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# Duality Properties in Von Neumann Algebras of Projective Unitary Representations

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**Abstract.** Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$. If the set $\mathcal{B}_\pi$ of Bessel vectors for $\pi$ is dense in $H$, then for any vector $x \in H$, the analysis operator $\theta_x$ makes sense as a densely defined operator from $\mathcal{B}_\pi$ to $l_2(G)$-space. If a projection $e \in M$ is equivalent to a projection $f_1 \in M$ with $f_1 \leq f \in M$, then we write $e \precsim f$. Let $P_x$ (resp. $P_y$) be the orthogonal projection from $\ell^2(G)$ onto $[\theta_x(\mathcal{B}_\pi)]$ (resp. $[\theta_y(\mathcal{B}_\pi)]$). Han and Larson have proved the duality properties of projective unitary representations, i.e. $P_x \leq P_y$ is equivalent to $Q_x \leq Q_y$. In this paper we prove that a similar result is true in the sense of von Neumann equivalence of projections, i.e. $P_x \precsim P_y$ in $\lambda(G)'$ is equivalent to $Q_x \precsim Q_y$ in $\pi(G)''$.

## 1. Introduction

Frame theory for special systems, including wavelet systems and Gabor systems, has close connections with group representations. The aim of this article is to give a general framework for exploring certain of these connections. The well-known (Ron-Shen) duality theorem (see Section 2) reveals the connection between the frame property of a Gabor family (built on a time-frequency lattice) and the Riesz sequence property of the associated Gabor family (built on the adjoint lattice). Han and Larson have presented some results on a duality property for orthogonal (that is, strongly disjoint) and weakly equivalent frame-generator vectors for group representations and, more generally, projective unitary representations in [1]. This duality theorem also indicates some duality connections between the so-called orthogonality or strong disjointness of Gabor families and the commutant of the Gabor operator system.

It is now very natural to ask whether we can extend the main duality property in [1] to von Neumann algebras. This is a motivation of this paper. Our main results is Theorem 2.6. Here we use the method that from special to general.

A projective unitary representation [2] $\pi$ for a countable discrete group $G$ is a mapping $g \to \pi(g)$ from $G$ into the group $U(H)$ of all the unitary operators on a separable Hilbert space $H$ such that $\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g, h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group $\mathbb{T}$. This function $\mu(g,h)$ is then called a multiplier of $\pi$. In this case we also say that $\pi$ is a $\mu$-projective unitary representation. It is clear from the definition that we have

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(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$,

(ii) $\mu(g, e) = \mu(e, g) = 1$ for all $g \in G$, where e denotes the group unit of G.

Any function $\mu : G \times G \to \mathbb{T}$ satisfying (i)-(ii) above will be called a multiplier for $G$. It follows from (i) and (ii) that we also have

(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ holds for all $g \in G$.

Typical examples of projective unitary representations are unitary group representation.

Similar to the group unitary representation case, the left and right regular projective representations with a prescribed multiplier $\mu$ for $G$ play important roles here. Let $\mu$ be a multiplier for $G$. For each $g \in G$, we define

$$\lambda_g x_h = \mu(g, h)x_{gh}, h \in G,$$

and

$$r_g x_h = \mu(h, g^{-1})x_{hg^{-1}}, h \in G,$$

where $\{x_g : g \in G\}$ is the standard orthonormal basis for $\ell^2(G)$. Clearly, $\lambda_g$ and $r_g$ are unitary operators on $\ell^2(G)$. Moreover, $\lambda$ is a $\mu$-projective unitary representation of $G$ with multiplier $\mu(g, h)$ and $r$ is a projective unitary representation of $G$ with multiplier $\overline{\mu(g, h)}$. The representation $\lambda$ and r are called the left regular $\mu$-projective representation and right regular $\mu$-projective representation, respectively, of G. Let $\mathcal{L}$ and $\mathcal{R}$ be the von Neumann algebras generated by $\lambda$ and $r$, respectively. We summarize a few basic properties in the following proposition:

**Proposition** [3] (i) The von Neumann algebra $\mathcal{R}$ is the commutant of $\mathcal{L}$.

(ii) Both $\mathcal{L}$ and $\mathcal{R}$ are finite von Neumann algebras.

(iii) If for each $e \neq u \in G$, either $\{vuv^{-1} : v \in G\}$ is a infinite set or $\{\mu(vuv^{-1}, v)\overline{\mu(v, u)} : v \in G\}$ is a infinite set, then both $\mathcal{L}$ and $\mathcal{R}$ are factor von Neumann algebras.

Since the basic techniques used in this paper involve von Neumann algebra theory, we first introduce some notation in [4].

A von Neumann algebra $M$ is a $*$-subalgebra of $B(H)$ such that $I \in M$ and $M$ is closed in the weak topology, where $B(H)$ is the algebra of all bounded operators acting on a separable Hilbert space $H$. By the double commutant theorem, a $*$-subalgebra $M$ of $B(H)$ is a von Neumann algebra if and only if $M = M''$, where $M'$ is the commutant of $M$. A von Neumann algebra is said to be finite if every isometry in the algebra is unitary.

Two projections $e$ and $f$ in a von Neumann algebra $M$ are said to be equivalent if there exists an element $u \in M$ with $u^*u = e$ and $uu^* = f$. we write this fact as $e \sim f$. The projections $e$ and $f$ are called, respectively, the initial projection and the final projection of $u$. If a projection $e \in M$ is equivalent to a projection $f_1 \in M$ with $f_1 \leq f \in M$, then we write $e \precsim f$. Clearly, the relation $e \sim f$ is an equivalence relation. We shall also use these notations for subspaces of the underlying Hilbert space $H$ of $M$ when $M$ is represented on it. In other words, if $\mathfrak{M}$ and $\mathfrak{N}$ are ranges of projections $e$ and $f$ in $M$, respectively, $\mathfrak{M} \precsim \mathfrak{N}$ means that $e \precsim f$ in $M$.

Recall that a frame for a Hilbert space $H$ is a sequence $\{x_n\}$ in $H$ with the property that there exist two positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n\in N} | < x, x_n > |^2 \leq B\|x\|^2, \tag{1}$$

holds for every $x \in H$. The optimal constants (maximal for $A$ and minimal for $B$) are called frame bounds. The frame $\{x_n\}$ is called tight frame if $A = B$. When $A = B = 1$, $\{x_n\}$ is called a normalized tight frame. In the case that (1) hold only for all the $x \in \overline{span}\{x_n\}$, then we say that $\{x_n\}$ is a frame sequence. If we only require the right-hand side of the inequality (1), then $\{x_n\}$ is called a Bessel sequence.

Given a projective unitary representation $\pi$ of a countable discrete group $G$ on a Hilbert space $H$. A vector $\xi \in H$ is called a complete frame vector (resp. tight frame vector, normalized tight frame vector) for $\pi$ if $\{\pi(g)\xi : g \in G\}$ (here we view this as a sequence indexed by $G$) is a frame (resp. tight frame,normalized tight frame) for $H$, and is just called a frame vector for $\pi$ if $\{\pi(g)\xi : g \in G\}$ is a frame sequence. A Bessel vector for $\pi$ is a vector $\xi \in H$ such that $\{\pi(g)\xi : g \in G\}$ is Bessel. We will use $\mathcal{B}_\pi$ to denote the set of all the

Bessel vector of $\pi$ and $\pi(G)$ is the von Neumann algebra generated by $\{\pi(g) : g \in G\}$. Then $\mathcal{B}_\pi$ is a linear subspace invariant under $\pi(G)$ and $\pi(G)'$.

For any projective representation $\pi$ of a countable discrete group on a Hilbert space $H$ and $x \in H$, the analysis operator $\theta_x$ for $x$ from $\mathcal{D}(\theta_x)(\subseteq H)$ to $\ell^2(G)$ is defined by

$$\theta_x(y) = \sum_{g\in G} < y, \pi(g)x > x_g,$$

where $\mathcal{D}(\theta_x) = \{y \in H : \sum_{g\in G} | < y, \pi(g)x > |^2 < \infty\}$.

Clearly $\mathcal{B}_\pi \subseteq \mathcal{D}(\theta_x)$ holds for every $x \in H$. In fact:
For all $\xi \in \mathcal{B}_\pi$, we have $\sum_{g\in G} | < y, \pi(g)\xi > |^2 \le B\|y\|^2$ for every $y \in H$, where $B$ is a positive constant. So

$$\begin{aligned}
&\sum_{g\in G} | < \xi, \pi(g)x > |^2 = \sum_{g\in G} | < x, \pi(g)^*\xi > |^2 \\
&= \sum_{g\in G} | < x, \pi(g)^{-1}\xi > |^2 = \sum_{g\in G} | < x, \overline{\mu(g, g^{-1})}\pi(g^{-1})\xi > |^2 \\
&= \sum_{g\in G} | < \mu(g, g^{-1})x, \pi(g^{-1})\xi > |^2 = \sum_{g\in G} | < x, \pi(g^{-1})\xi > |^2 \\
&\le B\|x\|^2 < +\infty.
\end{aligned}$$

In the case that $\mathcal{B}_\pi$ is dense in $H$, we have that $\theta_x$ is a densely defined and closable linear operator in [5]. Moreover, $x \in \mathcal{B}_\pi$ if and only if $\theta_x$ is a bounded linear operator on $H$, which is equivalence to the condition that $\mathcal{D}(\theta_x) = H$. Also, we have $\theta_x^* x_g = \pi(g)x$ for all $g \in G$, and $x$ is a normalized tight frame vector if and only if $\theta_x^*\theta_x$ is a projection. Moreover, $x$ is a complete frame vector (respectively, complete normalized tight frame vector) if and only if $\theta_x$ is injective with closed range (respectively, isometry).

## 2. Main Results

**Lemma 2.1**[3],[5] Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$ such that $\mathcal{B}_\pi$ is dense in $H$. Then for any $x \in H$, there exists $\xi \in \mathcal{B}_\pi$ such that

(i) $\{\pi(g)\xi : g \in G\}$ is a normalized tight frame for $[\pi(G)x]$;

(ii) $\theta_\xi(H) = [\theta_x(\mathcal{B}_\pi)]$.

**Lemma 2.2**[1] Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$ such that $\mathcal{B}_\pi$ is dense in $H$, and let $x, y \in H$. Then the following are equivalent:

(i) $[\theta_x(\mathcal{B}\pi)] = [\theta_y(\mathcal{B}\pi)]$

(ii) $[\pi(G)'x] = [\pi(G)'y]$

**Lemma 2.3**[1] Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$ such that $\mathcal{B}_\pi$ is dense in $H$, and let $x, y \in H$. Then the following are equivalent:

(i) $[\theta_x(\mathcal{B}\pi)] \subseteq [\theta_y(\mathcal{B}\pi)]$

(ii) $[\pi(G)'x] \subseteq [\pi(G)'y]$

Let $P_x$ (resp. $P_y$) be the orthogonal projection from $\ell^2(G)$ onto $[\theta_x(\mathcal{B}_\pi)]$ (resp. $[\theta_y(\mathcal{B}_\pi)]$), and let $\lambda$ be the left regular $\mu$-projection representation of $G$, where $\mu$ is the multiplier of $\pi$. It is routine to check that both $[\theta_x(\mathcal{B}_\pi)]$ and $[\theta_y(\mathcal{B}_\pi)]$) are invariant under $\lambda$. In fact:

For all $\xi \in \mathcal{B}_\pi$, for every $g \in G$, we have

$$\begin{aligned}
\lambda_g(\theta_x(\xi)) &= \lambda_g(\sum_{h\in G} < \xi, \pi(h)x > x_h) \\
&= \sum_{h\in G} < \xi, \pi(h)x > \mu(g, h)x_{gh} \\
&= \sum_{h\in G} < \pi(g)\xi, \pi(g)\pi(h)x > \mu(g, h)x_{gh} \\
&= \sum_{h\in G} < \pi(g)\xi, \mu(g, h)\pi(gh)x > \mu(g, h)x_{gh} \\
&= \sum_{h\in G} < \pi(g)\xi, \pi(gh)x > x_{gh} \\
&= \theta_x(\pi(g)\xi) \subseteq \theta_x(\mathcal{B}_\pi).
\end{aligned}$$

So $P_x, P_y \in \lambda(G)'$. Let $Q_x$ (resp. $Q_y$) be the orthogonal projection from $H$ onto $[\pi(G)'x]$ (resp. $[\pi(G)'x]$), so $Q_x, Q_y \in \pi(G)'' = \pi(G)$.

From Lemma 2.2 and Lemma 2.3 (i.e. Han's results), we know that $P_x \leq P_y$ is equivalent to $Q_x \leq Q_y$. Then we begin to prove the main result.

**Lemma 2.4**[4] Suppose $\mathcal{R}$ is a von Neumann algebra acting on the Hilbert space $H$ and $x, y$ are in $H$. Then $[\mathcal{R}'(x)] \precsim [\mathcal{R}'(y)]$ (resp. $[\mathcal{R}'(x)] \succsim [\mathcal{R}'(y)]$) if and only if $[\mathcal{R}(x)] \precsim [\mathcal{R}(y)]$ (resp. $[\mathcal{R}(x)] \succsim [\mathcal{R}(y)]$).

The following lemma is the main ingredient in proving the main result.

**Lemma 2.5** Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$ such that $\mathcal{B}_\pi$ is dense in $H$, and let $x, y \in H$. Then the following are equivalent:

(i) $P_x \sim P_y$ in $\lambda(G)'$,

(ii) $Q_x \sim Q_y$ in $\pi(G)''$.

Proof. In $\pi(G)$, by Lemma 2.4, we have $[\pi(G)'x] \sim [\pi(G)'y]$ is equivalent to $[\pi(G)''x] \sim [\pi(G)''y]$, i.e. $[\pi(G)'x] \sim [\pi(G)'y]$ is equivalent to $[\pi(G)x] \sim [\pi(G)y]$. So we let $Q'_x$ (resp. $Q'_y$) be the orthogonal projection from $H$ onto $[\pi(G)x]$ (resp. $[\pi(G)y]$). It suffices to prove $P_x \sim P_y$ is equivalent to $Q'_x \sim Q'_y$.

Assume that $P_x \sim P_y$, let $u \in \lambda(G)'$ be the partial isometry such that $u^*u = P_x$, $uu^* = P_y$. For $x \in H$, by Lemma 2.1, there exists $\xi \in \mathcal{B}_\pi$ such that $\{\pi(g)\xi : g \in G\}$ is a normalized tight frame for $[\pi(G)x]$ and $\theta_\xi(H) = [\theta_x(\mathcal{B}_\pi)]$; For $y \in H$, there exists $\eta \in \mathcal{B}_\pi$ such that $\{\pi(g)\eta : g \in G\}$ is a normalized tight frame for $[\pi(G)y]$ and $\theta_\eta(H) = [\theta_y(\mathcal{B}_\pi)]$. So we have $\theta_\xi\theta_\xi^* = P_x$ and $\theta_\eta\theta_\eta^* = P_y$. By the definition of projection, we also have $\theta_\xi^*\theta_\xi = Q'_x$ and $\theta_\eta^*\theta_\eta = Q'_y$.

Let $v = \theta_\eta^* u\theta_\xi$, then $v^* = \theta_\xi^* u^*\theta_\eta$. So

$$v^*v = \theta_\xi^* u^*\theta_\eta\theta_\eta^* u\theta_\xi = \theta_\xi^* u^* P_y u\theta_\xi = \theta_\xi^* u^* uu^* u\theta_\xi = \theta_\xi^* P_x\theta_\xi = \theta_\xi^*\theta_\xi = Q'_x.$$

In the fifth equality, we use the fact that $P_x\theta_\xi(H) = \theta_\xi(H)$. The proof of $vv^* = Q'_y$ is similar. So we have the expected result $Q'_x \sim Q'_y$.

By symmetry, the other hand is obvious.

**Theorem 2.6** Let $\pi$ be a projective representation of a countable discrete group $G$ on a Hilbert space $H$ such that $\mathcal{B}_\pi$ is dense in $H$, and let $x, y \in H$. Then the following are equivalent:

(i) $P_x \precsim P_y$ in $\lambda(G)'$,

(ii) $Q_x \precsim Q_y$ in $\pi(G)''$.

Proof. $(ii) \Rightarrow (i)$: Assume that $Q_x \precsim Q_y$ i.e. $[\pi(G)'x] \precsim [\pi(G)'y]$, then $[\pi(G)'x] \sim [\pi(G)'z] < [\pi(G)'y]$, where $z = Qy$ and $Q$ is the projection (in $\pi(G)$) with range $[\pi(G)'z]$. By Lemma 2.2 and Lemma 2.3, we have $[\theta_x(\mathcal{B}_\pi)] \sim [\theta_z(\mathcal{B}_\pi)] < [\theta_y(\mathcal{B}_\pi)]$, so that $P_x \precsim P_y$ in $\lambda(G)'$.

$(i) \Rightarrow (ii)$: If we are given that $P_x \precsim P_y$ in $\lambda(G)'$, so $P_x$ is equivalent to a subprojection $P$ of $P_y$ where $P \in \lambda(G)'$. Suppose the range of $P$ is $K_1$ which is contained in $[\theta_y(\mathcal{B}_\pi)]$. Clearly $[\theta_x(\mathcal{B}_\pi)]$, $[\theta_y(\mathcal{B}_\pi)]$ are invariant under $\lambda$. Thus we have $P_x\lambda_g = \lambda_g P_x$ and $P_y\lambda_g = \lambda_g P_y$ for all $g \in G$. Obviously $P_yP \neq 0$, we say that $P_yPx_e \neq 0$. In fact, since otherwise we would have $P_yPx_g = \lambda_g P_yPx_e = 0$ and so $P_yP = 0$, where $e$ is the group unit of $G$.

For $y \in H$, by Lemma 2.1 there exists $\eta \in \mathcal{B}_\pi$ such that $\{\pi(g)\eta : g \in G\}$ is a normalized tight frame for $[\pi(G)y]$ and $\theta_\eta(H) = [\theta_y(\mathcal{B}_\pi)]$. Let $z = \theta_\eta^* P_yPx_e$, then we have $z \in \mathcal{B}_\pi$ and $\pi(g)\theta_\eta^* P_yPx_e = \theta_\eta^*\lambda_g P_yPx_e$ for all $g \in G$. In fact:

Since $P_yPx_e \in [\theta_y(\mathcal{B}_\pi)] = \theta_\eta(H)$, so there exists $x' \in H$ such that $P_yPx_e = \sum_{g\in G} < x', \pi(h)\eta > x_h$. Then

$$\begin{aligned}
&\pi(g)\theta_\eta^* P_yPx_e = \pi(g)(\sum_{h\in G} < x', \pi(h)\eta > \pi(h)\eta) \\
&= \sum_{h\in G} < x', \pi(h)\eta > \pi(g)\pi(h)\eta \\
&= \sum_{h\in G} \mu(g,h) < x', \pi(h)\eta > \pi(gh)\eta \\
&= \theta_\eta^*(\sum_{h\in G} \mu(g,h) < x', \pi(h)\eta > x_{gh}) \\
&= \theta_\eta^*(\sum_{h\in G} < x', \pi(h)\eta > \lambda_g x_h) \\
&= \theta_\eta^*\lambda_g P_yPx_e.
\end{aligned}$$

About $z \in \mathcal{B}_\pi$: for any $y \in H$,

$$
\begin{aligned}
&\sum_{g\in G} | < y, \pi(g)z > |^2 = \sum_{g\in G} | < y, \pi(g)\theta_\eta^* P_y P x_e > |^2 \\
&= \sum_{g\in G} | < y, \theta_\eta^* \lambda_g P_y P x_e > |^2 = \sum_{g\in G} | < P P_y \theta_\eta(y), x_g > |^2 \\
&= ||P P_y \theta_\eta(y)||^2 \leq ||\theta_\eta(y)||^2 \leq ||y||^2.
\end{aligned}
$$

The last inequality is from the fact that $\theta_\eta$ is partial isometry.

Then for any $\omega \in H$, we have

$$
\begin{aligned}
&\theta_z(\omega) = \sum_{g\in G} < \omega, \pi(g)\theta_\eta^* P_y P x_e > x_g \\
&= \sum_{g\in G} < \omega, \theta_\eta^* \lambda_g P_y P x_e > x_g \\
&= \sum_{g\in G} < P P_y \theta_\eta \omega, x_g > x_g \\
&= P P_y \theta_\eta(\omega) = P\theta_\eta(\omega).
\end{aligned}
$$

So $K_1 = \theta_z(H) = [\theta_z(\mathcal{B}_\pi)]$ (since $z \in \mathcal{B}_\pi$), so that $P$ is the orthogonal projection from $\ell^2(G)$ onto $[\theta_z(\mathcal{B}_\pi)]$.

Then $[\theta_x(\mathcal{B}_\pi)] \sim [\theta_z(\mathcal{B}_\pi)] < [\theta_y(\mathcal{B}_\pi)]$. So we have $[\pi(G)'x] \sim [\pi(G)'z] < [\pi(G)'y]$. Therefore we get (ii).

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