Lindelöf spaces

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Abstract. A space X is said to be *neighborhood star-Lindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$. In this paper, we continue to investigate the relationship between neighborhood star-Lindelöf spaces and related spaces, and study topological properties of neighborhood star-Lindelöf spaces.

1. Introduction

By a space, we mean a topological space. In the rest of this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x, \mathcal{U})$ for $St(\{x\}, \mathcal{U})$.

Recall that a space X is *strongly starcompact* (see [5,7,8] - under different name) if for every open cover $\mathcal{U}$ of X there exists a finite subset A of X such that $X = St(A, \mathcal{U})$; A space X is *strongly star-Lindelöf* (see [1, 2, 5, 8, 9] - under different name) if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that $X = St(A, \mathcal{U})$; A space X is *starcompact* (resp., *star-Lindelöf*)(see [5, 8] - under different name) if for every open cover $\mathcal{U}$ of X there exists a finite (resp., countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = St(\bigcup \mathcal{V}, \mathcal{U})$. Clearly, every strongly starcompact space is strongly star-Lindelöf, every strongly starcompact space starcompact, every strongly star-Lindelöf space is star-Lindelöf and every strongly star-Lindelöf space is star-Lindelöf. It is known that every countably compact space is strongly starcompact, and every Hausdorff strongly starcompact space is countably compact (see [5, 8]).

It is natural in this context to introduce the following definitions:

Definition 1.1. ([3]) A space X is said to be *weakly starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$.

Definition 1.2. ([4]) A space X is said to be *neighborhood star-Lindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$.

From the definitions, it is clear that every weakly starcompact space is neighborhood star-Lindelöf, every strongly star-Lindelöf space is neighborhood star-Lindelöf space and every neighborhood star-Lindelöf space is star-Lindelöf.

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The purpose of this note is to investigate the relationship between neighborhood star-Lindelöf spaces and related spaces, and study topological properties of neighborhood star-Lindelöf spaces.

Throughout this paper, let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \le \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \le \gamma \le \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [6].

2. Neighborhood star-Lindelöf spaces and related spaces

In this section, we give some examples to clarify the relationship between neighborhood star-Lindelöf spaces and related spaces. Recall that a space is called *Urysohn* if every two distinct points have neighborhood with disjoint closures. Clearly, the property is between the Hausdorff condition and regularity. Bonanzinga et al. in [3] showed that the three properties, countable compactness, strongly starcompactness, and weak starcompactness are equivalent for Urysonn spaces.

Example 2.1. *There exists a Tychonoff neighborhood star-Lindelöf space X that is not weakly starcompact.*

Proof. Let $X = \omega$ be the countably infinite discrete space. Clearly, X is not weakly starcompact. Since X is countable, then X is strongly star-Lindelöf, hence X is neighborhood star-Lindelöf, since every strongly star-Lindelöf space is neighborhood star-Lindelöf. □

For the next example, we need the following Lemmas.

Lemma 2.2. *A space X having a dense Lindelöf subspace is star-Lindelöf.*

Proof. Let X have a dense Lindelöf subspace D. We show that X is star-Lindelöf. Let $\mathcal{U}$ be an open cover of X. Since D is a dense Lindelöf subset of X. Then there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $D \subseteq \bigcup \mathcal{V}$. Hence $St(\bigcup \mathcal{V}, \mathcal{U}) = X$, which shows that X is star-Lindelöf. □

Lemma 2.3. ([4]) *A space X is neighborhood star-Lindelöf if and only if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that $\overline{St(x, \mathcal{U})} \cap A \ne \emptyset$ for each $x \in X$.*

Example 2.4. *There exists a Tychonoff star-Lindelöf space that is not neighborhood star-Lindelöf.*

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{y_\infty\}$ be one-point compactification of D.

Let

$$X = (Y \times [0, \omega)) \cup (D \times \{\omega\})$$

be the subspace of the product space $Y \times [0, \omega]$. Then that X is star-Lindelöf by lemma 2.2, since $Y \times [0, \omega)$ is a dense Lindelöf subset of X.

Next we show that X is not neighborhood star-Lindelöf. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{d_\alpha\} \times [0, \omega].$$

Then

$$U_\alpha \cap U_{\alpha'} = \emptyset \text{ for } \alpha \ne \alpha'.$$

Let

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{Y \times [0, \omega)\}.$$

Then $\mathcal{U}$ is an open cover of X. Let us consider the open cover $\mathcal{U}$ of X. It suffices to show that for any countable subset A of X, there exists a point $x \in X$ such that $\overline{St(x, \mathcal{U})} \cap A = \emptyset$ by Lemma 2.3. Let A be any countable subset of X. Then $\{\alpha : A \cap U_\alpha \ne \emptyset\}$ is countable. Pick $\alpha_0 < \mathfrak{c}$ such that $A \cap U_{\alpha_0} = \emptyset$. Since U_{α_0} is the only element of $\mathcal{U}$ containing the point $\langle d_{\alpha_0}, \omega\rangle$, then $St(\langle d_{\alpha_0}, \omega\rangle, \mathcal{U}) = U_{\alpha_0}$. By the constructions of the topology of X and the open cover $\mathcal{U}$, we have $\overline{St(\langle d_{\alpha_0}, \omega\rangle, \mathcal{U})} = U_{\alpha_0}$. Thus we complete the proof. □

Remark 2.5. Bonanzinga et al. in [4] showed that there exists a Urysohn neighborhood star-Lindelöf space that is not strongly star-Lindelöf. But the author does not know if there exists a Tychonoff example.

3. Properties of neighborhood star-Lindelöf spaces

In this section, we study topological properties of neighborhood star-Lindelöf spaces. The Isbell-Mrówka space is $X = \omega \cup \mathcal{R}$ (see [10]), where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. The space X is strongly star-Lindelöf, since ω is a countable dense subset of X. Thus X is neighborhood star-Lindelöf. The space X shows that a closed subset of a Tychonoff neighborhood star-Lindelöf space X need not be neighborhood star-Lindelöf, since $\mathcal{R}$ is a discrete closed subset of cardinality $\mathfrak{c}$. Now we give a stronger example showing that a regular-closed subset of a Tychonoff neighborhood star-Lindelöf space X need not be neighborhood star-Lindelöf. Here a subset A of a space X is said to be *regular-closed* in X if $cl_X int_X A = A$.

Example 3.1. *There exists a Tychonoff neighborhood star-Lindelöf space having a regular-closed subspace which is not neighborhood star-Lindelöf.*

Proof. Let S_1 be the same space X in the proof of Example 2.4. Then S_1 is Tychonoff, not neighborhood star-Lindelöf.

Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Let

$$S_2 = \mathcal{R} \cup ([0, \mathfrak{c}^+) \times \omega).$$

We topologize S_2 as follows: $[0, \mathfrak{c}^+) \times \omega$ has the usual product topology and is an open subspace of X, and a basic neighborhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}^+\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}^+$ and a finite subset K of ω. To show that S_2 is neighborhood star-Lindelöf. We need only show that S_2 is strongly star-Lindelöf, since every strongly star-Lindelöf space is neighborhood star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of S_2. For each $n \in \omega$, since $[0, \mathfrak{c}^+) \times \{n\}$ is countably compact, there exists a finite subset $F_n \subseteq [0, \mathfrak{c}^+) \times \{n\}$ such that

$$[0, \mathfrak{c}^+) \times \{n\} \subseteq St(F_n, \mathcal{U}).$$

Let $F' = \bigcup_{n\in\omega} F_n$. Then

$$[0, \mathfrak{c}^+) \times \omega \subseteq St(F', \mathcal{U}).$$

On the other hand, for each $r \in \mathcal{R}$, take $U_r \in \mathcal{U}$ with $r \in U_r$, and fix $\alpha_r < \mathfrak{c}^+$ and $n_r \in r$ such that

$$\{\langle \alpha, n_r \rangle : \alpha_r < \alpha < \mathfrak{c}^+\} \subseteq U_r.$$

For each $n \in \omega$, let

$$\mathcal{R}_n = \{r \in \mathcal{R} : n_r = n\} \text{ and } \alpha'_n = \sup\{\alpha_r : r \in \mathcal{R}_n\}.$$

Then $\alpha'_n < \mathfrak{c}$, since $|\mathcal{R}_n| \leq \mathfrak{c}$. Pick $\alpha_n > \alpha'_n$. Then $\mathcal{R}_n \subseteq St(\langle \alpha_n, n \rangle, \mathcal{U})$. Thus, if we put $F'' = \{\langle \alpha_n, n \rangle : n \in \omega\}$, then $\mathcal{R} \subseteq St(F'', \mathcal{U})$. Let $F = F' \cup F''$. Then F is a countable subset of S_2 such that $S_2 = St(F, \mathcal{U})$, which completes the proof.

We assume $S_1 \cap S_2 = \emptyset$. Let $\pi : D \times \{\omega\} \to \mathcal{R}$ be a bijection. Let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ obtained by identifying $\langle d_\alpha, \omega \rangle$ of S_1 with $\pi(\langle d_\alpha, \omega \rangle\})$ of S_2 for every $\alpha < \mathfrak{c}$. Let $\varphi : S_1 \oplus S_2 \to X$ be the quotient map. It is clear that $\varphi(S_1)$ is a regular-closed subspace of X which is not neighborhood star-Lindelöf, since it is homeomorphic to S_1.

Finally we show that X is neighborhood star-Lindelöf. We need only show that X is strongly star-Lindelöf. To this end, let $\mathcal{U}$ be an open covers of X. Since $\varphi(S_2)$ is homeomorphic to S_2, then $\varphi(S_2)$ is strongly star-Lindelóf, there exists a countable subset $F' \subseteq \varphi(S_2)$ such that

$$\varphi(S_2) \subseteq St(F_1, \mathcal{U}).$$

On the other hand, for each $n \in \omega$, since $\varphi(Y \times \{n\})$ is homeomorphic to $Y \times \{n\}$, then $\varphi(Y \times \{n\})$ is compact, we can find a finite subset $F_n \subseteq \varphi(Y \times \{n\})$ such that

$$\varphi(Y \times \{n\}) \subseteq St(F_n, \mathcal{U}).$$

Let $F = F' \cup \bigcup_{n\in\omega} F_n$. Then F is a countable subset of X such that $X = St(F, \mathcal{U})$, which completes the proof. $\square$

It is known that a continuous image of a strongly star-Lindelöf space is strongly star-Lindelöf. Similarly, we show that neighborhood star-Lindelöfness is preserved by continuous mappings.

Theorem 3.2. *A continuous image of a neighborhood star-Lindelöf space is neighborhood star-Lindelöf.*

Proof. Let $f : X \to Y$ be a continuous mapping from a neighborhood star-Lindelöf space X onto a space Y. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X. Since X is neighborhood star-Lindelöf, there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, f^{-1}(\mathcal{U}))$. Then $f(A)$ is a countable subset of Y such that for every open $W \supseteq f(A)$, $Y = St(W, \mathcal{U})$. In fact, let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Let W be an open subset of Y such that $f(A) \subseteq W$. Then $f^{-1}(W)$ is an open subset of X such that $A \subseteq f^{-1}(W)$, $St(f^{-1}(W), f^{-1}(\mathcal{U})) = X$, Hence there exists $U \in \mathcal{U}$ such that $x \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(W) \neq \emptyset$. Thus $y = f(x) \in f(f^{-1}(U)) = U$ and $U \cap W \neq \emptyset$. This means that $y \in St(W, \mathcal{U})$. $\square$

Next we turn to consider preimages. To show that the preimage of a neighborhood star-Lindelöf space under a closed 2-to-1 continuous map need not be neighborhood star-Lindelöf, we use the the Alexandorff duplicate $A(X)$ of a space X. The underlying set $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 0\rangle\})$, where U is a neighborhood of x in X.

Example 3.3. *There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is a neighborhood star-Lindelöf space, but X is not neighborhood star-Lindelöf.*

Proof. Let Y be the same space S_2 in the proof of Example 3.1. As we proved in Example 3.1 above, Y is neighborhood star-Lindelöf. Let X be the Alexandorff duplicate $A(Y)$ of the space Y. Then X is not neighborhood star-Lindelöf. In fact, let $A = \{\langle r, 1\rangle : r \in \mathcal{R}\}$. Then A is an open and closed subset of X with $|A| = \mathfrak{c}$, and each point $\langle r, 1\rangle$ is isolated. Hence $A(X)$ is not neighborhood star-Lindelöf, since every open and closed subset of a neighborhood star-Lindelöf space is neighborhood star-Lindelöf and A is not neighborhood star-Lindelöf. Let $f : X \to Y$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof. $\square$

Example 3.4. *There exist a neighborhood star-Lindelöf space X and a compact space X such that $X \times Y$ is not neighborhood star-Lindelöf .*

Proof. Let $X = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [10], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Then X is neighborhood star-Lindelöf.

Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{y_\infty\}$ be the one-point compactification of D.

We show that $X \times Y$ is not neighborhood star-Lindelöf. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$.

Let

$$U_n = \{n\} \times Y \text{ for each } n \in \omega,$$

$$V_\alpha = X \times \{d_\alpha\} \text{ for each } \alpha < \mathfrak{c}$$

and

$$W_\alpha = (\{r_\alpha\} \cup \omega) \times (Y \setminus \{d_\alpha\}) \text{ for each } \alpha < \mathfrak{c}.$$

Let

$$\mathcal{U} = \{U_n : n \in \omega\} \cup \{V_\alpha : \alpha < \mathfrak{c}\} \cup \{W_\alpha : \alpha < \mathfrak{c}\}.$$

Then $\mathcal{U}$ is an open cover of $X \times Y$. Observe that $\langle r_\alpha, d_\alpha\rangle \in U \in \mathcal{U}$ if and only if $U = V_\alpha$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. It suffices to show that for any countable subset A of $X \times Y$, there exists a point $\langle x, y\rangle \in X \times Y$ such that $\overline{St(\langle x, y\rangle, \mathcal{U})} \cap A = \emptyset$ by Lemma 2.3. Let A be any countable subset of $X \times Y$. Then there exists $\alpha < \mathfrak{c}$ such that $A \cap V_\alpha = \emptyset$. Since V_α is the only element of $\mathcal{U}$ containing the point $\langle r_\alpha, d_\alpha\rangle$, then $St(\langle r_\alpha, d_\alpha\rangle, \mathcal{U}) = V_\alpha$. By the constructions of the topology of $X \times Y$ and the open cover $\mathcal{U}$, we have $\overline{St(\langle r_\alpha, d_\alpha\rangle, \mathcal{U})} = V_\alpha$, which shows that $X \times Y$ is not neighborhood star-Lindelöf. Thus we complete the proof. □

Remark 3.5. Example 3.4 shows that the preimage of a neighborhood star-Lindelöf space under an open perfect map need not be neighborhood star-Lindelöf.

The following well-known example shows that the product of two countably compact(and hence neighborhood star-Lindelöf) spaces need not be neighborhood star-Lindelöf. Here we give the proof roughly for the sake of completeness. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 3.6. *There exist two countably compact spaces X and Y such that $X \times Y$ is not neighborhood star-Lindelöf.*

Proof. Let D be a discrete space of cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha<\omega_1} E_\alpha$ and $Y = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of βD which are defined inductively so as to satisfy the following conditions (1),(2) and (3):

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;

(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;

(3) every infinite subset of E_α(resp., F_α) has an accumulation point in $E_{\alpha+1}$(resp.,$F_{\alpha+1}$).

These sets E_α and F_α are well-defined since every infinite closed set in βD has cardinality $2^{\mathfrak{c}}$ (see [11]). Then $X \times Y$ is not neighborhood star-Lindelöf, because the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete open and closed subset of $X \times Y$ with cardinality $\mathfrak{c}$ and the open and closed subsets of neighborhood star-Lindelöf spaces are neighborhood star-Lindelöf. □

In [5, Example 3.3.3], van Douwen-Reed-Roscoe-Tree gave an example showing that there exist a countably compact space X and a Lindelöf space Y such that $X \times Y$ is not strongly star-Lindelöf. Now, we shall show that the product space $X \times Y$ is not neighborhood star-Lindelöf.

Example 3.7. *There exist a countably compact (and neighborhood star-Lindelöf) space X and a Lindelöf space Y such that $X \times Y$ is not neighborhood star-Lindelöf.*

Proof. Let $X = [0, \omega_1)$ with the usual order topology. Then X is countably compact. Let $Y = [0, \omega_1]$ with the following topology: each point α with $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then Y is Lindelöf.

Now, we show that $X \times Y$ is not neighborhood star-Lindelöf. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha] \times [\alpha, \omega_1] \text{ and } V_\alpha = (\alpha, \omega_1) \times \{\alpha\}.$$

Then

$$V_\alpha \cap V'_\alpha = \emptyset \text{ if } \alpha \neq \alpha' \text{ and } U_\alpha \cap V_\beta = \emptyset \text{ for any } \alpha < \mathfrak{c}, \beta < \mathfrak{c}.$$

let

$$\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$

Then $\mathcal{U}$ is an open cover of $X \times Y$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. It suffices to show that for any countable subset A of $X \times Y$, there exists a point $\langle x, y\rangle \in X \times Y$ such that $\overline{St(\langle x, y\rangle, \mathcal{U})} \cap A = \emptyset$ by Lemma 2.3. Let A be any countable subset of $X \times Y$. Then there exists $\alpha < \mathfrak{c}$ such that $A \cap V_\alpha = \emptyset$. Since V_α is the only element of $\mathcal{U}$ containing the point $\langle \alpha + 1, \alpha\rangle$, then $St(\langle \alpha + 1, \alpha\rangle, \mathcal{U}) = V_\alpha$ and $V_\alpha \cap A = \emptyset$. By the constructions of the topology of X and the open cover $\mathcal{U}$, we have $\overline{St(\langle \alpha + 1, \alpha\rangle, \mathcal{U})} = V_\alpha$, which shows that $X \times Y$ is not neighborhood star-Lindelöf. Thus we complete the proof. □

Now we give some conditions under which neighborhood star-Lindelöfness implies strongly star-Lindelöfness. Recall that a space X is *paraLindelöf* if every open cover $\mathcal{U}$ of X has a locally countable open refinement.

Theorem 3.8. *Every paraLindelöf neighborhood star-Lindelöf space is Lindelöf (hence star-Lindelöf).*

Proof. Let X be a paraLindelöf neighborhood star-Lindelöf space and $\mathcal{U}$ be an open cover of X. Then there exists a locally countable open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists an open neighborhood V_x of x such that $V_x \subseteq V$ for some $V \in \mathcal{V}$ and $\{V \in \mathcal{V} : V_x \cap V \neq \emptyset\}$ is countable. Let $\mathcal{V}' = \{V_x : x \in X\}$. Then $\mathcal{V}'$ is an open refinement of $\mathcal{V}$. Since X is neighborhood star-Lindelöf, there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{V})$.

Let

$$O = \bigcup\{V_x \in \mathcal{V}' : x \in A\}.$$

Then O is an open subset of X and $A \subseteq O$. Thus $St(O, \mathcal{V}) = X$.

Let

$$\mathcal{V}'' = \{V \in \mathcal{V} : V \cap O \neq \emptyset\}.$$

Then $\mathcal{V}''$ is a countable open cover of X. For each $V \in \mathcal{V}''$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{V}''\}$ is a countable subcover of $\mathcal{U}$, which shows that X is Lindelöf. Thus we complete the proof. □

Since every strongly star-Lindelöf space is neighborhood star-Lindelöf, the following corollary follows from Theorem 3.8.

Corollary 3.9. *A paraLindelöf space X is neighborhood star-Lindelöf iff X is strongly star-Lindelöf.*

Since every paracompact space is paraLindelöf, the following Corollary follows from Corollary 3.9.

Corollary 3.10. *A paracompact space X is neighborhood star-Lindelöf iff X is strongly star-Lindelöf.*

Recall that a space X is *locally separable* if x has a separable neighborhood at every point $x \in X$.

Theorem 3.11. *Every locally separable neighborhood star-Lindelöf space is star-Lindel-öf .*

Proof. Let X be a locally separable neighborhood star-Lindelöf space and $\mathcal{U}$ be an open cover of X. For each $x \in X$, there exists an open separable subspace V_x of X such that $x \in V_x \subseteq U$ for some $U \in \mathcal{U}$, since X is locally separable. Let $\mathcal{V} = \{V_x : x \in X\}$. Then $\mathcal{V}$ is an open cover of X. Since X is neighborhood star-Lindelöf, there exists a countable subset A of X such that for every open $O \supseteq A$, $X = St(O, \mathcal{U})$.

Let

$$O = \bigcup\{V_x \in \mathcal{V} : x \in A\}.$$

Then O is an open subset of X and $A \subseteq O$. Thus $St(O, \mathcal{U}) = X$. For each $x \in A$, since V_x is separable, there exists a countable dense subset D_x of V_x.

Let

$$F = \bigcup\{D_x : x \in A\}.$$

Then F is a countable subset of X and $St(F, \mathcal{U}) = X$, which shows that X is star-Lindelöf. □

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