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On some types of convergence of sequences of functions in ideal context

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Abstract. In this paper we consider the notion of $\mathcal{I}^*$-uniform equal convergence introduced by Das, Dutta and Pal [15] and two related notions of convergence, namely, $\mathcal{I}^*$-uniform discrete and $\mathcal{I}^*$-strong uniform equal convergence. We then investigate some lattice properties of $\Phi^{\mathcal{I}^*-ue}$, $\Phi^{\mathcal{I}^*-ud}$ and $\Phi^{\mathcal{I}^*-sue}$, the classes of all functions defined on a non-empty set X, which are $\mathcal{I}^*$-uniform equal limits, $\mathcal{I}^*$-uniform discrete limits and $\mathcal{I}^*$-strong uniform equal limits of sequences of functions belonging to a class of functions Φ respectively.

1. Introduction

The concept of convergence of a sequence of real numbers had been extended to statistical convergence independently by Fast [17], Steinhaus [30] and Schoenberg [29]. A lot of developments have been made on this interesting notion of convergence and related areas after the pioneering works of Šalát [28] and Fridy [18]. The concept of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et. al.[20] as a generalization of statistical convergence using the notion of ideals. In [20], the concept of $\mathcal{I}^*$-convergence was also introduced and a detailed study was carried out to explore its relation with $\mathcal{I}$-convergence. For the last ten years several works have been done on $\mathcal{I}$-convergence (see for example [10–13, 22–24]). Recently some significant investigations have been done on sequences of real functions by using the idea of statistical and $\mathcal{I}$-convergence (see [2, 5, 6, 15, 21, 25]).

On the other hand in [8], Császár and Laczkovich introduced two new types of convergence of sequences of real valued functions under the name of Equal convergence and Discrete convergence(see also [7, 9]) and studied the lattice properties of these classes of functions. Later Bukovská [3] also studied equal convergence under the name of Quasi-normal convergence. In [26], Papanastassiou defined and studied the notions of uniform equal convergence, uniform discrete convergence and strong uniform equal convergence for sequences of real valued functions. Later Das and Papanastassiou [16] studied several properties of these classes of functions, in particular lattice properties following the line of investigation of [8]. Very recently the above notion of equal convergence was generalized using ideals and the notion of $\mathcal{I}^*$-uniform equal convergence of sequences of real valued functions was introduced by Das, Dutta and Pal [15].

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In the present paper we consider the notion of $\mathcal{I}^*$-uniform equal convergence and introduce two related notions of convergence, namely, $\mathcal{I}^*$-uniform discrete convergence and $\mathcal{I}^*$-strong uniform equal convergence which is stronger than $\mathcal{I}^*$-uniform equal convergence for sequence of real valued functions. We then investigate some lattice properties of these classes of functions mainly following the line of investigation of [8] and [16].

2. Preliminaries

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; $(ii) A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$), then the family of sets $F(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$.

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ [20]. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in F(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim\limits_{k\to\infty} x_{m_k} = x$ [20]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-divergent to ∞ or $-\infty$ if for any positive real number G, $\{n \in \mathbb{N} : x_n \leq G\} \in \mathcal{I}$ or $\{n \in \mathbb{N} : x_n \geq -G\} \in \mathcal{I}$ [24] (though in [24] the terms $\mathcal{I}$-convergent to $+\infty$ and $\mathcal{I}$-convergent to $-\infty$ were used).

We now recall the following types of convergence introduced in [8] which we generalized using the notion of ideals in [15]. Let X be a non-empty set and let f, f_n, $n = 1, 2, 3, ...$ be real valued functions defined on X. f is called the discrete limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if for every $x \in X$, there exists $n_0 = n_0(x)$ such that $f(x) = f_n(x)$ for $n \geq n_0$. The terminology is motivated by the fact that this condition means precisely the convergence of the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ to $f(x)$ with respect to the discrete topology of the real line. f is said to be the equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ tending to zero such that for every $x \in X$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq n_0$.

We say that f is the $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim\limits_{n\to\infty} \varepsilon_n = 0$ such that for any $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. f is said to be the $\mathcal{I}^*$-equal limit of $\{f_n\}_{n\in\mathbb{N}}$ if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in F(\mathcal{I})$ such that for all $x \in X$, $f(x)$ is the equal limit of the subsequence $\{f_{m_k}(x)\}_{k\in\mathbb{N}}$.

We also recall the following ideas of convergence of a sequence of functions from [2]. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions is said to be $\mathcal{I}$-pointwise convergent to f if for all $x \in X$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $f(x)$ and in this case we write $f_n \xrightarrow{\mathcal{I}} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for all $n \in A^c$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. f is said to be the $\mathcal{I}^*$-uniform limit of $\{f_n\}_{n\in\mathbb{N}}$ if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in F(\mathcal{I})$ such that for all $x \in X$, $f(x)$ is the uniform limit of the subsequence $\{f_{m_k}(x)\}_{k\in\mathbb{N}}$.

3. Main results

We first recall the following definition from the recent work of Das, Dutta and Pal [15].

Definition 3.1. $\{f_n\}_{n\in\mathbb{N}}$ is said to be *$\mathcal{I}^*$-uniformly equally convergent to f* if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim\limits_n \varepsilon_n = 0$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}|$ is at most $k = k(\{\varepsilon_n\})$ for all $x \in X$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-ue} f$.

Clearly $\mathcal{I}^*$-equal convergence is weaker than $\mathcal{I}^*$-uniform equal convergence which is again weaker than $\mathcal{I}^*$-uniform convergence.

Example 3.2. Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}$ must contain an infinite set A. Take a pairwise disjoint family $\{A_n\}_{n\in\mathbb{N}\setminus A}$ of non-empty subsets of $\mathbb{R}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions on $\mathbb{R}$ defined by

$$\begin{aligned} f_n &= \chi_{A_n} \text{ for all } n \in \mathbb{N} \setminus A \\ &= 1 \text{ for all } n \in A. \end{aligned}$$

Now clearly $\sup_{x\in\mathbb{R}} |f_n(x)| = 1$ for all n and so $\{f_n\}_{n\in\mathbb{N}}$ cannot converge $\mathcal{I}^*$-uniformly to the constant function $f \equiv 0$. But since for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, the set $\{n \in \mathbb{N} \setminus A : f_n(x) \geq \varepsilon_n\}$ has cardinality at most 1 for all $x \in \mathbb{R}$, so $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{I}^*$-uniformly equally to $f \equiv 0$. Clearly $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly equally to $f \equiv 0$.

Example 3.3. Consider the intervals of the form $[m, m + \frac{j}{m}]$, $j = 1, 2, ..., m-1$ for each $m \in \mathbb{N}$ and $\{f_i\}_{i\in\mathbb{N}}$ be the enumeration of the characteristic functions of these intervals. Let $A \in \mathcal{I}$. Then $M = \mathbb{N} \setminus A \in F(\mathcal{I})$ and so M must be infinite (since $\mathcal{I}$ is an admissible ideal). Let $M = \{n_1 < n_2 < n_3 < ...\}$. Now consider the sequence $\{g_k\}_{k\in\mathbb{N}}$ of functions on $\mathbb{R}$

$$\begin{aligned} g_k &= 1 \text{ for all } k \in A \\ g_{n_i} &= f_i \text{ for all } i \in \mathbb{N}. \end{aligned}$$

It is now easy to see that $\{g_k\}_{k\in\mathbb{N}}$ converges $\mathcal{I}^*$-equally to zero function. But if $\lim_n \varepsilon_n = 0$ for a given sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, then $|\{n \in \mathbb{N} \setminus A : |g_n(x)| \geq \varepsilon_n\}| = x - 1$ for each $x \in \mathbb{N}$ which increases with x and also these n's overlap the whole set $\mathbb{N} \setminus A$ as x runs over $\mathbb{N}$. Hence $\{g_k\}_{k\in M}$ cannot converge $\mathcal{I}^*$-uniformly equally to $f \equiv 0$.

We first observe the following equivalent condition for $\mathcal{I}^*$-uniform equal convergence.

Theorem 3.4. *Let $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then $f_n \xrightarrow{\mathcal{I}^*-ue} f$ if and only if there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers $\mathcal{I}$-divergent to ∞ such that*

$$\rho_n |f_n - f| \xrightarrow{\mathcal{I}^*-ue} 0.$$

Proof. Suppose that $f_n \xrightarrow{\mathcal{I}^*-ue} f$. Then there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that

$$|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k \text{ for all } x \in X. \tag{1}$$

Now, define a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ as

$$\begin{aligned} \rho_n &= \left[\frac{1}{\sqrt{\varepsilon_n}}\right] \quad , n \in M \\ &= 1 \quad , n \notin M. \end{aligned}$$

Obviously $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-divergent to ∞. Hence from (1)

$$|\{n \in M : \rho_n |f_n(x) - f(x)| \geq \sqrt{\varepsilon_n}\}| \leq k \text{ for all } x \in X$$

which implies $\rho_n |f_n - f| \xrightarrow{\mathcal{I}^*-ue} 0$.

Conversely, if $\rho_n |f_n - f| \xrightarrow{\mathcal{I}^*-ue} 0$ where $\{\rho_n\}_{n\in\mathbb{N}}$ is a sequence of positive integers $\mathcal{I}$-divergent to ∞, then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \lambda_n = 0$ and $M = M(\{\lambda_n\}) \in F(\mathcal{I})$ and $k = k(\{\lambda_n\}) \in \mathbb{N}$ such that $|\{n \in M : \rho_n |f_n(x) - f(x)| \geq \lambda_n\}| \leq k$ for all $x \in X$. Define a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ by

$$\begin{aligned} \theta_n &= \frac{\lambda_n}{\rho_n} \quad , n \in M \\ &= \frac{1}{n} \quad , n \notin M. \end{aligned}$$

Then $\lim_n \theta_n = 0$ and $|\{n \in M : |f_n(x) - f(x)| \geq \theta_n\}| \leq k$ for all $x \in X$. This completes the proof. □

Lemma 3.5. *Let* $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. *If* $f_n \xrightarrow{\mathcal{I}^*-ue} 0$, *then* $f_n^2 \xrightarrow{\mathcal{I}^*-ue} 0$.

Proof. By definition, there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that

$$|\{n \in M : |f_n(x)| \geq \varepsilon_n\}| \leq k \text{ for all } x \in X.$$

Then we have

$$|\{n \in M : |f_n(x)|^2 \geq \varepsilon_n^2\}| \leq k \text{ for all } x \in X.$$

and so

$$|\{n \in M : |f_n^2(x)| \geq \varepsilon_n^2\}| \leq k \text{ for all } x \in X.$$

Therefore $f_n^2 \xrightarrow{\mathcal{I}^*-ue} 0$. □

Lemma 3.6. *Let* $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. *If* f *is bounded and* $f_n \xrightarrow{\mathcal{I}^*-ue} f$, *then* $f_n.f \xrightarrow{\mathcal{I}^*-ue} f^2$.

Proof. Let B be a positive real number such that $|f(x)| \leq B$ for all $x \in X$. Since $f_n \xrightarrow{\mathcal{I}^*-ue} f$, there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that

$$|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k \text{ for all } x \in X.$$

Since $|f(x)||f_n(x) - f(x)| \geq |(f_n.f)(x) - f^2(x)|$, we have

$$\begin{aligned}\{n \in M : |(f_n.f)(x) - f^2(x)| \geq \varepsilon_n.B\} &\subseteq \{n \in M : |f(x)||f_n(x) - f(x)| \geq \varepsilon_n.B\}\\ &\subseteq \{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}\end{aligned}$$

for each $x \in X$. Therefore $|\{n \in M : |(f_n.f)(x) - f^2(x)| \geq \varepsilon_n.B\}| \leq k$ for all $x \in X$. This proves the result. □

Theorem 3.7. *If* $f_n \xrightarrow{\mathcal{I}^*-ue} f$ *and* $g_n \xrightarrow{\mathcal{I}^*-ue} g$ *then* $f_n.g_n \xrightarrow{\mathcal{I}^*-ue} f.g$, *where* f *and* g *are bounded.*

Proof. Using Lemma 3.5, Lemma 3.6 and writing $f_n.g_n = \frac{(f_n+g_n)^2-(f_n-g_n)^2}{4}$ we can deduce that $f_n.g_n \xrightarrow{\mathcal{I}^*-ue} f.g$. □

Let Φ be an arbitrary class of functions defined on a non-empty set X. We denote by $\Phi^{\mathcal{I}^*-ue}$, the class of all functions defined on X, which are $\mathcal{I}^*$-uniform equal limits of sequences of functions belonging to Φ. For any class of functions Φ on X we first recall the following definitions from [9].

Definition 3.8. (a) Φ is called a *lattice* if Φ contains all constants and $f, g \in \Phi$ implies $\max(f, g) \in \Phi$ and $\min(f, g) \in \Phi$.

(b) Φ is called a *translation lattice* if it is a lattice and $f \in \Phi, c \in \mathbb{R}$ implies $f + c \in \Phi$.

(c) Φ is called a *congruence lattice* if it is a translation lattice and $f \in \Phi$ implies $-f \in \Phi$.

(d) Φ is called a *weakly affine lattice* if it is a congruence lattice and there is a set $C \subset (0, \infty)$ such that C is not bounded and $f \in \Phi$, $c \in C$ implies $cf \in \Phi$.

(e) Φ is called an *affine lattice* if it is a congruence lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $cf \in \Phi$.

(f) Φ is called a *subtractive lattice* if it is a lattice and $f, g \in \Phi$ implies $f - g \in \Phi$.

(g) Φ is called an *ordinary class* if it is a subtractive lattice, $f, g \in \Phi$ implies $f.g \in \Phi$ and $f \in \Phi$, $f(x) \neq 0$, for all $x \in X$ implies $1/f \in \Phi$.

Theorem 3.9. *Let* Φ *be a class of functions on* X. *If* Φ *is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is* $\Phi^{\mathcal{I}^*-ue}$. *Further if* $f \in \Phi^{\mathcal{I}^*-ue}$ *is bounded, then* $f^2 \in \Phi^{\mathcal{I}^*-ue}$.

Proof. Let Φ be a lattice. Since Φ contains the constant functions, $\Phi^{\mathcal{I}^*-ue}$ contains the constant functions. Let $f_n \xrightarrow{\mathcal{I}^*-ue} f$. Then there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k$ for all $x \in X$. Now $\big||f_n|(x) - |f|(x)\big| \leq \big|f_n(x) - f(x)\big|$. Therefore $|\{n \in M : \big||f_n|(x) - |f|(x)\big| \geq \varepsilon_n\}| \leq k$ for each $x \in X$ i.e. $|f_n| \xrightarrow{\mathcal{I}^*-ue} |f|$.

Next we show that if $f_n \xrightarrow{\mathcal{I}^*-ue} f$, $g_n \xrightarrow{\mathcal{I}^*-ue} g$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}^*-ue} \alpha f + \beta g$. Indeed, by definition there exist $M_f, M_g \in F(\mathcal{I})$, $\lim_n \varepsilon_n = 0$, $\lim_n \lambda_n = 0$ and $n_f = n_f(\{\varepsilon_n\})$, $n_g = n_g(\{\lambda_n\}) \in \mathbb{N}$ such that

$$|\{n \in M_f : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_f$$

and

$$|\{n \in M_g : |g_n(x) - g(x)| \geq \lambda_n\}| \leq n_g.$$

Let us assume that $\theta_n = \max\{2|\alpha|\varepsilon_n, 2|\beta|\lambda_n\}$ and $k = n_f + n_g$. Hence we have

$$|\{n \in M_f \cap M_g : |\alpha(f_n - f)(x) + \beta(g_n - g)(x)| \geq \theta_n\}| \leq k$$

where $M_f \cap M_g \in F(\mathcal{I})$ and $\lim_n \theta_n = 0$. Hence $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}^*-ue} \alpha f + \beta g$.

Next observe that if $f, g \in \Phi^{\mathcal{I}^*-ue}$, $f_n \xrightarrow{\mathcal{I}^*-ue} f$ and $g_n \xrightarrow{\mathcal{I}^*-ue} g$, then, in view of above,

$$\frac{f_n+g_n}{2} + \frac{|f_n-g_n|}{2} \xrightarrow{\mathcal{I}^*-ue} \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f, g)$$

which implies that $\max(f, g) \in \Phi^{\mathcal{I}^*-ue}$. Similarly we can show that $\min(f, g) \in \Phi^{\mathcal{I}^*-ue}$. Thus $\Phi^{\mathcal{I}^*-ue}$ is a lattice. The proofs of the remaining assertions are straightforward. The last assertion follows from Lemma 3.6. □

Theorem 3.10. *Let Φ be an ordinary class of functions on X. Let $f \in \Phi^{\mathcal{I}^*-ue}$ be bounded and $f(x) \neq 0$ for each $x \in X$. If $\frac{1}{f}$ is bounded on X, then $\frac{1}{f} \in \Phi^{\mathcal{I}^*-ue}$.*

Proof. Assume that $\frac{1}{f}$ is bounded on X. Then there exists a $\lambda > 0$ be such that $f^2(x) > \lambda$ for each $x \in X$. Since $f \in \Phi^{\mathcal{I}^*-ue}$ and f is bounded then $f^2 \in \Phi^{\mathcal{I}^*-ue}$. Hence there exist a sequence $\{f_n\}_{n\in\mathbb{N}}$ of Φ, a set $M \in F(\mathcal{I})$ and $k \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq k$ for all $x \in X$. Let $g_n(x) = \max\{f_n(x), \frac{1}{n}\}$ for $x \in X$. Then $g_n \in \Phi$ for each $n \in \mathbb{N}$. Therefore

$$|\{n \in M : g_n(x) = f_n(x), |g_n(x) - f^2(x)| \geq \tfrac{1}{n^3}\}| \leq k$$

and

$$\begin{aligned}
&\{n \in M : g_n(x) = \tfrac{1}{n},\ |g_n(x) - f^2(x)| \geq \tfrac{1}{n^3}\} \\
&= \{n \in M : g_n(x) = \frac{1}{n},\ g_n(x) - f^2(x) \geq \frac{1}{n^3}\} \\
&\quad \cup \{n \in M : g_n(x) = \frac{1}{n},\ -g_n(x) + f^2(x) \geq \frac{1}{n^3}\} \\
&\subseteq \{n \in M : f^2(x) \leq \frac{1}{n} - \frac{1}{n^3}\} \cup \{n \in M : f^2(x) \geq f_n(x) + \frac{1}{n^3}\} \\
&\subseteq \{n \in M : f^2(x) < \frac{1}{n}\} \cup \{n \in M : f^2(x) \geq f_n(x) + \frac{1}{n^3}\}
\end{aligned}$$

Therefore $|\{n \in M : g_n(x) = \frac{1}{n},\ |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq k' + k = k_1$(say) where $k' = [\frac{1}{\lambda}] + 1$. Hence

$$\begin{aligned}
\{n \in M : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\} &= \{n \in M : g_n(x) = f_n(x), |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\} \\
&\quad \cup \{n \in M : g_n(x) = \frac{1}{n}, |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}.
\end{aligned}$$

This implies that $|\{n \in M : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq k_1 + k = k_2$ (say). Therefore

$$\begin{aligned} |\{n \in M : |\frac{1}{g_n(x)} - \frac{1}{f^2(x)}| \geq \frac{1}{n^3}.n.\frac{1}{\lambda}\}| &= |\{n \in M : \frac{|f^2(x) - g_n(x)|}{|g_n(x)||f^2(x)|} \geq \frac{1}{n^3}.n.\frac{1}{\lambda}\}| \\ &\leq |\{n \in M : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \\ &\leq k_2. \end{aligned}$$

Hence $f^{-2} \in \Phi^{\mathcal{I}^*-ue}$ and so $f.f^{-2} = f^{-1} \in \Phi^{\mathcal{I}^*-ue}$. □

We now introduce the following notion.

Definition 3.11. $\{f_n\}_{n\in\mathbb{N}}$ is said to be *$\mathcal{I}^*$-uniformly discretely convergent to f* if there exist a set $M \in F(\mathcal{I})$ and a natural number $k \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| > 0\}|$ is at most k for all $x \in X$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-ud} f$.

We denote by $\Phi^{\mathcal{I}^*-ud}$, the class of all functions defined on X, which are $\mathcal{I}^*$-uniform discrete limits of sequences of functions belonging to Φ.

Now, we study some properties of the class $\Phi^{\mathcal{I}^*-ud}$.

Theorem 3.12. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{\mathcal{I}^*-ud}$.*

Proof. This theorem readily follows from Definition 3.11. □

Theorem 3.13. *Let Φ be an ordinary class of functions on X. Then $f, g \in \Phi^{\mathcal{I}^*-ud}$ implies $f.g \in \Phi^{\mathcal{I}^*-ud}$. Also if $f \in \Phi^{\mathcal{I}^*-ud}$ is such that $f(x) \neq 0$ for each $x \in X$ and $\frac{1}{f}$ is bounded on X, then $\frac{1}{f} \in \Phi^{\mathcal{I}^*-ud}$.*

Proof. Let $f, g \in \Phi^{\mathcal{I}^*-ud}$. Then there exist sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \xrightarrow{\mathcal{I}^*-ud} f$ and $g_n \xrightarrow{\mathcal{I}^*-ud} g$. Then from definition, we can prove that $f_n.g_n \xrightarrow{\mathcal{I}^*-ud} f.g$.

Let $f \in \Phi^{\mathcal{I}^*-ud}$ be such that $f(x) \neq 0$ for each $x \in X$ and $\frac{1}{f}$ is bounded on X. Choose $\mu > 0$ such that $f^2(x) > \mu > 0$ for each $x \in X$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in Φ such that $f_n \xrightarrow{\mathcal{I}^*-ud} f$. Since Φ is an ordinary class, $f_n^2 \in \Phi$ for each $n \in \mathbb{N}$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive reals converging to zero and $g_n = \max\{f_n^2, \lambda_n\}$. Then $g_n \in \Phi$. Since $f_n \xrightarrow{\mathcal{I}^*-ud} f$, then by definition

$$|\{n \in M : f_n(x) \neq f(x)\}| \leq k \text{ for all } x \in X$$

which implies that

$$|\{n \in M : g_n(x) \neq \max\{f^2(x), \lambda_n\}\}| \leq k \text{ for all } x \in X$$

i.e.,

$$|\{n \in M : \frac{1}{g_n(x)} \neq \frac{1}{\max\{f^2(x), \lambda_n\}}\}| \leq k \text{ for all } x \in X. \tag{2}$$

Now since $\lim_n \lambda_n = 0$, there exists a $k' \in \mathbb{N}$ such that $\lambda_n < \mu$ for all $n \in M$ such that $n \geq k'$. Therefore (2) becomes

$$|\{n \in M : \tfrac{1}{g_n(x)} \neq \tfrac{1}{f^2(x)}\}| \leq k + k' \text{ for each } x \in X.$$

Hence $f^{-2} \in \Phi^{\mathcal{I}^*-ud}$ and consequently $f.f^{-2} = f^{-1} \in \Phi^{\mathcal{I}^*-ud}$. □

Finally we introduce the following notion of convergence for a sequence of real valued functions.

Definition 3.14. $\{f_n\}_{n\in\mathbb{N}}$ is said to be *$\mathcal{I}^*$-strongly uniformly equally convergent to f* if there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}|$ is at most $k = k(\{\varepsilon_n\})$ for all $x \in X$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-sue} f$.

We denote by $\Phi^{\mathcal{I}^*-sue}$, the class of all $\mathcal{I}^*$-strong uniform equal limits of a class of functions Φ defined on X.

Example 3.15. Let $\mathcal{I}$ be a non-trivial proper admissible ideal. So there exists an infinite set $M \in F(\mathcal{I})$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions on $\mathbb{R}$ defined by

$$\begin{aligned} f_n(x) &= \frac{1}{n},\ n \in M \\ &= 0,\ n \notin M \end{aligned}$$

for all $x \in X$. Then $f_n \xrightarrow{\mathcal{I}^*-ue} 0$ but $f_n \overset{\mathcal{I}^*-sue}{\nrightarrow} 0$.

From the definition and the above example it follows that $\mathcal{I}^*$-strong uniform equal convergence is stronger than $\mathcal{I}^*$-uniform equal convergence. As in the case of $\mathcal{I}^*$-uniform equal convergence we can easily prove the following results.

Lemma 3.16. *Let $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. If $f_n \xrightarrow{\mathcal{I}^*-sue} 0$, then $f_n^2 \xrightarrow{\mathcal{I}^*-sue} 0$.*

Lemma 3.17. *Let $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. If f is bounded and $f_n \xrightarrow{\mathcal{I}^*-sue} f$, then $f_n.f \xrightarrow{\mathcal{I}^*-sue} f^2$.*

Theorem 3.18. *If $f_n \xrightarrow{\mathcal{I}^*-sue} f$ and $g_n \xrightarrow{\mathcal{I}^*-sue} g$ then $f_n.g_n \xrightarrow{\mathcal{I}^*-sue} f.g$, where f and g are bounded.*

Theorem 3.19. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{\mathcal{I}^*-sue}$.*

Proof. Let Φ be a lattice. Since Φ contains the constant functions, $\Phi^{\mathcal{I}^*-ue}$ also contains the constant functions. Let $f_n \xrightarrow{\mathcal{I}^*-sue} f$. Then there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, a set $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k$ for all $x \in X$. Now $\big||f_n|(x) - |f|(x)\big| \leq \big|f_n(x) - f(x)\big|$. Therefore $|\{n \in M : \big||f_n|(x) - |f|(x)\big| \geq \varepsilon_n| \leq k$ for each $x \in X$ i.e. $|f_n| \xrightarrow{\mathcal{I}^*-sue} |f|$.

Now we show that if $f_n \xrightarrow{\mathcal{I}^*-sue} f$, $g_n \xrightarrow{\mathcal{I}^*-sue} g$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}^*-sue} \alpha f + \beta g$. To see this, by definition there exist $M_f, M_g \in F(\mathcal{I})$, $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $n_f = n_f(\{\varepsilon_n\})$, $n_g = n_g(\{\lambda_n\}) \in \mathbb{N}$ such that

$$|\{n \in M_f : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_f$$

and

$$|\{n \in M_g : |g_n(x) - g(x)| \geq \lambda_n\}| \leq n_g.$$

Let us choose $\theta_n = \max\{2|\alpha|\varepsilon_n, 2|\beta|\lambda_n\}$ and $k = n_f + n_g$. Then we have

$$|\{n \in M_f \cap M_g : |\alpha(f_n - f)(x) + \beta(g_n - g)(x)| \geq \theta_n\}| \leq k$$

where

$$\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}\max\{2|\alpha|\varepsilon_n, 2|\beta|\lambda_n\}$$
$$\leq \sum_{n=1}^{\infty}(2|\alpha|\varepsilon_n + 2|\beta|\lambda_n)$$
$$< \infty$$

and $M_f \cap M_g \in F(\mathcal{I})$. Hence $\alpha f_n + \beta g_n \xrightarrow{\mathcal{I}^*-sue} \alpha f + \beta g$.

Therefore $f, g \in \Phi^{\mathcal{I}^*-sue}$, $f_n \xrightarrow{\mathcal{I}^*-sue} f$ and $g_n \xrightarrow{\mathcal{I}^*-sue} g$ implies that

$$\frac{f_n+g_n}{2} + \frac{|f_n-g_n|}{2} \xrightarrow{\mathcal{I}^*-sue} \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f,g)$$

i.e. $\max(f,g) \in \Phi^{\mathcal{I}^*-sue}$. Similarly, $\min(f,g) \in \Phi^{\mathcal{I}^*-sue}$. Thus $\Phi^{\mathcal{I}^*-sue}$ is a lattice. It is easy to check the remaining assertions. □

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