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ω-continuous multifunctions

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Abstract. The purpose of this paper is to study ω-continuous multifunctions. Basic characterizations, preservation theorems and several properties concerning upper and lower ω-continuous multifunctions are investigated. Furthermore, some characterizations of ω-connectedness and its relations with ω-continuous multifunctions are given.

1. Introduction

The concepts of upper and lower continuity for multifunctions were firstly introduced by Berge [3]. After this work several authors have given the several weak and strong forms of continuity of multifunctions ([1, 4, 5, 8, 10, 11, 16]). On the other hand, a generalization of the notion of the classical open sets which has received much attention lately is the so-called ω-open sets. In this direction, we will introduce the concept of ω-continuous multifunctions and studied some propeties of ω-continuous multifunctions. Also we have obtained some results on ω-connectedness and its relations with ω-continuous multifunctions.

All through this paper, (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. Let (X,τ) be a space and let A be a subset of X. A point $x \in X$ is called a condensation point of A [12] if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [6] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. These sets are characterized as follows [6]: a subset W of a topological space (X,τ) is an ω-open set if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The ω-closure and ω-interior, that can be defined in a manner to $Cl(A)$ and $Int(A)$, respectively, will be denoted by $\omega Cl(A)$ and $\omega Int(A)$, respectively. Several characterizations and properties of ω-closed subsets were provided in [6, 7, 17]. We set $\omega O(X,x) = \{U : x \in U \text{ and } U \in \tau_\omega\}$

A multifunction $F : X \to Y$ is a point to set correspondence, and we always assume that $F(x) \neq \emptyset$ for every point $x \in X$. For each subset A of X and each subset B of Y, let $F(A) = \cup\{F(x) : x \in A\}$, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Then $F^- : Y \to P(X)$ and if $y \in Y$, then $F^-(y) = \{x \in X : y \in F(x)\}$ where $P(X)$ be the collection of the subsets of X. Thus for $B \subseteq Y$, $F^-(B) = \cup\{F^-(y) : y \in B\}$. F is said to be a surjection if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. A multifunction $F : X \to Y$ is called upper semi continuous [3], abbreviated as u.s.c., (resp. lower semi continuous [3], or l.s.c.) at $x \in X$ if for each open $V \subseteq Y$ with $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there is an open

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neighbourhood U of x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for all $z \in U$). F is u.s.c. (resp. l.s.c.) if and only if it is u.s.c. (resp. l.s.c.) at each point of X. Then F is called semi continuous if and only if it is both u.s.c.and l.s.c. A multifunction $F : X \to Y$ is image-P if $F(x)$ has property P for every $x \in X$.

2. Characterizations

Definition 2.1. A multifunction $F : X \to Y$ is called

(a) *upper ω-continuous* (briefly, u.ω-c.) at a point $x \in X$ if for each open subset V of Y with $F(x) \subseteq V$, there is an ω-open set U containing x such that $F(U) \subseteq V$.

(b) *lower ω-continuous* (briefly, l.ω-c.) at a point $x \in X$ if for each open subset V of Y with $F(x) \cap V \neq \emptyset$, there is an ω-open set U containing x such that $F(z) \cap V \neq \emptyset$ for every point $z \in U$.

(c) *ω-continuous at $x \in X$* if it is both u.ω-c. and l.ω-c. at $x \in X$.

(d) *ω-continuous* if it is ω-continuous at each point $x \in X$.

The following examples show that u.ω-c. and l.ω-c. are independent.

Example 2.2. Let $X = \mathbb{R}$ with the usual topology τ and let $Y = \{a, b, c\}$ with the topology $\sigma = \{\emptyset, Y, \{a\}\}$.

(a) Define a multifunction $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ by $F(x) = \begin{cases} \{a\} & ; x < 0 \\ \{a, b\} & ; x = 0 \\ \{c\} & ; x > 0 \end{cases}$. Then F is u.ω-c., but it is not l.ω-c.

(b) Define a multifunction $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ by $F(x) = \begin{cases} \{a\} & ; x \leq 0 \\ \{a, c\} & ; x > 0 \end{cases}$. Then F is l.ω-c., but it is not u.ω-c.

Theorem 2.3. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent;*

(1) F is l.ω-c.;

(2) For each open subset V of Y, $F^-(V)$ is ω-open;

(3) For each closed subset K of Y, $F^+(K)$ is ω-closed;

(4) For any subset B of Y, $\omega Cl(F^+(B)) \subseteq F^+(Cl(B))$;

(5) For any subset B of Y, $F^-(Int(B)) \subseteq \omega Int(F^-(B))$;

(6) For any subset A of X, $F(\omega Cl(A)) \subseteq Cl(F(A))$;

(7) $F : (X, \tau_\omega) \to (Y, \sigma)$ is l.s.c.

Proof. (1)⇔(2) It is obvious.

(2)⇔(3) These follow from equality $F^-(Y \backslash K) = X \backslash F^+(K)$ for each subset K of Y.

(3)⇒(4) Let B be any subset of Y. Then by (3) $F^+(Cl(B))$ is ω-closed subset of X. Since $F^+(B) \subseteq F^+(Cl(B))$, then $\omega Cl(F^+(B)) \subseteq \omega Cl(F^+(Cl(B))) = F^+(Cl(B))$.

(4)⇔(5) These follow from the facts that $F^-(Y \backslash K) = X \backslash F^+(K)$, $Y \backslash (Cl(B)) = Int(Y \backslash B)$ for $B \subseteq Y$ and $X \backslash (\omega Cl(A)) = \omega Int(X \backslash A)$ for each subset A of X.

(5)⇒(6) Under the assumption (5), suppose (6) is not true i.e. for some $A \subseteq X$, $F(\omega Cl(A)) \nsubseteq Cl(F(A))$. Then there exists a $y_0 \in Y$ such that $y_0 \in F(\omega Cl(A))$ but $y_0 \notin Cl(F(A))$. So $Y \backslash Cl(F(A))$ is an open set containing y_0. By (5), we have $F^-(Y \backslash Cl(F(A))) = F^-(Int(Y \backslash Cl(F(A)))) \subseteq \omega Int(F^-(Y \backslash Cl(F(A))))$ and $F^-(y_0) \subseteq F^-(Y \backslash Cl(F(A)))$. Since $F^-(Y \backslash Cl(F(A))) \cap F^+(F(A)) = \emptyset$ and $A \subset F^+(F(A))$, we have $F^-(Y \backslash Cl(F(A))) \cap A = \emptyset$. Since $F^-(Y \backslash Cl(F(A)))$ is ω-open set, clearly we have that $F^-(Y \backslash Cl(F(A))) \cap \omega Cl(A) = \emptyset$. On the other hand, because of $y_0 \in F(\omega Cl(A))$, we have $F^-(y_0) \cap \omega Cl(A) \neq \emptyset$. But this is a contradiction with $F^-(Y \backslash Cl(F(A))) \cap \omega Cl(A) = \emptyset$. Thus $y \in F(\omega Cl(A))$ implies $y \in Cl(F(A))$. Consequently $\omega Cl(F(A)) \subseteq Cl(F(A))$.

(6)⇒(3) Let $K \subseteq Y$ be a closed set. Since we always have $F(F^+(K)) \subset K$, $Cl(F(F^+(K))) \subseteq Cl(K)$ and by (6), $F(\omega Cl(F^+(K))) \subseteq Cl(F(F^+(K))) \subseteq Cl(K) = K$. Therefore, $\omega Cl(F^+(K)) \subseteq F^+(F(\omega Cl(F^+(K)))) \subset F^+(K)$ and so $F^+(K)$ is ω-closed in X.

(1)⇔(7) It is clear. □

Theorem 2.4. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent;*
(1) F is u.ω-c.;
(2) For each open subset V of Y, $F^+(V)$ is ω-open;
(3) For each closed subset K of Y, $F^-(K)$ is ω-closed;
(4) $F : (X, \tau_\omega) \to (Y, \sigma)$ is u.s.c.;

The proof is similar to that of Theorem 2.3, and is omitted.

Definition 2.5. The net $(x_\alpha)_{\alpha\in I}$ is *ω-convergent* to x if for each ω-open set U containing x, there exists an $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$.

Theorem 2.6. *The multifunction $F : X \to Y$ is l.ω-c. at $x \in X$ if and only if for each $y \in F(x)$ and for every net $(x_\alpha)_{\alpha\in I}$ ω-converging to x, there exists a subnet $(z_\beta)_{\beta\in\xi}$ of the net $(x_\alpha)_{\alpha\in I}$ and a net $(y_\beta)_{(\beta,V)\in\xi}$ in Y with $y_\beta \in F(z_\beta)$ is convergent to y.*

Proof. (⇒) Suppose F is l.ω-c. at x_0. Let $(x_\alpha)_{\alpha\in I}$ be a net ω-converging to x_0. Let $y \in F(x_0)$ and V be any open set containing y. So we have $F(x_0) \cap V \neq \emptyset$. Since F is l.ω-c. at x_0, there exists an ω-open set U such that $x_0 \in U \subseteq F^-(V)$. Since the net $(x_\alpha)_{\alpha\in I}$ is ω-convergent to x_0, for this U, there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$. Therefore, we have the implication $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(V)$. For each open set $V \subseteq Y$ containing y, define the sets $I_V = \{\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow x_\alpha \in F^-(V)\}$ and $\xi = \{(\alpha, V) : \alpha \in I_V, y \in V$ and V is open $\}$ and order " $\geq$ " on ξ as follows: "$(\acute{\alpha}, \acute{V}) \geq (\alpha, V) \Leftrightarrow \acute{V} \subseteq V$ and $\acute{\alpha} \geq \alpha$". Define $\varphi : \xi \longrightarrow I$, by $\varphi((\beta, V)) = \beta$. Then φ is increasing and cofinal in I, so φ defines a subnet of $(x_\alpha)_{\alpha\in I}$. We denote the subnet $(z_\beta)_{(\beta,V)\in\xi}$. On the other hand, for any $(\beta, V) \in \xi$, if $\beta \geq \beta_0 \Rightarrow x_\beta \in F^-(V)$ and we have $F(z_\beta) \cap V = F(x_\beta) \cap V \neq \phi$. Pick $y_\beta \in F(z_\beta) \cap V \neq \phi$. Then the net $(y_\beta)_{(\beta,V)\in\xi}$ is convergent to y. To see this, let V_0 be an open set containing y. Then there exists $\beta_0 \in I$ such that $\varphi((\beta_0, V_0)) = \beta_0$ and $y_{\beta_0} \in V$. If $(\beta, V) \geq (\beta_0, V_0)$ this means that $\beta \geq \beta_0$ and $V \subseteq V_0$. Therefore, $y_\beta \in F(z_\beta) \cap V = F(x_\beta) \cap V \subseteq F(x_\beta) \cap V_0$, so $y_\beta \in V_0$. Thus $(y_\beta)_{(\beta,V)\in\xi}$ is convergent to y.

(⇐) Suppose F is not l.ω-c. at x_0. Then there exists an open set $V \subseteq Y$ so that $x_0 \in F^-(V)$ and for each ω-open set $U \subseteq X$ containing x_0, there is a point $x_U \in U$ for which $x_U \notin F^-(V)$. Let us consider the net $(x_U)_{U\in\omega O(X,x_0)}$. Obviously $(x_U)_{U\in\omega O(X,x_0)}$ is ω-convergent to x_0. Let $y_0 \in F(x_0) \cap V$. By hypothesis, there is a subnet $(z_w)_{w\in W}$ of $(x_U)_{U\in\omega O(X,x_0)}$ and $y_w \in F(z_w)$ such that $(y_w)_{w\in W}$ is convergent to y_0. As $y_0 \in V$ and $V \subseteq Y$ is an open set, there is $w'_0 \in W$ so that $w \geq w'_0$ implies $y_w \in V$. On the other hand, $(z_w)_{w\in W}$ is a subnet of the net $(x_U)_{U\in\omega O(X,x_0)}$ and so there is a function $h : W \longrightarrow \omega O(X, x_0)$ such that $z_w = x_{h(w)}$. By the definition of the net $(x_U)_{U\in\omega O(X,x_0)}$, we have $F(z_w) \cap V = F(x_{h(w)}) \cap V = \emptyset$ and this means that $y_w \notin V$. This is a contradiction and so F is l.ω-c. at x_0. □

Theorem 2.7. *The multifunction $F : X \to Y$ is l.ω-c. (resp. u.ω-c.) at $x \in X$ if and only if for each net $(x_\alpha)_{\alpha\in I}$ ω-convergent to x and for each open subset V of Y with $F(x) \cap V \neq \emptyset$ (resp. $F(x) \subseteq V$), there is an $\alpha_0 \in I$ such that $F(x_\alpha) \cap V \neq \emptyset$ (resp. $F(x_\alpha) \subseteq V$) for all $\alpha \geq \alpha_0$.*

Proof. We prove only for lower ω-continuity. The other is entirely analogous.

(⇒) Let $(x_\alpha)_{\alpha\in I}$ be a net which ω-converges to x in X and let V be any open set in Y such that $x \in F^-(V)$. Since F is l.ω-c. multifunction, it follows that there exists an ω-open set U in X containing x such that $U \subseteq F^-(V)$. Since (x_α) ω-converges to x, it follows that there exists an index $\alpha_0 \in I$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. So we obtain that $x_\alpha \in F^-(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(V)$.

(⇐) Suppose that F is not l.ω-c. Then there is an open set V in Y with $x \in F^-(V)$ such that for each ω-open set U of X containing x, $x \in U \nsubseteq F^-(V)$ i.e. there is a $x_U \in U$ such that $x_U \notin F^-(V)$. Define $D = \{(x_U, U) : U \in \omega O(X), x_U \in U, x_U \notin F^-(V)\}$. Now the order " $\leq$ " defined by $(x_{U_1}, U_1) \leq (x_U, U) \Leftrightarrow U \subseteq U_1$ is a direction on D and g defined by $g : D \longrightarrow X$, $g((x_U, U)) = x_U$ is a net on X. The net $(x_U)_{(x_U,U)\in D}$ is ω-convergent to x. But $F(x_U) \cap V = \emptyset$ for all $(x_U, U) \in D$. This is a contradiction. □

From the definitions, it is obvious that upper (lower) semi-continuity implies upper (lower) ω-continuity. But the converse is not true in general.

Example 2.8. Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Define a multifunction $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ by $F(x) = \begin{cases} \mathbb{Q} & ; x \in \mathbb{R} - \mathbb{Q} \\ \mathbb{R} - \mathbb{Q} & ; x \in \mathbb{Q} \end{cases}$. Then F is u.ω-c. and l.ω-c. But it is neither u.s.c nor l.s.c.

Definition 2.9. ([17]) A space X is *anti-locally countable* if each non-empty open set is uncountable.

Corollary 2.10. *Let X be an anti-locally countable space. Then the multifunction $F : X \to Y$ is u(l).ω-c iff F u(l).s.c.*

Recall that A multifunction $F : X \to Y$ is called open if for each open subset U of X, $F(U)$ is open in Y.

Definition 2.11. A multifunction $F : X \to Y$ is called
(a) *ω-open* if for each open subset U of X, $F(U)$ is ω-open in Y.
(b) *pre-ω-open* if for each ω-open subset U of X, $F(U)$ is ω-open in Y.

The proofs of the following two lemmas follow from the fact that $\tau \subseteq \tau_\omega$ and definitions.

Lemma 2.12. *Let $F : X \to Y$ be a multifunction.*
(1) If F is image-open, then F is open, ω-open;
(2) If F is image-ω-open, then F is both ω-open and pre-ω-open.

Lemma 2.13. *Let $F : X \to Y$ be a multifunction.*
(1) If F^- is image-open, then F l.ω-c.;
(2) If F^- is image-ω-open, then F is l.ω-c.

Lemma 2.14. *If $F : X \to Y$ is image-open and u.ω-c., then $F^-(B)$ is ω-closed in X for any $B \subseteq Y$. In particular; F^- is image-ω-closed.*

Proof. Let $x \in X - F^-(B) = F^+(Y - B)$. Then $F(x) \subseteq Y - B$. Since $F(x)$ is open and F is u.ω-c., $F^+(F(x))$ is an ω-open set in X and $x \in F^+(F(x)) \subseteq F^+(Y - B) = X - F^-(B)$. This shows that $X - F^-(B)$ is an ω-open and hence $F^+(B)$ is an ω-closed in X. □

A multifunction $F : X \to Y$ is said to be have *nonmingled point images* [14] provided that for $x_1, x_2 \in X$ with $x_1 \neq x_2$, the image sets $F(x_1)$ and $F(x_2)$ are either disjoint or identical.

Note that for a multifunction F, F is image-nonmingled if and only if $F \circ F^- \circ F = F$ [14].

Theorem 2.15. *Let $F : X \to Y$ be image-nonmingled such that F is either image-open and l.ω-c. or F^- image-ω-open. Then F is u.ω-c.*

Proof. Let $x \in X$ and V be an open set with $F(x) \subseteq V$. Firstly, suppose that F is image-open and l.ω-c. Then $F^-(F(x))$ is ω-open in X and $x \in F^-(F(x))$. Put $U = F^-(F(x))$. Thus we have an ω-open set U containing x such that $F(U) = F(F^-(F(x))) = F(x) \subseteq V$ by above note. This shows that F is u.ω-c.

Now suppose that F^- is image-ω-open. Then $F^-(F(x))$ is an ω-open set in X containing x. On the other hand, by Lemma 2.13(2), F is l.ω-c. and proceed as above. □

Theorem 2.16. *Let $F : X \to Y$ be image-open, image-nonmingled and u.ω-c. Then F is l.ω-c.*

Proof. Let $x \in X$ and V be an open set with $F(x) \cap V \neq \emptyset$. Then $F^+(F(x))$ is ω-open in X and $x \in F^+(F(x))$. Put $U = F^+(F(x))$. Thus we have an ω-open set U containing x such that if $z \in U$ then $F(z) = F(x)$ and $F(z) \cap V \neq \emptyset$. This shows that F is l.ω-c. □

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 2.17. ([10]) *For a multifunction $F : X \to Y$, the following hold:*
(1) $G_F^+(A \times B) = A \cap F^+(B)$,
(2) $G_F^-(A \times B) = A \cap F^-(B)$
for any subsets $A \subseteq X$ and $B \subseteq Y$.

Theorem 2.18. *Let $F : X \to Y$ be an image-compact multifunction. Then the graph multifunction of F is u.ω-c. if and only if F is u.ω-c.*

Proof. ($\Rightarrow$) Suppose that $G_F : X \to X \times Y$ is u.ω-c. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists $U \in \omega O(X, x)$ such that $G_F(U) \subseteq X \times V$. By the previous lemma, we have $U \subseteq G_F^+(X \times V) = F^+(V)$ and $F(U) \subseteq V$. This shows that F is u.ω-c.

($\Leftarrow$) Suppose that F is u.ω-c. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist open sets $U(y) \subseteq X$ and $V(y) \subseteq Y$ such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family of $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$. Since $F(x)$ is compact, it follows that there exists a finite number of points, says $y_1, y_2, ..., y_n$ in $F(x)$ such that $F(x) \subseteq \{V(y_i) : i = 1, 2, ..., n\}$. Take $U = \cap\{U(y_i) : i = 1, 2, ..., n\}$ and $V = \cup\{V(y_i) : i = 1, 2, ..., n\}$. Then U and V are open sets in X and Y, respectively, and $\{x\} \times F(x) \subseteq U \times V \subseteq W$. Since F is u.ω-c., there exists $U_0 \in \omega O(X, x)$ such that $F(U_0) \subseteq V$. By the previous lemma, we have $U \cap U_0 \subseteq U \cap F^+(V) = G_F^+(U \times V) \subseteq G_F^+(W)$. Therefore, we obtain $U \cap U_0 \in \omega O(X, x)$ and $G_F(U \cap U_0) \subseteq W$. This shows that G_F is u.ω-c. □

Theorem 2.19. *A multifunction $F : X \to Y$ is l.ω-c. if and only if the graph multifunction G_F is l.ω-c.*

Proof. ($\Rightarrow$) Suppose that F is l.ω-c. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$ and hence $(x, y) \in U \times V \subseteq W$ for some open sets U and V of X and Y, respectively. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \omega O(X, x)$ such that $G \subseteq F^-(V)$. By Lemma 2.17, $U \cap G \subseteq U \cap F^-(V) = G_F^-(U \times V) \subseteq G_F^-(W)$. Therefore, we obtain $x \in U \cap G \in \omega O(X, x)$ and hence G_F is l.ω-c.

($\Leftarrow$) Suppose that G_F is l.ω-c. Let $x \in X$ and V be any open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is l.ω-c., there exists an ω-open set U containing x such that $U \subseteq G_F^-(X \times V)$. By Lemma 2.17, we have $U \subseteq F^-(V)$. This shows that F is l.ω-c. □

Lemma 2.20. ([17]) *Let A be a subset of a space (X, τ). Then $(\tau_\omega)_A = (\tau_A)_\omega$.*

Theorem 2.21. *For a multifunction $F : X \to Y$, the following statements are true.*

a) If F is u(l).ω-c. and $A \subseteq X$, then $F|_A : A \to Y$ is u(l).ω-c.;

b) Let $\{A_\alpha : \alpha \in I\}$ be open cover of X. Then a multifunction $F : X \to Y$ is u(l).ω-c. iff the restrictions $F|_{A_\alpha} : A_\alpha \to Y$ are u(l).ω-c. for every $\alpha \in I$.

The proof is obvious from the above lemma and we omit it.

3. Some applications

Theorem 3.1. *Let F and G be u.ω-c. and image-closed multifunctions from a topological space X to a normal topological space Y. Then the set $A = \{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

Proof. Let $x \in X - A$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are image-closed multifunctions and Y is a normal space, then there exist disjoint open sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are u.ω-c., then the sets $F^+(U)$ and $G^+(V)$ are ω-open and contain x. Put $W = F^+(U) \cap G^+(V)$. Then W is an ω-open set containing x and $W \cap A = \emptyset$. Hence, A is closed in X. □

Definition 3.2. ([2]) A space X is said to be ω-T_2 if for each pair of distinct points x and y in X, there exist $U \in \omega O(X, x)$ and $V \in \omega O(X, y)$ such that $U \cap V = \emptyset$.

Theorem 3.3. *Let $F : X \to Y$ be an u.ω-c. multifunction and image-closed from a topological space X to a normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is an ω-T_2 space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a normal space, then there exists disjoint open sets U and V containing $F(x)$ and $F(y)$, respectively. Thus, $F^+(U)$ and $F^+(V)$ are disjoint ω-open sets containing x and y, respectively. Thus, X is ω-T_2. □

Definition 3.4. The graph $G(F)$ of the multifunction $F : X \to Y$ is *ω-closed with respect to* X if for each $(x, y) \notin G(F)$, there exist an ω-open set U containing x and an open set V containing y such that $(U \times V) \cap G(F) = \emptyset$.

Definition 3.5. A subset A of a topological space X is called *α-paracompact* [15] if every open cover of A in X has a locally finite open refinement in X which covers A.

Theorem 3.6. *If $F : X \to Y$ is u.ω-c. and image-α-paracompact multifunction into a Hausdorff space Y, then the graph $G(F)$ is ω-closed with respect to X.*

Proof. Let $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Therefore, for every $y \in F(x_0)$, there exists an open set $V(y)$ and an open set $W(y)$ in Y containing y and y_0 respectively, such that $V(y) \cap W(y) = \emptyset$. Then $\{V(y)|y \in F(x_0)\}$ is a open cover of $F(x_0)$, thus there is a locally finite open cover $\Psi = \{U_\beta|\beta \in \Delta\}$ of $F(x_0)$ which refines $\{V(y)|y \in F(x_0)\}$. So there exists an open neighborhood W_0 of y_0 such that W_0 intersect only finitely many members $U_{\beta_1}, U_{\beta_2}, ..., U_{\beta_n}$ of Ψ. Chose finitely many points $y_1, y_2, ..., y_n$ of $F(x_0)$ such that $U_{\beta_k} \subset V(y_k)$ of each $1 \leq k \leq n$ and set $W = W_0 \cap [\bigcap_{k=1}^{n} W(y_k)]$. Then W is an open neighborhood of y_0 such that $W \cap (\cup\Psi) = \emptyset$. Since F is u.ω-c., then there exists an ω-open set U containing x_0 such that $F(U) \subset \cup\Psi$. Therefore, we have that $(U \times W) \cap G(F) = \emptyset$. Thus, $G(F)$ is ω-closed set with respect to X. □

In the above theorem, for upper ω-continuous multifunction F, if F is taken as a image-closed multifunction and Y is taken as a regular space, then we get also same result.

Definition 3.7. A space X is called *ω-compact* [2] if every ω-open cover of X has a finite subcover.

Theorem 3.8. *Let $F : X \to Y$ be a image-compact and u.ω-c. multifunction. If X is ω-compact and F is surjective, then Y is compact.*

Proof. Let Φ be an open cover of Y. If $x \in X$, then we have $F(x) \subseteq \cup\Phi$. Thus Φ is an open cover of $F(x)$. Since $F(x)$ is compact, there exists a finite subfamily $\Phi_{n(x)}$ of Φ such that $F(x) \subseteq \cup\Phi_{n(x)} = V_x$. Then V_x is an open set in Y. Since F is u.ω-c., $F^+(V_x)$ is an ω-open set in X. Therefore, $\Omega = \{F^+(V_x) : x \in X\}$ is an ω-open cover of X. Since X is ω-compact, there exists points $x_1, x_2, ..., x_n \in X$ such that $X \subset \cup\{F^+(V_{x_i}) : x_i \in X, i = 1, 2, ..., n\}$. So we obtain $Y = F(X) \subseteq F(\cup\{F^+(V_{x_i}) : i = 1, 2, ..., n\}) \subset \cup\{V_{x_i} : i = 1, 2, ..., n\} \subset \cup\{\Phi_{n(x_i)} : i = 1, 2, ..., n\}$. Thus Y is compact. □

In [[6], Theorem 4.1], Hdeib showed that a space (X, τ) is Lindelöf if and only if (X, τ_ω) is Lindelöf.

Theorem 3.9. *Let $F : (X, \tau) \to (Y, \sigma)$ be an image-Lindelöf or image-compact and u.ω-c. multifunction. If X is Lindelöf and F is surjective, then Y is Lindelöf.*

Proof. Let Φ be an open cover of Y. If $x \in X$, then we have $F(x) \subseteq \cup\Phi$. Thus Φ is an open cover of $F(x)$.

When $F(x)$ is Lindelöf, there exists a countable subfamily Φ_x of Φ such that $F(x) \subseteq \cup\Phi_x = V_x$. Then V_x is an open set in Y. Since F is u.ω-c., $F^+(V_x)$ is an ω-open set in X. Therefore, $\Omega = \{F^+(V_x) : x \in X\}$ is an ω-open cover of X. By Theorem 4.1 of [6], there exists points $x_1, x_2, ..., x_n, ... \in X$ such that $X \subseteq \cup\{F^+(V_{x_i}) : x_i \in X, i = 1, 2, ..., n, ...\}$. So we obtain $Y = F(X) \subseteq F(\cup\{F^+(V_{x_i}) : i = 1, 2, ..., n, ...\}) \subseteq \cup\{V_{x_i} : i = 1, 2, ..., n, ...\} \subseteq \cup\{\Phi_{x_i} : i = 1, 2, ..., n, ...\}$. Thus Y is Lindelöf.

When $F(x)$ is compact, there exists a finite subfamily Φ_x of Φ such that $F(x) \subseteq \cup\Phi_x = V_x$. Then V_x is an open set in Y. Since F is u.ω-c., $F^+(V_x)$ is an ω-open set in X. Therefore, $\Omega = \{F^+(V_x) : x \in X\}$ is an ω-open cover of X. By Theorem 4.1 of [6], there exists points $x_1, x_2, ..., x_n, ... \in X$ such that $X \subseteq \cup\{F^+(V_{x_i}) : x_i \in X, i = 1, 2, ..., n, ...\}$. So we obtain $Y = F(X) \subseteq F(\cup\{F^+(V_{x_i}) : i = 1, 2, ..., n, ...\}) \subseteq \cup\{V_{x_i} : i = 1, 2, ..., n, ...\} \subseteq \cup\{\Phi_{x_i} : i = 1, 2, ..., n, ...\}$. Thus Y is Lindelöf. □

4. ω-connectedness

Definition 4.1. ([2]) If a space X can not be written as the union of two nonempty disjoint ω-open sets, then X is said to be *ω-connected*.

Definition 4.2. Two non-empty subsets A and B of X are said to be *ω-separated* if $\omega Cl(A) \cap B = \emptyset = A \cap \omega Cl(B)$.

The proof of the following theorem is obtained by ordinary arguments.

Theorem 4.3. *For every topological space X, the following conditions are equivalent:*
(1) X is ω-connected;
(2) $\emptyset$ and X are the only ω-open and ω-closed subsets of X;
(3) If $X = A \cup B$ and the sets A and B are ω-separated, then one of them is empty.

Theorem 4.4. *Let X be ω-connected, $F : X \to Y$ be ω-continuous multifunction on X and V be a subset of Y such that at least one of the following conditions is fulfilled:*
(1) V is clopen;
(2) F is image-open and V is closed;
(3) F^- is image-ω-open and V is open;
(4) F is image-open and F^- is image-ω-open.
Then either $F^+(V) = X$ or $F^-(Y - V) = X$.

Proof. (1) Let V be clopen set in Y. Since F is l.ω-c. and u.ω-c., $F^+(V)$ is ω-open and ω-closed in X by Theorems 2.3 and 2.4. Then by Theorem 4.3, $F^+(V) = X$ or $X - F^+(V) = X$. Hence, $F^+(V) = X$ or $F^-(Y - V) = X$.

(2) Let F be image-open and V be closed. Since F is l.ω-c., $F^-(Y - V)$ is ω-open in X. By Lemma 2.14, $F^-(Y - V)$ is ω-closed. Since $F^-(Y - V) = X - F^+(V)$, the result follows.

(3) Let F^- be image-ω-open and V be open. Since F is u.ω-c., $F^-(Y - V)$ is ω-closed in X. On the other hand, since F^- is image-ω-open, $F^-(Y - V) = \cup\{F^-(y) : y \in Y - V\}$ is ω-open in X. Hence, the result follows.

(4) Let F be image-open and F^- be image-ω-open. By Lemma 2.14, $F^-(Y - V)$ is ω-closed for any open set $V \subseteq Y$. On the other hand, since F^- image-ω-open, $F^-(Y - V) = \cup\{F^-(y) : y \in Y - V\}$ is ω-open in X. Hence, the result follows. □

Corollary 4.5. *Let X be ω-connected and $F : X \to Y$ be an ω-continuous multifunction onto Y such that $F(x)$ is connected in Y for some $x \in X$. Then Y is connected.*

Proof. Let V be a clopen set in Y. Then V and $Y - V$ are separated. Since $F(x)$ is connected, either $F(x) \subseteq V$ or $F(x) \subseteq Y - V$. By Theorem 4.4(1), either $F(X) \subseteq V$ or $F(X) \subseteq Y - V$. Since F is onto, it follows that $V = Y$ or $V = \emptyset$. This implies that Y is connected. □

Corollary 4.6. *Let X be ω-connected and $F : X \to Y$ be an ω-continuous image-open multifunction such that either F is image-closed or F^- is image-ω-open. Then F is constant.*

Proof. Let $x \in X$ and $F(x) = V$. Suppose that F is image-closed. By Theorem 4.4(1), $F(X) \subseteq V$, thus $F(x) = F(X)$. Now suppose that F^- is image-ω-open. By Theorem 4.4(3), $F(X) \subseteq V$, thus $F(x) = F(X)$. This completes the proof. □

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