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On the approximations of solutions to stochastic differential delay equations with Poisson random measure via Taylor series

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Abstract. The subject of this paper is the analytic approximation of solution to stochastic differential delay equations with Poisson jump. We introduce approximate methods for stochastic differential equations driven by Poisson random measure, as well as for those driven by Poisson process. In both cases, approximate equations are defined on equidistant partitions of the time interval, and their coefficients are Taylor approximations of the coefficients of the initial equation. It will be shown that the approximate solutions converge in the L^p-sense and almost surely to the solutions of the corresponding initial equations. The order of the L^p-convergence of the approximate solutions to the solution of the initial equation is established and it increases when the number of degrees in Taylor approximations of coefficients increases.

1. Introduction

Recently there is an increasing interest in the study of stochastic differential equations with jumps (see [1, 2]). Namely, models which incorporate jumps have become popular in finance and some areas of science and engineering. There is evidence that the dynamics of prices of financial instruments exhibit jumps which cannot be adequately described solely by diffusion processes (see,for example, [3]). Also, there are empirical studies, such as [4, 5], which demonstrate the existence of jumps in stock markets, the foreign exchange market and bond markets. Since only a limited class of stochastic differential delay equations admit explicit solutions, there is a need for the development of approximate methods. Some of the results in this area can be found in [6–8] where convergence of explicit numerical methods is considered. On the other hand, in [9], stability and convergence of the semi-implicit Euler method for linear stochastic differential delay equations are studied.

The subject of this paper are stochastic differential delay equations with Poisson jump which present a natural extension of stochastic differential delay equations. The fact that explicit solutions can hardly be obtained for stochastic differential delay equations, either with Poisson random measure or with Poisson process, was the main motivation for the approximate methods which will be presented in this paper. These methods could give explicitly solvable approximate equations or those suitable for the application of numerical methods. Some of the existing results related to this class of equations can be found in [10–12]

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where the semi-implicit Euler method is developed, in [13] where convergence of the Euler-Maruyama method is considered, as well as in [14].

The fundamentals of the approximate method considered here go back to papers [15, 16] by M.A. Atalla and [17, 18] by S. Janković and D. Ilić. In [16], the approximate solution to the solution of an ordinary stochastic differential equation is constructed on the basis of Taylor approximations of the coefficients of the initial equation, up to the first derivative. The rate of this approximation in the L^p-sense was $O(\delta_n^p)$ when $n \to \infty$ and $\delta_n \to 0$. This concept is appropriately extended in [17] in the sense that drift and diffusion coefficients of the approximate equations are taken to be Taylor approximations of coefficients of the initial equation up to the m_1th and m_2th derivatives, respectively. In this case, the closeness between the solutions in L^p-sense was measured as $O(\delta_n^{(m+1)p/2})$ when $n \to \infty$ and $\delta_n \to 0$, where $m = \min\{m_1, m_2\}$. In [18] this idea was extended to stochastic integrodifferential equations, in [19] to stochastic functional differential equations, in [20] to stochastic pantograph differential equations with Markovian switching and in [21] to stochastic differential equations with time-dependent delay. In a similar way, solutions of stochastic differential delay equations with Markovian switching are approximated in [22]. Moreover, in [23] the authors considered the application of Taylor expansion in approximation of solution to stochastic differential delay equations with Poisson process. However, it remains open problem under which assumptions the order of the L^p-convergence of the approximate solution to the exact solution increases when the number of degrees in Taylor approximation increases.

Our main goal is the development of the approximate method for stochastic differential delay equations driven by Poisson random measure which generalizes the results from [23] related to stochastic differential equations driven by Poisson process. In Section 2 of the present paper the solution of the initial equation is approximated by a sequence of solutions to stochastic differential equations with drift, diffusion and jump coefficients which are Taylor approximations of coefficients of the corresponding initial equation, up to arbitrary derivatives. We will show, under the appropriate conditions, that the rate of the L^p-closeness between the approximate solution and the solution of the initial equation increases when the number of degrees in Taylor approximations of coefficients increases. Moreover, we will show the almost sure convergence of the approximate solution to the exact solution. In Section 3 we will present the results analogous to those in Section 2 for stochastic differential delay equations with Poisson process. In that way we improve the results from [23] where the rate of the L^p-convergence of the approximate solution to the exact solution of this type of equation is $\gamma \in (0, 1]$.

Before stating the main results, we present the essential notations and definitions which are necessary for further consideration. The initial assumption is that all random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T$, $t \geq 0$ be an m-dimensional standard Brownian motion, $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Let the Euclidean norm be denoted by $|\cdot|$ and, for simplicity, $trace[B^T B] = |B|^2$ for matrix B, where B^T is the transpose of a vector or a matrix. For a fixed delay $\tau > 0$, let $C([-\tau, 0]; R^d)$ be the family of continuous functions $\varphi : [-\tau, 0] \to R^d$ with the norm $\|\varphi\| = \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|$ and $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$, the family of all bounded $\mathcal{F}_0$-measurable $C([-\tau, 0]; R^d)$-valued random variables.

2. Stochastic differential delay equations driven by Poisson random measure

Let us begin with discussion of the following stochastic differential delay equation driven by Poisson random measure

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t) + \int_{R^d} h(x(t), x(t-\tau), u, t)\tilde{\nu}(du, dt),\ \ t \in [t_0, T], \tag{1}$$

$$x_{t_0} = \xi = \{\xi(\theta) : \theta \in [-\tau, 0]\}, \tag{2}$$

where

$$f : R^d \times R^d \times [t_0, T] \to R^d,\ \ g : R^d \times R^d \times [t_0, T] \to R^{d\times m},\ \ h : R^d \times R^d \times R^d \times [t_0, T] \to R^d,$$

$x(t)$ is a d-dimensional state process, $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ and

$$\tilde{\nu}(du, dt) = \nu(du, dt) - \Pi(du)dt$$

is a compensated Poisson random measure on $R^d \times [t_0, T]$ which is independent of w.

A d-dimensional stochastic process $\{x(t), t \in [t_0 - \tau, T]\}$ is said to be a solution to Eq. (1) if it is a.s. *cádlág*, $\{x(t), t \in [t_0, T]\}$ is $\mathcal{F}_t$-adapted, $\int_{t_0}^T |f(x(t), x(t-\tau), t)|dt < \infty$ a.s., $\int_{t_0}^T |g(x(t), x(t-\tau), t)|^2 dt < \infty$ a.s., $\int_{t_0}^T \int_{R^d} |h(x(t), x(t-\tau), u, t)|^2 \Pi(du)dt < \infty$ a.s., $x_{t_0} = \xi$ a.s. and for every $t \in [t_0, T]$, the integral form of Eq. (1) holds a.s.

A solution $\{x(t), t \in [t_0 - \tau, T]\}$ is said to be unique if any other solution $\{\tilde{x}(t) : t \in [t_0 - \tau, T]\}$ is indistinguishable from it, in the sense that $P\{x(t) = \tilde{x}(t), t \in [t_0 - \tau, T]\} = 1$.

If one assumes that the global Lipschitz condition and the linear growth condition are satisfied, that is, there exists a constant $\bar{K} > 0$ such that for all $x_1, x_2, y_1, y_2, u \in R^d$ and $t \in [t_0, T]$,

$$|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 \vee |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \vee \int_{R^d} |h(x_1, y_1, u, t) - h(x_2, y_2, u, t)|^2 \Pi(du) \le \bar{K}\Big(|x_1 - x_2|^2 + |y_1 - y_2|^2\Big), \tag{3}$$

and also there exists a constant $K > 0$ such that for all $x, y, u \in R^d$ and $t \in [t_0, T]$,

$$|f(x, y, t)|^2 \vee |g(x, y, t)|^2 \vee \int_{R^d} |h(x, y, u, t)|^2 \Pi(du) \le K(1 + |x|^2 + |y|^2), \tag{4}$$

then there exists a unique solution $\{x(t), t \in [t_0 - \tau, T]\}$ to Eq. (1). The idea for the proof can be found in [24].

Let us present Eq. (1) in its equivalent integral form, that is, for $t \in [t_0, T]$,

$$x(t) = \xi(0) + \int_{t_0}^t f(x(s), x(s-\tau), s)ds + \int_{t_0}^t g(x(s), x(s-\tau), s)dw(s) + \int_{t_0}^t \int_{R^d} h(x(s), x(s-\tau), u, s)\tilde{\nu}(du, ds). \tag{5}$$

We will approximate the solution of the equation (5) on the equidistant partition

$$t_0 < t_1 < \ldots < t_n = T \tag{6}$$

of the interval $[t_0, T]$, where n is chosen in a way that $\delta_n = \frac{T - t_0}{n} < 1$ and also there exists an integer n_* such that $\tau = n_* \delta_n$. So, the partitioning points of the interval $[t_0 - \tau, T]$ are

$$t_k = t_0 + k\delta_n, \quad k = -n_*, -n_* + 1, \ldots, -1, 0, 1, \ldots, n.$$

The solution $x = \{x(t), t \in [t_0, T]\}$ to Eq. (5) will be approximated on the partition (6) by the solutions $\{x^n(t), t \in [t_k, t_{k+1}]\}, k = 0, 1, \ldots, n-1$ of the equations

$$x^n(t) = x^n(t_k) + \int_{t_k}^t \sum_{i=0}^{m_1} \frac{d^i f(x^n(t_k), x^n(t_{k-n_*}), s)}{i!} ds + \int_{t_k}^t \sum_{i=0}^{m_2} \frac{d^i g(x^n(t_k), x^n(t_{k-n_*}), s)}{i!} dw(s) + \int_{t_k}^t \int_{R^d} \sum_{i=0}^{m_3} \frac{d^i h(x^n(t_k), x^n(t_{k-n_*}), u, s)}{i!} \tilde{\nu}(du, dt), \tag{7}$$

satisfying the initial condition $x^n_{t_0} = \xi$ a.s. In this equation coefficients are Taylor approximations of f, g and h in the first argument in the neighbourhood of the points $x^n(t_k)$, and also in the second argument in the neighbourhood of the points $x^n(t_{k-n_*}), k = 0, 1, ..., n-1$, up to the m_1-th, m_2-th and m_3-th derivatives, respectively, while

$$d^i f(x^n(t_k), x^n(t_{k-n_*}), s) = \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^i f(x^n(t_k), x^n(t_{k-n_*}), s)}{\partial^j x^n(s) \partial^{i-j} x^n(s-\tau)} (\Delta x^n_{t_k})^j (\Delta x^n_{t_{k-n_*}})^{i-j},$$

$$d^i g(x^n(t_k), x^n(t_{k-n_*}), s) = \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^i g(x^n(t_k), x^n(t_{k-n_*}), s)}{\partial^j x^n(s) \partial^{i-j} x^n(s-\tau)} (\Delta x^n_{t_k})^j (\Delta x^n_{t_{k-n_*}})^{i-j},$$

$$d^i h(x^n(t_k), x^n(t_{k-n_*}), u, s) = \sum_{j=0}^{i} \binom{i}{j} \frac{\partial^i h(x^n(t_k), x^n(t_{k-n_*}), u, s)}{\partial^j x^n(s) \partial^{i-j} x^n(s-\tau)} (\Delta x^n_{t_k})^j (\Delta x^n_{t_{k-n_*}})^{i-j},$$

for $\Delta x^n_{t_k} = x^n(s) - x^n(t_k)$ and $\Delta x^n_{t_{k-n_*}} = x^n(s-\tau) - x^n(t_{k-n_*})$.

The approximate solution $x^n = \{x^n(t), t \in [t_0 - \tau, T]\}$ is constructed as an a.s. *cádlág* process by connecting successively the initial condition $\{\xi(\theta) : \theta \in [-\tau, 0]\}$ and processes $\{x^n(t), t \in [t_k, t_{k+1}]\}$ at the points t_k whenever $k = 0, 1, ..., n-1$.

Obviously, it must be required that f, g and h satisfy appropriate conditions. With no particular emphasis on conditions, we suppose the existence and uniqueness of the solutions explicitly used in our discussion. In addition to the Lipschitz condition (3) and the linear growth condition (4), we introduce the following assumptions:

$\mathcal{A}_1$: The functions f, g and h have Taylor expansions in the first and second arguments up to the m_1-th, m_2-th and m_3-th derivatives, respectively.

$\mathcal{A}_2$: Partial derivatives of the order $m_1 + 1$ of f, of the order $m_2 + 1$ of g and $m_3 + 1$ of h are uniformly bounded, i.e. there exists a positive constant L such that

$$\sup_{R^d \times R^d \times [t_0, T]} \left| \frac{\partial^{m_1+1} f(x, y, t)}{\partial x^j \partial y^{m_1+1-j}} \right| \le L, \ j = 0, 1, ..., m_1 + 1$$

$$\sup_{R^d \times R^d \times [t_0, T]} \left| \frac{\partial^{m_2+1} g(x, y, t)}{\partial x^j \partial y^{m_2+1-j}} \right| \le L, \ j = 0, 1, ..., m_2 + 1,$$

$$\sup_{R^d \times R^d \times [t_0, T]} \int_{R^d} \left| \frac{\partial^{m_3+1} h(x, y, u, t)}{\partial x^j \partial y^{m_3+1-j}} \right|^2 \Pi(du) \le L^2, \ j = 0, 1, ..., m_3 + 1,$$

$\mathcal{A}_3$: There exist unique, a.s. *cádlág* solutions x and x^n to the equations (5) and (7), respectively, such that, for $p \ge 2$,

$$E \sup_{t \in [t_0 - \tau, T]} |x(t)|^p < \infty, \quad E \sup_{t \in [t_0 - \tau, T]} |x^n(t)|^{(M+1)^2 p} \le Q < \infty,$$

where $M = \max\{m_1, m_2, m_3\}$ and $Q > 0$ is a constant independent of n. Moreover, we suppose that all the Lebesque and Ito integrals, as well as the integrals with respect to Poisson measure, which will be used further are also well defined.

$\mathcal{A}_4$: For $p \ge 2$ there exists a constant $C_\xi > 0$ such that for $k = -n_*, -n_* + 1, ..., -1$,

$$E \sup_{s, t \in [t_k, t_{k+1}]} |\xi(t) - \xi(s)|^p \le C_\xi \cdot n^{-p/2}.$$

Remark 2.1. *Under the Lipschitz condition (3) and the linear growth condition (4), the solution x of Eq. (1) has uniformly finite moments of the order $p \ge 2$, that is,*

$$E \sup_{t \in [t_0 - \tau, T]} |x(t)|^p < \infty.$$

If $m_1 = m_2 = m_3 = 0$, then the conditions (3) and (4) guarantee that both exact and approximate solutions admit finite moments.

Further more, we will apply several times, without special emphasis, the elementary inequality $\left(\sum_{i=1}^m a_i\right)^q \leq m^{q-1}\sum_{i=1}^m a_i^q, a_i > 0, q \in N$, the Hölder inequality to Lebesgue integrals and the Burkholder-Davis-Gundy inequality to the integrals of two other types.

In order to estimate the closeness between the solutions x and x^n, we first state the following result which will be used in the proof of the main result.

Proposition 2.2. *Let $\{x^n(t), t \in [t_k, t_{k+1}]\}, k = 0, 1, ..., n-1$, be the solution to Eq. (7) and let the condition (4) and the assumptions $\mathcal{A}_1$,$\mathcal{A}_2$,$\mathcal{A}_3$ be satisfied. Then for every $2 \leq r \leq (M+1)p$,*

$$E\sup_{s\in[t_k,t]}|x^n(s)-x^n(t_k)|^r \leq C\cdot n^{-r/2}, \quad t\in[t_k,t_{k+1}], \quad k=0,1,...,n-1.$$

Moreover, if the assumption $\mathcal{A}_4$ is satisfied, then

$$E\sup_{s\in[t_k,t]}|x^n(s-\tau)-x^n(t_{k-n_*})|^r \leq \bar{C}\cdot n^{-r/2},\ t\in[t_k,t_{k+1}],\ k=0,1,...,n-1,$$

where C and $\bar{C}$ are positive constants, independent of n.

Proof. For reasons of notational simplicity, let us denote that

$$\begin{aligned}
F(x_t^n, x_{t-\tau}^n, t; x_{t_k}^n, x_{t_{k-n_*}}^n) &= \sum_{i=0}^{m_1}\frac{d^i f(x^n(t_k), x^n(t_{k-n_*}), t)}{i!},\\
G(x_t^n, x_{t-\tau}^n, t; x_{t_k}^n, x_{t_{k-n_*}}^n) &= \sum_{i=0}^{m_2}\frac{d^i g(x^n(t_k), x^n(t_{k-n_*}), t)}{i!},\\
H(x_t^n, x_{t-\tau}^n, u, t; x_{t_k}^n, x_{t_{k-n_*}}^n) &= \sum_{i=0}^{m_3}\frac{d^i h(x^n(t_k), x^n(t_{k-n_*}), u, t)}{i!},
\end{aligned}$$

whenever $t \in [t_k, t_{k+1}], k \in \{0, 1, ..., n-1\}$.

Then, in a view of the assumption $\mathcal{A}_1$, we have

$$\begin{aligned}
f(x^n(t), x^n(t-\tau), t) &= F(x_t^n, x_{t-\tau}^n, t; x_{t_k}^n, x_{t_{k-n_*}}^n) + r_{m_1}^f(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, t),\\
g(x^n(t), x^n(t-\tau), t) &= G(x_t^n, x_{t-\tau}^n, t; x_{t_k}^n, x_{t_{k-n_*}}^n) + r_{m_2}^g(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, t),\\
h(x^n(t), x^n(t-\tau), u, t) &= H(x_t^n, x_{t-\tau}^n, u, t; x_{t_k}^n, x_{t_{k-n_*}}^n) + r_{m_3}^h(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, u, t),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
r_{m_1}^f(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, t) &= \frac{d^{m_1+1} f(x^n(t_k)+\theta_f\Delta x_{t_k}^n, x^n(t_{k-n_*})+\theta_f\Delta x_{t_{k-n_*}}^n, t)}{(m_1+1)!},\\
r_{m_2}^g(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, t) &= \frac{d^{m_2+1} g(x^n(t_k)+\theta_g\Delta x_{t_k}^n, x^n(t_{k-n_*})+\theta_g\Delta x_{t_{k-n_*}}^n, t)}{(m_2+1)!},\\
r_{m_3}^h(\Delta x_{t_k}^n, \Delta x_{t_{k-n_*}}^n, u, t) &= \frac{d^{m_3+1} h(x^n(t_k)+\theta_h\Delta x_{t_k}^n, x^n(t_{k-n_*})+\theta_h\Delta x_{t_{k-n_*}}^n, u, t)}{(m_3+1)!},
\end{aligned}$$

for some $\theta_f, \theta_g, \theta_h \in (0,1)$, are the appropriate remainders in Taylor approximations of the functions f, g and h, respectively. Using the assumption $\mathcal{A}_2$, that is, the uniform boundedness of the $(m_1+1)th$, $(m_2+1)th$

and $(m_3+1)th$ partial derivatives of the functions f, g and h, respectively, as well as the Newton binomial formula, we find that

$$
\begin{aligned}
&|r^f_{m_1}(\Delta x^n_{t_k}, \Delta x^n_{t_{k-n_*}}, t)| \le \frac{L}{(m_1+1)!}\Big(|\Delta x^n_{t_k}| + |\Delta x^n_{t_{k-n_*}}|\Big)^{m_1+1}, \qquad (9)\\
&|r^g_{m_2}(\Delta x^n_{t_k}, \Delta x^n_{t_{k-n_*}}, t)| \le \frac{L}{(m_2+1)!}\Big(|\Delta x^n_{t_k}| + |\Delta x^n_{t_{k-n_*}}|\Big)^{m_2+1},\\
&\int_{R^d} |r^h_{m_3}(\Delta x^n_{t_k}, \Delta x^n_{t_{k-n_*}}, u, t)|^2 \Pi(du) \le \frac{L^2}{[(m_3+1)!]^2}\Big(|\Delta x^n_{t_k}| + |\Delta x^n_{t_{k-n_*}}|\Big)^{2(m_3+1)},
\end{aligned}
$$

for $t \in [t_k, t_{k+1}], k = 0, 1, ..., n-1$.

In order to estimate $E\sup_{s\in[t_k,t]}|x^n(s) - x^n(t_k)|^r$, we will apply the previously mentioned elementary inequality to Eq. (7), the Hölder inequality to the Lebesgue integral and the Burkholder-Davis-Gundy inequality to the Ito integral and also to the integral with respect to Poisson measure. Then, we get for all $t \in [t_k, t_{k+1}], k = 0, 1, ..., n-1$,

$$
\begin{aligned}
E\sup_{s\in[t_k,t]}|x^n(s) - x^n(t_k)|^r \le\ & 3^{r-1}(t-t_k)^{r-1}\int_{t_k}^t E|F(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^r ds \qquad (10)\\
&+3^{r-1}c_r(t-t_k)^{r/2-1}\int_{t_k}^t E|G(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^r ds\\
&+3^{r-1}c_r E\left(\int_{t_k}^t \int_{R^d} |H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^2 \Pi(du) ds\right)^{r/2}\\
\equiv\ & 3^{r-1}(t-t_k)^{r/2-1}[(t-t_k)^{r/2}J_1(t) + c_r J_2(t)] + 3^{r-1}c_r J_3(t),
\end{aligned}
$$

where $J_1(t), J_2(t)$ and $J_3(t)$ are the appropriate integrals, while c_r is a universal constant from the Burkholder-Davis-Gundy inequality.

On the basis of Taylor expansion (8), the growth condition (4), the assumptions $\mathcal{A}_1, \mathcal{A}_3$ and the estimate (9), we get

$$
\begin{aligned}
J_1(t) &= \int_{t_k}^t E|F(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^r ds \qquad (11)\\
&\le 2^{r-1}\int_{t_k}^t E|F(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}}) - f(x^n(s), x^n(s-\tau), s)|^r ds + 2^{r-1}\int_{t_k}^t E|f(x^n(s), x^n(s-\tau), s)|^r ds\\
&\le 2^{r-1}\int_{t_k}^t E\left|\frac{d^{m_1+1} f(x^n(t_k) + \theta_f \Delta x^n_{t_k}, x^n(t_{k-n_*}) + \theta_f \Delta x^n_{t_{k-n_*}}, s)}{(m_1+1)!}\right|^r ds\\
&\quad +2^{r-1}K^{r/2}\int_{t_k}^t E\Big[1 + |x^n(s)|^2 + |x^n(s-\tau)|^2\Big]^{r/2} ds\\
&\le \frac{2^{r-1}L^r}{[(m_1+1)!]^r}\int_{t_k}^t E\Big[|\Delta x^n_{t_k}| + |\Delta x^n_{t_{k-n_*}}|\Big]^{(m_1+1)r} ds + 2^{r-1}3^{r/2-1}K^{r/2}\int_{t_k}^t [1 + E|x^n(s)|^r + E|x^n(s-\tau)|^r]\, ds\\
&\le \frac{2^{r-1}4^{(m_1+1)r}L^r}{[(m_1+1)!]^r}\int_{t_k}^t E\sup_{u\in[t_0-\tau,T]}|x^n(u)|^{(m_1+1)r} ds + 2^{r-1}3^{r/2-1}K^{r/2}\int_{t_k}^t \Big[1 + 2E\sup_{u\in[t_0-\tau,T]}|x^n(u)|^r\Big] ds\\
&\le \frac{2^{(2m_1+3)r-1}L^r R}{[(m_1+1)!]^r}(t-t_k) + 2^{r-1}3^{r/2-1}K^{r/2}(1+2R)(t-t_k)\\
&\equiv C_1\cdot(t-t_k),
\end{aligned}
$$

where $C_1 \equiv C_1(K, L, R, r, m_1)$ is a generic constant and $R = 1 + Q$.

Similarly, by repeating completely the previous procedure, we see that

$$
J_2(t) \le C_2\cdot(t-t_k), \qquad (12)
$$

where $C_2 \equiv C_2(K, L, R, r, m_2)$ is a generic constant. In order to estimate the integral $J_3(t)$, observe that

$$\begin{aligned}\int_{t_k}^{t}\int_{R^d}|H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^2\Pi(du)ds \\ \leq 2\int_{t_k}^{t}\int_{R^d}|H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}}) - h(x^n(s), x^n(s-\tau), u, s)|^2\Pi(du)ds \\ +2\int_{t_k}^{t}\int_{R^d}|h(x^n(s), x^n(s-\tau), u, s)|^2\Pi(du)ds.\end{aligned} \tag{13}$$

Using the estimate (9) and the linear growth condition (4), the estimate (13) becomes

$$\begin{aligned}\int_{t_k}^{t}\int_{R^d}|H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^2\Pi(du)ds \\ \leq \frac{2L^2}{[(m_3+1)!]^2}\int_{t_k}^{t}\Big(|\Delta x^n_{t_k}| + |\Delta x^n_{t_{k-n_*}}|\Big)^{2(m_3+1)}ds + 2K\int_{t_k}^{t}[1+|x^n(s)|^2+|x^n(s-\tau)|^2]ds \\ \leq \left(\frac{2^{4(m_3+1)+1}L^2}{[(m_3+1)!]^2}\sup_{u\in[t_0-\tau,T]}|x^n(u)|^{2(m_3+1)} + 2K\Big[1+2\sup_{u\in[t_0-\tau,T]}|x^n(u)|^2\Big]\right)(t-t_k).\end{aligned} \tag{14}$$

Consequently, in view of the assumption $\mathcal{A}_3$, we obtain

$$\begin{aligned}J_3(t) &= E\left(\int_{t_k}^{t}\int_{R^d}|H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}})|^2\Pi(du)ds\right)^{r/2} \\ &\leq \left(\frac{2^{(2m_3+3)r-1}L^r}{[(m_3+1)!]^r}R + 2^{3r/2-2}K^{r/2}[1+2^{r/2}R]\right)(t-t_k)^{r/2} \\ &\equiv C_3\cdot(t-t_k)^{r/2},\end{aligned} \tag{15}$$

where $C_3 \equiv C_3(K, L, R, r, m_3)$ and $R = 1 + Q$.

Substituting the estimates (11), (12) and (15) into (10) we get

$$E\sup_{s\in[t_k,t]}|x^n(s) - x^n(t_k)|^r \leq C\cdot n^{-r/2},\ \ t\in[t_k, t_{k+1}],\ \ k = 0, 1, ..., n-1, \tag{16}$$

where C is a generic constant independent of n.

Let us now estimate $E\sup_{s\in[t_k,t]}|x^n(s-\tau) - x^n(t_{k-n_*})|^r$ bearing in mind the additional assumption $\mathcal{A}_4$. In further discussion we distinguish two cases depending on whether the approximate solution x^n coincide with the initial condition or not.

1. If $t-\tau < t_0$ for $t \in [t_k, t_{k+1}]$, then $t_{k-n_*} < t_0$. So, in these points the solution x^n coincide with the initial condition. On the basis of the assumption $\mathcal{A}_4$ we get

$$E\sup_{s\in[t_k,t]}|x^n(s-\tau) - x^n(t_{k-n_*})|^r = E\sup_{s\in[t_k,t]}|\xi(s-\tau-t_0) - \xi(t_{k-n_*}-t_0)|^r \leq C_\xi\cdot n^{-r/2}. \tag{17}$$

2. If $t-\tau \geq t_0$ for $t\in[t_k, t_{k+1}]$, then the way we defined partitioning points guarantees that $t_{k-n_*} \geq t_0$. Thus the first part of the proof yields

$$E\sup_{s\in[t_k,t]}|x^n(s-\tau) - x^n(t_{k-n_*})|^r \leq C\cdot n^{-r/2}. \tag{18}$$

Using the estimates (17) and (18) we obtain

$$E\sup_{s\in[t_k,t]}|x^n(s-\tau) - x^n(t_{k-n_*})|^r \leq \bar{C}\cdot n^{-r/2},\ t\in[t_k, t_{k+1}],\ k = 0, 1, ..., n-1.$$

where $\bar{C} = \max\{C_\xi, C\}$. This completes the proof. □

Now, we are in a position to state the main result related to the closeness between the solutions x and x^n in the L^p-sense.

Theorem 2.3. *Let x be the solution to Eq. (5) and x^n be its approximate solution determined by Eqs. (7). Let also the conditions of Proposition 2.2 and the Lipschitz condition (3) be satisfied. Then for $p \geq 2$,*

$$E \sup_{t\in[t_0-\tau,T]} |x(t)-x^n(t)|^p \leq \beta \cdot n^{-(m+1)p/2},$$

where $m=\min\{m_1,m_2,m_3\}$ and β is a generic constant independent of n.

Proof. For an arbitrary $t\in[t_0,T]$, by substituting Eqs. (5) and (7), it follows that

$$x(t)-x^n(t) = \int_{t_0}^t \sum_{k=0}^{n-1} J_{t_k,t_{k+1}\wedge t}(s)ds + \int_{t_0}^t \sum_{k=0}^{n-1} \tilde{J}_{t_k,t_{k+1}\wedge t}(s)dw(s) + \int_{t_0}^t \int_{R^d} \sum_{k=0}^{n-1} \hat{J}_{t_k,t_{k+1}\wedge t}(s)\tilde{\nu}(du,ds),$$

where

$$\begin{aligned} J_{t_k,t_{k+1}\wedge t}(s) &= [f(x(s),x(s-\tau),s) - F(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}})]I_{[t_k,t_{k+1}\wedge t)}(s), \\ \tilde{J}_{t_k,t_{k+1}\wedge t}(s) &= [g(x(s),x(s-\tau),s) - G(x^n_s, x^n_{s-\tau}, s; x^n_{t_k}, x^n_{t_{k-n_*}})]I_{[t_k,t_{k+1}\wedge t)}(s), \\ \hat{J}_{t_k,t_{k+1}\wedge t}(s) &= [h(x(s),x(s-\tau),u,s) - H(x^n_s, x^n_{s-\tau}, u, s; x^n_{t_k}, x^n_{t_{k-n_*}})]I_{[t_k,t_{k+1}\wedge t)}(s). \end{aligned} \tag{19}$$

Then, since both x and x^n satisfy the same initial condition, one obtains

$$\begin{aligned} E \sup_{s\in[t_0-\tau,t]} |x(s)-x^n(s)|^p &\leq E \sup_{s\in[t_0,t]} |x(s)-x^n(s)|^p \\ &\leq 3^{p-1}(t-t_0)^{p-1} \int_{t_0}^t E\Big|\sum_{k=0}^{n-1} J_{t_k,t_{k+1}\wedge t}(s)\Big|^p ds \\ &+3^{p-1}c_p(t-t_0)^{\frac{p}{2}-1} \int_{t_0}^t E\Big|\sum_{k=0}^{n-1} \tilde{J}_{t_k,t_{k+1}\wedge t}(s)\Big|^p ds \\ &+3^{p-1}c_p E\left(\int_{t_0}^t \int_{R^d} \Big|\sum_{k=0}^{n-1} \hat{J}_{t_k,t_{k+1}\wedge t}(s)\Big|^2 \Pi(du)ds\right)^{p/2}. \end{aligned} \tag{20}$$

Clearly, for $j=\max\{i\in\{0,1,...,n-1\}, t_i\leq t\}$, the inequality (20) can be written as

$$\begin{aligned} E \sup_{s\in[t_0-\tau,t]} |x(s)-x^n(s)|^p &\leq 3^{p-1}(t-t_0)^{p-1} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}\wedge t} E\Big|\sum_{k=0}^{n-1} J_{t_k,t_{k+1}\wedge t}(s)\Big|^p ds \\ &+ 3^{p-1}c_p(t-t_0)^{\frac{p}{2}-1} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}\wedge t} E\Big|\sum_{k=0}^{n-1} \tilde{J}_{t_k,t_{k+1}\wedge t}(s)\Big|^p ds \\ &+ 3^{p-1}c_p E\left(\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}\wedge t} \int_{R^d} \Big|\sum_{k=0}^{n-1} \hat{J}_{t_k,t_{k+1}\wedge t}(s)\Big|^2 \Pi(du)ds\right)^{p/2}. \end{aligned} \tag{21}$$

Then, the relation (21) becomes

$$\begin{aligned} &E \sup_{s\in[t_0-\tau,t]} |x(s)-x^n(s)|^p \\ &\leq 3^{p-1}(T-t_0)^{p-1} \sum_{i=0}^{j} J^1_{t_i,t_{i+1}\wedge t} + 3^{p-1}c_p(T-t_0)^{\frac{p}{2}-1} \sum_{i=0}^{j} J^2_{t_i,t_{i+1}\wedge t} + 3^{p-1}c_p E\Big(\sum_{i=0}^{j} J^3_{t_i,t_{i+1}\wedge t}\Big)^{p/2}, \end{aligned} \tag{22}$$

where

$$J^1_{t_i,t_{i+1}\wedge t} = \int_{t_i}^{t_{i+1}\wedge t} E|f(x(s),x(s-\tau),s) - F(x^n_s,x^n_{s-\tau},s;x^n_{t_i},x^n_{t_{i-n_*}})|^p ds, \tag{23}$$

$$J^2_{t_i,t_{i+1}\wedge t} = \int_{t_i}^{t_{i+1}\wedge t} E|g(x(s),x(s-\tau),s) - G(x^n_s,x^n_{s-\tau},s;x^n_{t_i},x^n_{t_{i-n_*}})|^p ds,$$

$$J^3_{t_i,t_{i+1}\wedge t} = \int_{t_i}^{t_{i+1}\wedge t}\int_{R^d} |h(x(s),x(s-\tau),u,s) - H(x^n_s,x^n_{s-\tau},u,s;x^n_{t_i},x^n_{t_{i-n_*}})|^2\Pi(du)ds.$$

Then, we could estimate the integral $J^1_{t_i,t_{i+1}\wedge t}$ in the following way

$$J^1_{t_i,t_{i+1}\wedge t} \le 2^{p-1}\bigg[\int_{t_i}^{t_{i+1}\wedge t} E|f(x(s),x(s-\tau),s) - f(x^n(s),x^n(s-\tau),s)|^p ds \tag{24}$$
$$+\int_{t_i}^{t_{i+1}\wedge t} E|f(x^n(s),x^n(s-\tau),s) - F(x^n_s,x^n_{s-\tau},s;x^n_{t_i},x^n_{t_{i-n_*}})|^p ds\bigg].$$

By applying the Lipschitz condition (3) to the first summand of (24), we obtain

$$\int_{t_i}^{t_{i+1}\wedge t} E|f(x(s),x(s-\tau),s) - f(x^n(s),x^n(s-\tau),s)|^p ds \tag{25}$$
$$\le 2^{p/2-1}\bar{K}^{p/2}\bigg[\int_{t_i}^{t_{i+1}\wedge t} E|x(s)-x^n(s)|^p ds + \int_{t_i}^{t_{i+1}\wedge t} E|x(s-\tau)-x^n(s-\tau)|^p ds\bigg].$$

For the estimation of the second summand of (24) we will use the assumption $\mathcal{A}_2$ and Proposition 2.2. Therefore,

$$\int_{t_i}^{t_{i+1}\wedge t} E|f(x^n(s),x^n(s-\tau),s) - F(x^n_s,x^n_{s-\tau},s;x^n_{t_i},x^n_{t_{i-n_*}})|^p ds \tag{26}$$
$$\equiv \int_{t_i}^{t_{i+1}\wedge t} E\left|\frac{d^{m_1+1}f(x^n(t_i)+\theta_f\Delta x^n_{t_i}, x^n(t_{i-n_*})+\theta_f\Delta x^n_{t_{i-n_*}},s)}{(m_1+1)!}\right|^p ds$$
$$\le \frac{2^{(m_1+1)p-1}L^p}{[(m_1+1)!]^p}\left[\int_{t_i}^{t_{i+1}\wedge t} E|x^n(s)-x^n(t_i)|^{(m_1+1)p}ds + \int_{t_i}^{t_{i+1}\wedge t} E|x^n(s-\tau)-x^n(t_{i-n_*})|^{(m_1+1)p}ds\right]$$
$$\le K_2\cdot n^{-(m_1+1)p/2}(t_{i+1}\wedge t - t_i),$$

where $K_2 = \frac{2^{(m_1+1)p-1}L^p(C+\bar{C})}{[(m_1+1)!]^p}$.

Substituting (25) and (26) into (24), we obtain

$$J^1_{t_i,t_{i+1}\wedge t} \le \varphi(i,t,m_1), \tag{27}$$

where

$$\varphi(i,t,m_1) = 2^{3p/2-2}\bar{K}^{p/2}\left[\int_{t_i}^{t_{i+1}\wedge t}\Big[E|x(s)-x^n(s)|^p ds + E|x(s-\tau)-x^n(s-\tau)|^p\Big]ds\right] \tag{28}$$
$$+2^{p-1}K_2\cdot n^{-(m_1+1)p/2}(t_{i+1}\wedge t - t_i).$$

Analogously,

$$J^2_{t_i,t_{i+1}\wedge t} \le \varphi(i,t,m_2). \tag{29}$$

On the other hand, we have

$$
\begin{aligned}
J^3_{t_i,t_{i+1}\wedge t} &\le 2\int_{t_i}^{t_{i+1}\wedge t}\int_{R^d}|h(x(s),x(s-\tau),u,s)-h(x^n(s),x^n(s-\tau),u,s)|^2\Pi(du)ds\\
&\quad+2\int_{t_i}^{t_{i+1}\wedge t}\int_{R^d}|h(x^n(s),x^n(s-\tau),u,s)-H(x^n_s,x^n_{s-\tau},u,s;x^n_{t_i},x^n_{t_{i-n_*}})|^2\Pi(du)ds\\
&\le 2\bar{K}\int_{t_i}^{t_{i+1}\wedge t}\Big[|x(s)-x^n(s)|^2+|x(s-\tau)-x^n(s-\tau)|^2\Big]ds\\
&\quad+K_3\int_{t_i}^{t_{i+1}\wedge t}\Big[|x^n(s)-x^n(t_i)|^{2(m_3+1)}+|x^n(s-\tau)-x^n(t_{i-n_*})|^{2(m_3+1)}\Big]ds,
\end{aligned}
$$

where $K_3=\frac{2^{2(m_3+1)}L^2}{[(m_3+1)!]^2}$. Consequently, we have that

$$
\begin{aligned}
\sum_{i=0}^{j}J^3_{t_i,t_{i+1}\wedge t} &\le 2\bar{K}\int_{t_0}^{t}\Big[|x(s)-x^n(s)|^2+|x(s-\tau)-x^n(s-\tau)|^2\Big]ds+K_3\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)|x^n(s)-x^n(t_i)|^{2(m_3+1)}ds \\
&\quad+K_3\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)|x^n(s-\tau)-x^n(t_{i-n_*})|^{2(m_3+1)}ds.
\end{aligned}
\tag{30}
$$

On the basis of Proposition 2.2 and the estimate (30), we get

$$
\begin{aligned}
E\Big(\sum_{i=0}^{j}J^3_{t_i,t_{i+1}\wedge t}\Big)^{p/2} &\le 3^{p/2-1}2^{p/2}\bar{K}^{p/2}E\left(\int_{t_0}^{t}[|x(s)-x^n(s)|^2+|x(s-\tau)-x^n(s-\tau)|^2]ds\right)^{p/2}\\
&\quad+3^{p/2-1}K_3^{p/2}\left[E\left(\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)|x^n(s)-x^n(t_i)|^{2(m_3+1)}ds\right)^{p/2}\right.\\
&\quad\left.+E\left(\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)|x^n(s-\tau)-x^n(t_{i-n_*})|^{2(m_3+1)}ds\right)^{p/2}\right]\\
&\le 3^{p/2-1}2^{p}\bar{K}^{p/2}(T-t_0)^{p/2-1}\int_{t_0}^{t}E\sup_{u\in[t_0-\tau,s]}|x(u)-x^n(u)|^p ds\\
&\quad+3^{p/2-1}K_3^{p/2}(T-t_0)^{p/2-1}\left[\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)E|x^n(s)-x^n(t_i)|^{(m_3+1)p}ds\right.\\
&\quad\left.+\int_{t_0}^{t}\sum_{i=0}^{j}I_{[t_i,t_{i+1}\wedge t)}(s)E|x^n(s-\tau)-x^n(t_{i-n_*})|^{(m_3+1)p}ds\right]\\
&\le 3^{p/2-1}2^{p}\bar{K}^{p/2}(T-t_0)^{p/2-1}\int_{t_0}^{t}E\sup_{u\in[t_0-\tau,s]}|x(u)-x^n(u)|^p ds\\
&\quad+3^{p/2-1}K_3^{p/2}(T-t_0)^{p/2}(C+\bar{C})\cdot n^{-(m_3+1)p/2}.
\end{aligned}
\tag{31}
$$

Now, the estimates (27), (29) and (31) together with (22) yield

$$
E\sup_{s\in[t_0-\tau,t]}|x(s)-x^n(s)|^p\le\alpha_1\int_{t_0}^{t}E\sup_{u\in[t_0-\tau,s]}|x(u)-x^n(u)|^p ds+\alpha_2 n^{-(m+1)p/2},
$$

where $m=\min\{m_1,m_2,m_3\}$ and α_1,α_2 are generic constants independent of n.

The application of the Gronwall-Bellman lemma gives

$$E \sup_{s\in[t_0-\tau,t]} |x(s) - x^n(s)|^p \le \alpha_2 n^{-(m+1)p/2} e^{\alpha_1(T-t_0)} \equiv \beta \cdot n^{-(m+1)p/2},$$

where β is a generic constant. Since the last inequality holds for all $t \in [t_0, T]$, it follows that

$$E \sup_{s\in[t_0-\tau,T]} |x(s) - x^n(s)|^p \le \beta \cdot n^{-(m+1)p/2},$$

which completes the proof. □

On the basis of the previous assertions, we can prove the almost sure convergence of the sequence of the approximate solutions $\{x^n, n \in N\}$ given by Eqs. (7) to the solution x of the initial equation (5) which states the following theorem.

Theorem 2.4. *Let the conditions of Theorem 2.3 be satisfied. Then, the sequence $\{x^n, n \in N\}$ of approximate solutions determined by Eqs. (7) converges with probability one to the solution x of Eq. (5).*

Proof. By applying the Chebyshev inequality and Theorem 2.3, we find for an arbitrary $\eta > 0$ that

$$\sum_{n=1}^{\infty} P\Big(\sup_{t\in[t_0-\tau,T]} |x(t) - x^n(t)|^{\frac{p}{2}} \ge n^{-\eta}\Big) \le \sum_{n=1}^{\infty} E \sup_{t\in[t_0-\tau,T]} |x(t) - x^n(t)|^p \cdot n^{2\eta} \le \beta \sum_{n=1}^{\infty} n^{-[(m+1)p-4\eta]/2}.$$

The series on the right-hand side converges if we choose, for example, $\eta < 1/2$ for $p = 2$ and $\eta < (p/2 - 1)/2$ for $p > 2$. Then, $x^n \xrightarrow{a.s.} x$ as $n \to \infty$, in view of the Borell-Cantelli lemma. □

3. Stochastic differential delay equations driven by Poisson process

In the sequel we will develop the approximate method analogous to the one from the previous section, for the following stochastic differential delay equations driven by Poisson process

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t) + h^*(x(t), x(t-\tau), t)dN(t), \ \ t \in [t_0, T], \tag{32}$$

$$x_{t_0} = \xi = \{\xi(\theta) : \theta \in [-\tau, 0]\}, \tag{33}$$

where $h^* : R^d \times R^d \times [t_0, T] \to R^d$, $x(t)$ is a d-dimensional state process, $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ and N is a scalar Poisson process with intensity λ.

A d-dimensional stochastic process $\{x(t), t \in [t_0 - \tau, T]\}$ is said to be a solution to Eq. (32) if it is a.s. *cádlág*, $x(t)$, $t \in [t_0, T]$ is $\mathcal{F}_t$-adapted, $\int_{t_0}^T |f(x(t), x(t-\tau), t)|dt < \infty$ a.s., $\int_{t_0}^T |g(x(t), x(t-\tau), t)|^2 dt < \infty$, a.s., $\int_{t_0}^T |h(x(t), x(t-\tau), t)|^2 dt < \infty$, a.s., $x_{t_0} = \xi$ a.s. and for every $t \in [t_0, T]$, the integral form of Eq. (32) holds a.s.

If one assumes that the global Lipschitz condition and the linear growth condition are satisfied, that is, there exists a constant $\bar{K}^* > 0$ such that for all $x_1, x_2, y_1, y_2 \in R^d$ and $t \in [t_0, T]$,

$$\begin{aligned} &|f(x_1, y_1, t) - f(x_2, y_2, t)|^2 \vee |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 \vee |h(x_1, y_1, t) - h(x_2, y_2, t)|^2 \\ &\qquad\qquad \le \bar{K}^*\Big(|x_1 - x_2|^2 + |y_1 - y_2|^2\Big), \end{aligned} \tag{34}$$

and also there exists a constant $K^* > 0$ such that for all $x, y \in R^d$ and $t \in [t_0, T]$,

$$|f(x, y, t)|^2 \vee |g(x, y, t)|^2 \vee |h(x, y, t)|^2 \le K^*(1 + |x|^2 + |y|^2), \tag{35}$$

then there exists a unique solution $\{x(t), t \in [t_0 - \tau, T]\}$ to Eq. (32). We refer the reader to [25].

Using the similar technique as the one used in the previous section as well as that used in [23], one can prove theorems analogue to Theorems 2.3 and 2.4. In that sense we will present the appropriate assumptions under which these theorems hold

The solution $x = \{x(t), t \in [t_0, T]\}$ to Eq. (32) will be approximated on the partition (6) by the solutions $\{x^n(t), t \in [t_k, t_{k+1}]\}, k = 0, 1, ..., n-1$ of the equations

$$x^n(t) = x^n(t_k) + \int_{t_k}^{t} \sum_{i=0}^{m_1} \frac{d^i f(x^n(t_k), x^n(t_{k-n_*}), s)}{i!} ds + \int_{t_k}^{t} \sum_{i=0}^{m_2} \frac{d^i g(x^n(t_k), x^n(t_{k-n_*}), s)}{i!} dw(s) \quad (36)$$
$$+ \int_{t_k}^{t} \sum_{i=0}^{m_3} \frac{d^i h^*(x^n(t_k), x^n(t_{k-n_*}), s)}{i!} dN(s),$$

satisfying the initial condition $x^n_{t_0} = \xi$ a.s.

In this case, we will prove the closeness between the solutions of Eqs. (32) and (36) under the Lipschitz condition (34), the linear growth condition (35) and the assumptions $\mathcal{A}_1$-$\mathcal{A}_4$ with modified assumption $\mathcal{A}_2$, in the sense that

$$\sup_{R^d \times R^d \times [t_0, T]} \left| \frac{\partial^{m_3+1} h^*(x, y, t)}{\partial x^j \partial y^{m_3+1-j}} \right| \le L, \ j = 0, 1, ..., m_3 + 1. \quad (37)$$

Before stating the main results we give the following useful proposition.

Proposition 3.1. *Let $\{x^n(t), t \in [t_k, t_{k+1}]\}, k = 0, 1, ..., n-1$, be the solution to Eq. (36) and let the condition (35) and the assumptions $\mathcal{A}_1$, $\mathcal{A}_3$ and modified $\mathcal{A}_2$, be satisfied. Then for every $2 \le r \le (M+1)p$,*

$$E \sup_{s \in [t_k, t]} |x^n(s) - x^n(t_k)|^r \le C^* \cdot n^{-r/2}, \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, ..., n-1.$$

Moreover, if the assumption $\mathcal{A}_4$ is satisfied, then

$$E \sup_{s \in [t_k, t]} |x^n(s - \tau) - x^n(t_{k-n_*})|^r \le \bar{C}^* \cdot n^{-r/2}, \ t \in [t_k, t_{k+1}], \ k = 0, 1, ..., n-1,$$

where C^* and $\bar{C}^*$ are positive constants, independent of n.

The proof is similar to that of Proposition 2.2 except the integral with respect to the Poisson process is treated differently then we treated the integral with respect to Poisson measure, as it was done in [23]. In that sense we omit the proof.

The previous proposition allows for the proving of the L^p-closeness between the solution x of Eq. (32) and the approximate solution x^n determined by Eqs. (36).

Theorem 3.2. *Let x be the solution to Eq. (32) and x^n be its approximate solution determined by Eqs. (36). Let also the conditions of Proposition 3.1 and the Lipschitz condition (34) be satisfied. Then, for $p \ge 2$,*

$$E \sup_{t \in [t_0 - \tau, T]} |x(t) - x^n(t)|^p \le \beta^* n^{-(m+1)p/2},$$

where $m = \min\{m_1, m_2, m_3\}$ and β^* is a generic constants independent of n.

In the same way as in Theorem 2.4, we can prove the almost sure convergence of the sequence of the approximate solutions $\{x^n, n \in N\}$ given by Eqs. (36) to the solution x of the initial equation (32). Because of that, we also give the following assertion without the proof.

Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied. Then, the sequence $\{x^n, n \in N\}$ of approximate solutions determined by Eqs. (36) converges with probability one to the solution x of Eq. (32).*

Remark 3.4. *If $m_1 = m_2 = m_3 = 0$, the approximate solutions (7) and (36) reduce to the well-known Euler-Maruyama solutions of stochastic differential delay equations with Poisson random measure and those with Poisson process, respectively. Moreover, by omitting the jump term and the delayed argument in Eqs. (1) and (32), our results reduce to those from [17].*

In order to illustrate the previous theory, we give an example.

Example 3.5. *Consider the following scalar stochastic differential delay equation with Poisson jump*

$$dx(t) = \sin x(t-1)dt + x(t)dw(t) + 2x(t)dN(t), \quad t \in [0,T], \tag{38}$$

satisfying the initial condition $\xi(\theta) = \theta + 1,\ \theta \in [-1,0]$, *where* N *is homogenous Poisson process with intensity* $\lambda = 2$ *and* $T = 1$. *Clearly, the Lipschitz condition (34) and the linear growth condition (35) hold, so there exists unique solution to Eq. (38). We simulated* 2000 *trajectories of both Brownian motion and Poisson process on the partition of the time interval* $[0,1]$ *with* $n = 2^9$ *points. Then, we applied the approximation given by (36), for* $m_1 = m_2 = m_3 = 1$, *which yields*

$$dx^n(t) = [\sin x^n(t_k - 1) + \cos x^n(t_k - 1)(x^n(t-1) - x^n(t_k - 1))]dt + x(t)dw(t) + 2x(t)dN(t), \tag{39}$$

whenever $t \in [t_k, t_{k+1}), k \in \{0, 1, ..., 2^{9-i}\}, i \in \{0, 1, ..., 4\}$.

On the basis of Theorem 3.2, we have that

$$E|x(T) - x^n(T)|^p \le \beta^* \cdot n^{-p}, \quad p \ge 2.$$

Taking logarithms, we obtain that

$$\log E|x(T) - x^n(T)| \approx p^{-1} \log C + \log \delta_n, \quad \delta_n = n^{-1},$$

where C *is a generic constant, independent of* n.

Furtheron, we approximated $\log E|x(T) - x^n(T)|$ *by the sample average based on* 2000 *trajectories, and plotted it against* $\log \delta_n$, $n = 2^{9-i}$, $i \in \{0, 1, ..., 4\}$, *which is represented in Figure 1. As a reference, we added a dashed line with slope* 1. *Thus, one can conclude that the closeness in the* L^p*-sense between the exact solution* x *of Eq. (38) and the approximate solution* x^n *given by (39) is of the order* p, *since* $m = 1$.

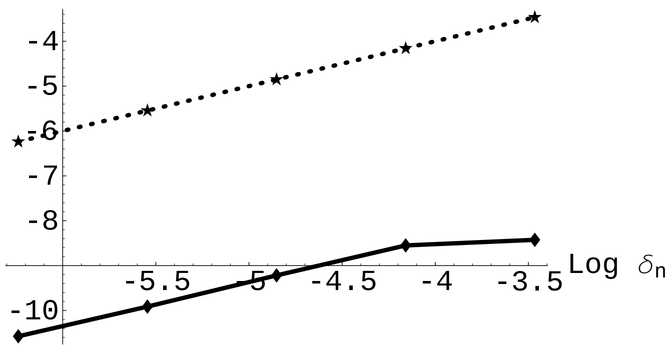

Figure 1: $\log E|x(T) - x^n(T)|$ and reference line of slope 1 (dashed line)

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