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Asymptotic enumeration of independent sets on the Sierpinski gasket

Shu-Chiuan Chang[a,b], Lung-Chi Chen[c], Weigen Yan[d]

[a]Department of Physics
National Cheng Kung University
Tainan 70101, Taiwan
[b]Physics Division
National Center for Theoretical Science
National Taiwan University
Taipei 10617, Taiwan
[c]Department of Mathematics
Fu Jen Catholic University
Taipei 24205, Taiwan
[d]School of Sciences
Jimei University
Xiamen 361021, China

Abstract. The number of independent sets is equivalent to the partition function of the hard-core lattice gas model with nearest-neighbor exclusion and unit activity. We study the number of independent sets $m_{d,b}(n)$ on the generalized Sierpinski gasket $SG_{d,b}(n)$ at stage n with dimension d equal to two, three and four for $b = 2$, and layer b equal to three for $d = 2$. Upper and lower bounds for the asymptotic growth constant, defined as $z_{SG_{d,b}} = \lim_{v\to\infty} \ln m_{d,b}(n)/v$ where v is the number of vertices, on these Sierpinski gaskets are derived in terms of the numbers at a certain stage. The numerical values of these $z_{SG_{d,b}}$ are evaluated with more than a hundred significant figures accurate. We also conjecture upper and lower bounds for the asymptotic growth constant $z_{SG_{d,2}}$ with general d, and an approximation of $z_{SG_{d,2}}$ when d is large.

1. Introduction

The lattice gas with repulsive pair interaction is an important model in statistical mechanics [1–4]. For the special case with hard-core nearest-neighbor exclusion such that each site can be occupied by at most one particle and no pair of adjacent sites can be simultaneously occupied, the partition function of the lattice gas coincides with the independence polynomial in combinatorics [5, 6]. This model is a problem of interest in mathematics [7–10]. While an activity (or fugacity) λ can be associated to each occupied site, the special case with $\lambda = 1$ counts the number of independent (vertex) sets $N_{IS}(G)$ on a graph G [11]. Kaplansky

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Email addresses: scchang@mail.ncku.edu.tw (Shu-Chiuan Chang), lcchen@math.fju.edu.tw (Lung-Chi Chen), weigenyan@263.net (Weigen Yan)

considered the number of k-element independent sets on the path and circuit graphs almost 70 years ago [12]. For a graph G with $v(G)$ vertices, the number of independent sets can grow exponentially when $v(G)$ is large. [1)] For the $m \times n$ grid graph, i.e. the square lattice (sq), it was shown that the limit $\lim_{m,n\to\infty} N_{IS}(sq)^{1/mn}$ exists and its upper and lower bounds were estimated [13]. Baxter has obtained the numerical value for the square lattice to 43 decimal places [14]. The number of independent sets and its bounds had been considered on various graphs [15–17].

It is of interest to consider independent sets on self-similar fractal lattices which have scaling invariance rather than translational invariance [18]. Fractals are geometric structures of (generally noninteger) Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [19, 20]. A well-known example of a fractal is the Sierpinski gasket which has been extensively studied in several contexts [21–37].

We shall derive the recursion relations for the numbers of independent sets on the Sierpinski gasket with dimension equal to two, three and four, and determine the asymptotic growth constants. We shall also consider the number of independent sets on the generalized two-dimensional Sierpinski gasket with layer equal to three.

2. Preliminaries

We first recall some relevant definitions for graphs and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [38, 39]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. An independent set is a subset of the vertices such that any two of them are not adjacent.

When the number of independent sets $N_{IS}(G)$ grows exponentially with $v(G)$ as $v(G) \to \infty$, let us define a constant z_G describing this exponential growth:

$$z_G = \lim_{v(G)\to\infty} \frac{\ln N_{IS}(G)}{v(G)} , \tag{1}$$

where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit. We will see that the limit in Eq. (1) exists for the Sierpinski gasket considered in this paper.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $(n + 1)$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimension $D = \ln(d + 1)/\ln 2$ [22]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1} , \tag{2}$$

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1] . \tag{3}$$

Except the $(d + 1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.

The Sierpinski gasket can be generalized, denoted as $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [40]. The generalized Sierpinski gasket at stage $n + 1$ is constructed from b layers of stage n hypertetrahedrons (the generalization of a tetrahedron to d dimensions). The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$ are illustrated in Fig. 2. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln\binom{b+d-1}{d}/\ln b$ [40]. Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

[1)] For certain graphs, e.g. complete graph, the number of independent sets do not grow exponentially.

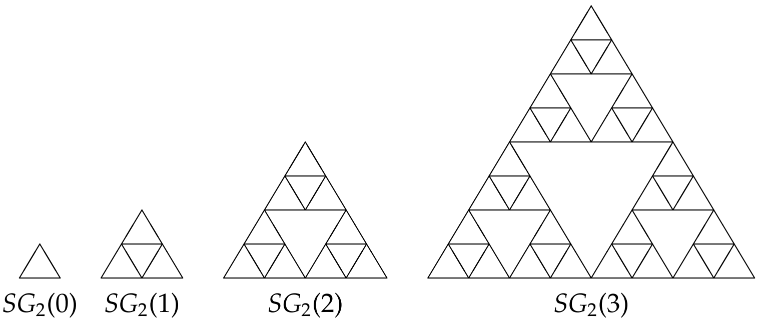


Figure 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$.

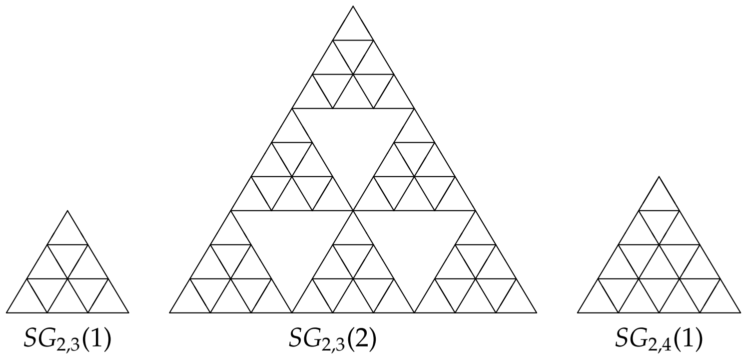


Figure 2: The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$.

3. The number of independent sets on $SG_2(n)$

In this section we derive the asymptotic growth constant for the number of independent sets on the two-dimensional Sierpinski gasket $SG_2(n)$ in detail. Let us start with the definitions of the quantities to be used.

Definition 3.1. *Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage n. (i) Define $m_{2,b}(n) \equiv N_{IS}(SG_{2,b}(n))$ as the number of independent sets. (ii) Define $f_{2,b}(n)$ as the number of independent sets such that all three outmost vertices are not in the vertex subset. (ii) Define $g_{2,b}(n)$ as the number of independent sets such that only one specified vertex of the three outmost vertices (illustrated in Fig. 3) is in the vertex subset. (iii) Define $h_{2,b}(n)$ as the number of independent sets such that exact two specified vertices of the three outmost vertices (illustrated in Fig. 3) are in the vertex subset. (iv) Define $p_{2,b}(n)$ as the number of independent sets such that all three outmost vertices are in the vertex subset.*

Since we only consider the ordinary Sierpinski gasket in this section, we use the notations $m_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$ and $p_2(n)$ for simplicity. They are illustrated in Fig. 3, where only the outmost vertices are shown. Because of rotational symmetry, there are three possible $g_2(n)$ and three possible $h_2(n)$ such that

$$m_2(n) = f_2(n) + 3g_2(n) + 3h_2(n) + p_2(n) \; . \tag{4}$$

The initial values at stage zero are $f_2(0) = 1$, $g_2(0) = 1$, $h_2(0) = 0$, $p_2(0) = 0$ and $m_2(0) = 4$. The purpose of this section is to obtain the asymptotic behavior of $m_2(n)$ as follows. The four quantities $f_2(n)$, $g_2(n)$, $h_2(n)$ and $p_2(n)$ satisfy recursion relations.

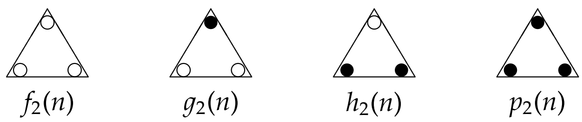


Figure 3: Illustration for the configurations $f_2(n)$, $g_2(n)$, $h_2(n)$, and $p_2(n)$. Only the three outmost vertices are shown explicitly, where a solid circle is in the vertex subset and an open circle is not.

Lemma 3.2. *For any non-negative integer n,*

$$f_2(n+1) = f_2^3(n) + 3f_2(n)g_2^2(n) + 3g_2^2(n)h_2(n) + h_2^3(n)\ , \tag{5}$$

$$g_2(n+1) = f_2^2(n)g_2(n) + 2f_2(n)g_2(n)h_2(n) + g_2^3(n) + 2g_2(n)h_2^2(n) + g_2^2(n)p_2(n) + h_2^2(n)p_2(n)\ , \tag{6}$$

$$h_2(n+1) = f_2(n)g_2^2(n) + f_2(n)h_2^2(n) + 2g_2^2(n)h_2(n) + h_2^3(n) + 2g_2(n)h_2(n)p_2(n) + h_2(n)p_2^2(n)\ , \tag{7}$$

$$p_2(n+1) = g_2^3(n) + 3g_2(n)h_2^2(n) + 3h_2^2(n)p_2(n) + p_2^3(n)\ . \tag{8}$$

Proof The Sierpinski gasket $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. The number $f_2(n+1)$ consists of (i) one configuration where all three $SG_2(n)$ belong to the class that is enumerated by $f_2(n)$; (ii) three configurations where one of the $SG_2(n)$ belongs to the class enumerated by $f_2(n)$ and the other two belong to the class enumerated by $g_2(n)$; (iii) three configurations where two of the $SG_2(n)$ belong to the class enumerated by $g_2(n)$ and the other one belongs to the class enumerated by $h_2(n)$; (iv) one configuration where all three $SG_2(n)$ belongs to the class enumerated by $h_2(n)$ as illustrated in Fig. 4. Eq. (5) is verified by adding these configurations.

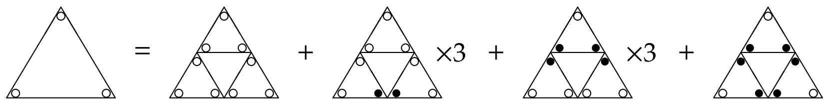


Figure 4: Illustration for the expression of $f_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_2(n+1)$.

Similarly, $g_2(n+1)$, $h_2(n+1)$ and $p_2(n+1)$ for $SG_2(n+1)$ can be obtained with appropriate configurations of its three constituting $SG_2(n)$ as illustrated in Figs. 5, 6 and 7 to verify Eqs. (6), (7) and (8), respectively. There are always $8 = 2^3$ terms (counting multiplicity) in Eqs. (5) - (8) because for each of the three pairs of identified vertices it can be either in the vertex subset or not. □

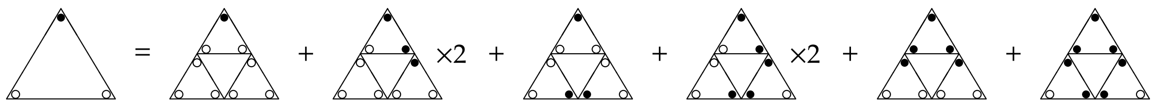


Figure 5: Illustration for the expression of $g_2(n+1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

Alternatively, it is known that the number of dimer-monomers (known as a matching in combinatorics) on a graph G is the same as the number of independent sets on the associated line graph $L(G)$ [41]. Consider the sequence of graphs $H(n)$ shown in Fig. 8 that is obtained by adding an extra edge to each of the outmost vertices of the Hanoi graph. As $H(n)$ has $SG_2(n)$ as its line graph, the enumeration of the number of independent sets on $SG_2(n)$ is equivalent to the enumeration of the number of dimer-monomers on these $H(n)$. One can define corresponding quantities of $f_2(n)$, $g_2(n)$, $h_2(n)$, $p_2(n)$ on $H(n)$ that satisfy the same recursion relations as in Lemma 3.2.

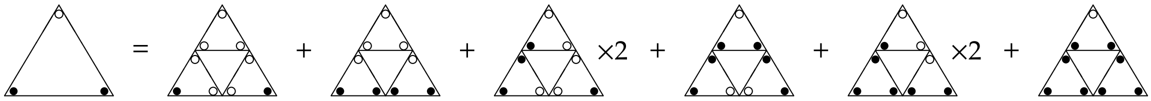


Figure 6: Illustration for the expression of $h_2(n+1)$. The multiplication of two on the right-hand-side corresponds to the reflection symmetry with respect to the central vertical axis.

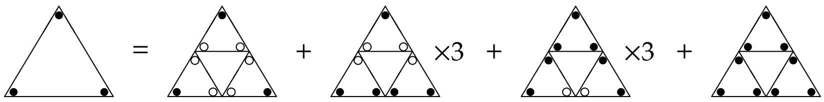


Figure 7: Illustration for the expression of $p_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_2(n+1)$.

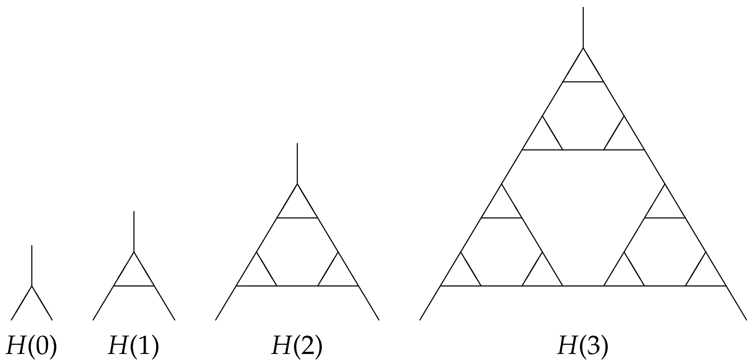


Figure 8: The first four stages $n = 0, 1, 2, 3$ of the graph $H(n)$.

The values of $f_2(n)$, $g_2(n)$, $h_2(n)$, $p_2(n)$ for small n can be evaluated recursively by Eqs. (5) - (8) as listed in Table 1. These numbers grow exponentially, and do not have simple integer factorizations. To estimate the value of the asymptotic growth constant defined in Eq. (1), we need the following lemmas.

For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$, define the ratios

$$\alpha_{2,b}(n) = \frac{g_{2,b}(n)}{f_{2,b}(n)} , \qquad \beta_{2,b}(n) = \frac{h_{2,b}(n)}{g_{2,b}(n)} , \qquad \gamma_{2,b}(n) = \frac{p_{2,b}(n)}{h_{2,b}(n)} , \tag{9}$$

where n is a positive integer. For the ordinary Sierpinski gasket in this section, they are simplified to be $\alpha_2(n), \beta_2(n), \gamma_2(n)$ and their values for small n are listed in Table 2. From the initial values of $f_2(n)$, $g_2(n)$, $h_2(n)$, $p_2(n)$, it is easy to see that $f_2(n) \ge g_2(n) \ge h_2(n) \ge p_2(n)$ for all non-negative n by induction. Alternatively, these inequalities can be obtained by an injection. For instance, if one of the independent sets enumerated by $g_2(n)$ is given, one can remove the corner vertex to obtain another independent set that is among those that are enumerated by $f_2(n)$ such that $f_2(n) \ge g_2(n)$ is established. Similarly, the other two inequalities can be established. It follows that $\alpha_2(n), \beta_2(n), \gamma_2(n) \in (0, 1]$.

Table 1: The first few values of $f_2(n)$, $g_2(n)$, $h_2(n)$, $p_2(n)$, $m_2(n)$.

n	0	1	2	3	4
$f_2(n)$	1	4	125	4,007,754	132,460,031,222,098,852,477
$g_2(n)$	1	2	65	2,089,888	69,073,020,285,472,159,669
$h_2(n)$	0	1	34	1,089,805	36,019,032,212,213,865,476
$p_2(n)$	0	1	18	568,301	18,782,596,680,434,060,148
$m_2(n)$	4	14	440	14,115,134	466,518,785,395,590,988,060

Table 2: The first few values of $\alpha_2(n)$, $\beta_2(n)$, $\gamma_2(n)$. The last digits given are rounded off.

n	1	2	3	4
$\alpha_2(n)$	0.5	0.52	0.521461147565444	0.521463113425180
$\beta_2(n)$	0.5	0.523076923076923	0.521465743618797	0.521463113431998
$\gamma_2(n)$	1	0.529411764705882	0.521470354788242	0.521463113438816

Lemma 3.3. *For any positive integer n, the ratios satisfy*

$$\alpha_2(n) \le \beta_2(n) \le \gamma_2(n) . \tag{10}$$

When n increases, the ratio $\alpha_2(n)$ increases monotonically while $\gamma_2(n)$ decreases monotonically. The three ratios in the large n limit are equal to each other

$$\lim_{n\to\infty} \alpha_2(n) = \lim_{n\to\infty} \beta_2(n) = \lim_{n\to\infty} \gamma_2(n) . \tag{11}$$

Proof It is clear that Eq. (10) is valid for small values of n given in Table 2. In order to save space, we will use α_n, β_n, γ_n to denote $\alpha_2(n)$, $\beta_2(n)$, $\gamma_2(n)$ for the lengthy equations in this Lemma. By definition, we have

$$\alpha_{n+1} = \alpha_n \frac{B_n}{A_n} , \qquad \beta_{n+1} = \alpha_n \frac{C_n}{B_n} , \qquad \gamma_{n+1} = \alpha_n \frac{D_n}{C_n} \tag{12}$$

for a positive n, where

$$\begin{aligned} A_n &= 1+3\alpha_n^2+3\alpha_n^3\beta_n+\alpha_n^3\beta_n^3\ , \\ B_n &= 1+2\alpha_n\beta_n+\alpha_n^2+2\alpha_n^2\beta_n^2+\alpha_n^2\beta_n\gamma_n+\alpha_n^2\beta_n^3\gamma_n\ , \\ C_n &= 1+\beta_n^2+2\alpha_n\beta_n+\alpha_n\beta_n^3+2\alpha_n\beta_n^2\gamma_n+\alpha_n\beta_n^3\gamma_n^2\ , \\ D_n &= 1+3\beta_n^2+3\beta_n^3\gamma_n+\beta_n^3\gamma_n^3\ , \end{aligned} \tag{13}$$

such that

$$\alpha_{n+1}-\alpha_n=\frac{1}{A_n}\Big\{2\alpha_n^2(1+\alpha_n\beta_n)(\beta_n-\alpha_n)+\alpha_n^3\beta_n(1+\beta_n^2)(\gamma_n-\alpha_n)\Big\}\ , \tag{14}$$

$$\beta_{n+1}-\alpha_n=\frac{1}{B_n}\Big\{\alpha_n(\beta_n+\alpha_n+\alpha_n\beta_n^2+\alpha_n\beta_n\gamma_n)(\beta_n-\alpha_n)+\alpha_n^2\beta_n^2(1+\beta_n\gamma_n)(\gamma_n-\alpha_n)\Big\}\ . \tag{15}$$

It follows that

$$\begin{aligned} \beta_{n+1}-\alpha_{n+1} &= \frac{\beta_n-\alpha_n}{A_nB_n}\Big\{\alpha_n(\beta_n+\alpha_n+\alpha_n\beta_n^2+\alpha_n\beta_n\gamma_n)A_n-2\alpha_n^2(1+\alpha_n\beta_n)B_n\Big\} \\ &\quad+\frac{\gamma_n-\alpha_n}{A_nB_n}\Big\{\alpha_n^2\beta_n^2(1+\beta_n\gamma_n)A_n-\alpha_n^3\beta_n(1+\beta_n^2)B_n\Big\} \\ &= \frac{\beta_n-\alpha_n}{A_nB_n}\alpha_n(1+\alpha_n\beta_n)\Big\{(1-\alpha_n^2-\alpha_n^2\beta_n\gamma_n)(\beta_n-\alpha_n)+\alpha_n\beta_n(\gamma_n-\alpha_n)-\alpha_n^3\beta_n^3(\gamma_n-\beta_n)\Big\} \\ &\quad+\frac{\gamma_n-\alpha_n}{A_nB_n}\alpha_n^2\beta_n\Big\{(1+\alpha_n^2+\alpha_n^2\beta_n\gamma_n)(\beta_n-\alpha_n)+\beta_n^2(\gamma_n-\alpha_n)+\alpha_n^2\beta_n^2(2+\alpha_n\beta_n)(\gamma_n-\beta_n)\Big\}\ , \end{aligned} \tag{16}$$

where

$$\begin{aligned} A_nB_n &= 1+4\alpha_n^2+2\alpha_n\beta_n+3\alpha_n^4+9\alpha_n^3\beta_n+2\alpha_n^2\beta_n^2+\alpha_n^2\beta_n\gamma_n+3\alpha_n^5\beta_n+12\alpha_n^4\beta_n^2+\alpha_n^3\beta_n^3+3\alpha_n^4\beta_n\gamma_n+\alpha_n^2\beta_n^3\gamma_n \\ &\quad+7\alpha_n^5\beta_n^3+2\alpha_n^4\beta_n^4+3\alpha_n^5\beta_n^2\gamma_n+3\alpha_n^4\beta_n^3\gamma_n+2\alpha_n^5\beta_n^5+4\alpha_n^5\beta_n^4\gamma_n+\alpha_n^5\beta_n^6\gamma_n\ . \end{aligned} \tag{17}$$

Since $\gamma_n-\alpha_n=(\gamma_n-\beta_n)+(\beta_n-\alpha_n)$, Eq. (17) leads to

$$\begin{aligned} \beta_{n+1}-\alpha_{n+1} &= \frac{\beta_n-\alpha_n}{A_nB_n}\alpha_n(1+\alpha_n\beta_n)\Big\{(1+\alpha_n\beta_n-\alpha_n^2-\alpha_n^2\beta_n\gamma_n)(\beta_n-\alpha_n)+(\alpha_n\beta_n-\alpha_n^3\beta_n^3)(\gamma_n-\beta_n)\Big\} \\ &\quad+\frac{\gamma_n-\alpha_n}{A_nB_n}\alpha_n^2\beta_n\Big\{(1+\alpha_n^2+\alpha_n^2\beta_n\gamma_n)(\beta_n-\alpha_n)+\beta_n^2(\gamma_n-\alpha_n)+\alpha_n^2\beta_n^2(2+\alpha_n\beta_n)(\gamma_n-\beta_n)\Big\}\ . \end{aligned} \tag{18}$$

Using the fact that $\alpha_n,\beta_n,\gamma_n\in(0,1]$ and the inequality $\beta_n\le\gamma_n$ to be shown below, $\alpha_n\le\beta_n$ is proved by induction. Define $\epsilon_n=\gamma_n-\alpha_n$, which is larger than $\gamma_n-\beta_n$ and $\beta_n-\alpha_n$ as we shall prove $\beta_n\le\gamma_n$, then

$$\begin{aligned} \beta_{n+1}-\alpha_{n+1} &\le \frac{\epsilon_n^2}{A_nB_n}\Big\{\alpha_n(1+\alpha_n\beta_n)^2+\alpha_n^2\beta_n(1+\alpha_n^2+\alpha_n^2\beta_n\gamma_n+\beta_n^2+2\alpha_n^2\beta_n^2+\alpha_n^3\beta_n^3)\Big\} \\ &= \frac{\epsilon_n^2}{A_nB_n}\Big\{\alpha_n+3\alpha_n^2\beta_n+\alpha_n^3\beta_n^2+\alpha_n^4\beta_n+\alpha_n^2\beta_n^3+2\alpha_n^4\beta_n^3+\alpha_n^4\beta_n^2\gamma_n+\alpha_n^5\beta_n^4\Big\} \\ &\le \frac{\epsilon_n^2}{A_nB_n}\Big\{1+3\alpha_n^2+2\alpha_n^3\beta_n+\alpha_n^2\beta_n^2+3\alpha_n^4\beta_n^2+\alpha_n^5\beta_n^3\Big\}\le\epsilon_n^2\ . \end{aligned} \tag{19}$$

Similarly, we have

$$\gamma_{n+1}-\alpha_n=\frac{1}{C_n}\Big\{2\alpha_n\beta_n(1+\beta_n\gamma_n)(\beta_n-\alpha_n)+\alpha_n\beta_n^3(1+\gamma_n^2)(\gamma_n-\alpha_n)\Big\}\ , \tag{20}$$

and

$$\gamma_{n+1}-\beta_{n+1} = \frac{\beta_n-\alpha_n}{B_nC_n}\Big\{2\alpha_n\beta_n(1+\beta_n\gamma_n)B_n-\alpha_n(\beta_n+\alpha_n+\alpha_n\beta_n^2+\alpha_n\beta_n\gamma_n)C_n\Big\}$$

$$
\begin{aligned}
&+\frac{\gamma_n-\alpha_n}{B_nC_n}\Big\{\alpha_n\beta_n^3(1+\gamma_n^2)B_n-\alpha_n^2\beta_n^2(1+\beta_n\gamma_n)C_n\Big\}\\
=\ &\frac{\beta_n-\alpha_n}{B_nC_n}\alpha_n\Big\{(1+\beta_n\gamma_n-\alpha_n\beta_n^3)(\beta_n-\alpha_n)\\
&+\beta_n(\beta_n+\alpha_n\beta_n^2+\alpha_n^2\beta_n^3-\alpha_n^2\beta_n^2\gamma_n-\alpha_n^2\beta_n^3\gamma_n^2)(\gamma_n-\beta_n)-\alpha_n\beta_n^4\gamma_n(\gamma_n-\alpha_n)\Big\}\\
&+\frac{\gamma_n-\alpha_n}{B_nC_n}\alpha_n\beta_n^2\Big\{(1+\alpha_n\beta_n)(\beta_n-\alpha_n)+\beta_n(\gamma_n-\alpha_n\beta_n^2+2\alpha_n\beta_n\gamma_n)(\gamma_n-\alpha_n)\\
&+\alpha_n^2\beta_n\gamma_n(1+\beta_n\gamma_n)(\gamma_n-\beta_n)\Big\}\\
=\ &\frac{\beta_n-\alpha_n}{B_nC_n}\alpha_n\Big\{(1+\beta_n\gamma_n-\alpha_n\beta_n^3-\alpha_n\beta_n^4\gamma_n)(\beta_n-\alpha_n)\\
&+\beta_n(\beta_n+\alpha_n\beta_n^2+\alpha_n^2\beta_n^3-\alpha_n^2\beta_n^2\gamma_n-\alpha_n^2\beta_n^3\gamma_n^2-\alpha_n\beta_n^3\gamma_n)(\gamma_n-\beta_n)\Big\}\\
&+\frac{\gamma_n-\alpha_n}{B_nC_n}\alpha_n\beta_n^2\Big\{(1+\alpha_n\beta_n)(\beta_n-\alpha_n)+\beta_n(\gamma_n-\alpha_n\beta_n^2+2\alpha_n\beta_n\gamma_n)(\gamma_n-\alpha_n)\\
&+\alpha_n^2\beta_n\gamma_n(1+\beta_n\gamma_n)(\gamma_n-\beta_n)\Big\}\,, \qquad (21)
\end{aligned}
$$

where the last equality holds using $\gamma_n-\alpha_n=(\gamma_n-\beta_n)+(\beta_n-\alpha_n)$, and

$$
\begin{aligned}
B_nC_n\ =\ &1+\alpha_n^2+4\alpha_n\beta_n+\beta_n^2+2\alpha_n^3\beta_n+7\alpha_n^2\beta_n^2+3\alpha_n\beta_n^3+\alpha_n^2\beta_n\gamma_n+2\alpha_n\beta_n^2\gamma_n+5\alpha_n^3\beta_n^3+4\alpha_n^2\beta_n^4+4\alpha_n^3\beta_n^2\gamma_n\\
&+6\alpha_n^2\beta_n^3\gamma_n+\alpha_n\beta_n^3\gamma_n^2+2\alpha_n^3\beta_n^5+7\alpha_n^3\beta_n^4\gamma_n+\alpha_n^2\beta_n^5\gamma_n+3\alpha_n^3\beta_n^3\gamma_n^2+2\alpha_n^2\beta_n^4\gamma_n^2+\alpha_n^3\beta_n^6\gamma_n+4\alpha_n^3\beta_n^5\gamma_n^2\\
&+\alpha_n^3\beta_n^4\gamma_n^3+\alpha_n^3\beta_n^6\gamma_n^3\,. \qquad (22)
\end{aligned}
$$

Using the fact that $\alpha_n,\beta_n,\gamma_n\in(0,1]$ and the inequality $\alpha_n\le\beta_n$ proven above, $\beta_n\le\gamma_n$ is proved by induction. We also have

$$
\begin{aligned}
\gamma_{n+1}-\beta_{n+1}\ \le\ &\frac{\epsilon_n^2}{B_nC_n}\Big\{\alpha_n(1+\beta_n\gamma_n+\beta_n^2+\alpha_n\beta_n^3+\alpha_n^2\beta_n^4)\\
&+\alpha_n\beta_n^2(1+\alpha_n\beta_n+\beta_n\gamma_n+2\alpha_n\beta_n^2\gamma_n+\alpha_n^2\beta_n\gamma_n+\alpha_n^2\beta_n^2\gamma_n^2)\Big\}\\
=\ &\frac{\epsilon_n^2}{B_nC_n}\Big\{\alpha_n+2\alpha_n\beta_n^2+\alpha_n\beta_n\gamma_n+2\alpha_n^2\beta_n^3+\alpha_n^3\beta_n^4+\alpha_n\beta_n^3\gamma_n+\alpha_n^3\beta_n^3\gamma_n+2\alpha_n^2\beta_n^4\gamma_n+\alpha_n^3\beta_n^4\gamma_n^2\Big\}\\
\le\ &\frac{\epsilon_n^2}{B_nC_n}\Big\{1+3\alpha_n\beta_n+3\alpha_n\beta_n^3+2\alpha_n^3\beta_n^3+2\alpha_n^2\beta_n^3\gamma_n+\alpha_n^3\beta_n^4\gamma_n\Big\}\le\epsilon_n^2\,. \qquad (23)
\end{aligned}
$$

From Eqs. (19) and (23), we obtain $\epsilon_{n+1}\le 2\epsilon_n^2$ for all positive n by induction. It follows that for any positive integer $m\le n$,

$$
\epsilon_n\le 2\epsilon_{n-1}^2\le 2\Big[2\epsilon_{n-2}^2\Big]^2\le\cdots\le\frac{1}{2}\Big[2\epsilon_m\Big]^{2^{n-m}}. \qquad (24)
$$

Taking m as an integer larger than one so that $\epsilon_m<1/2$, then we have the values of α_n, β_n, γ_n are close to each other when n becomes large.

Finally, it is clear that $\alpha_2(n)$ increases monotonically as n increases by Eq. (14). As

$$
\gamma_n-\gamma_{n+1}=\frac{1}{C_n}\Big\{(1+\beta_n^2)(\gamma_n-\alpha_n)+2\alpha_n\beta_n(1+\beta_n\gamma_n)(\gamma_n-\beta_n)\Big\}\,, \qquad (25)
$$

we know $\gamma_2(n)$ decreases monotonically as n increases, and the proof is completed. □

Numerically, we find

$$
\lim_{n\to\infty}\alpha_2(n)=\lim_{n\to\infty}\beta_2(n)=\lim_{n\to\infty}\gamma_2(n)=0.52146311342809496 5776... \qquad (26)
$$

From the above lemma, we have the following bounds for the asymptotic growth constant.

Lemma 3.4. *The asymptotic growth constant for the number of independent sets on $SG_2(n)$ is bounded:*

$$\frac{2}{3^{m+1}}\ln[f_2(m)] + \frac{1}{3^m}\ln[1+\alpha_2^2(m)] \le z_{SG_2} \le \frac{2}{3^{m+1}}\ln[f_2(m)] + \frac{1}{3^m}\ln[1+\beta_2^2(m)] \,, \tag{27}$$

where m is a positive integer.

Proof From Eq. (5) and Lemma 3.3, we have the upper bound for $f_2(n)$,

$$\begin{aligned} f_2(n) &= f_2^3(n-1)\Big[1+3\alpha_2^2(n-1)+3\alpha_2^3(n-1)\beta_2(n-1)+\alpha_2^3(n-1)\beta_2^3(n-1)\Big] \\ &\le f_2^3(n-1)\Big[1+\beta_2^2(n-1)\Big]^3 \\ &\le \Big\{f_2^3(n-2)\Big[1+\beta_2^2(n-2)\Big]^3\Big\}^3\Big[1+\beta_2^2(n-1)\Big]^3 \le \cdots \\ &\le \Big[f_2(m)\Big]^{3^{n-m}}\Big[1+\beta_2^2(m)\Big]^{\frac{3}{2}(3^{n-m}-1)} \,. \end{aligned} \tag{28}$$

From Eq. (4), the number of independent sets has the upper bound

$$m_2(n) = f_2(n)\Big[1+3\alpha_2(n)+3\alpha_2(n)\beta_2(n)+\alpha_2(n)\beta_2(n)\gamma_2(n)\Big] \le \Big[f_2(m)\Big]^{3^{n-m}}\Big[1+\beta_2^2(m)\Big]^{\frac{3}{2}(3^{n-m}-1)}\Big[1+\gamma_2(n)\Big]^3 \,. \tag{29}$$

As the number of vertices of $SG_2(n)$ is $3(3^n+1)/2$ by Eq. (3), the upper bound for z_{SG_2} defined in Eq. (1) follows. The lower bound for z_{SG_2} can be derived similarly. □

As m increases, the difference between the upper and lower bounds in Eq. (27) becomes small and the convergence is rapid. The numerical value of z_{SG_2} can be obtained with more than a hundred significant figures accurate when m is equal to eight.

Proposition 3.5. *The asymptotic growth constant for the number of independent sets on the two-dimensional Sierpinski gasket $SG_2(n)$ in the large n limit is* $z_{SG_2} = 0.38430953443368558352....$

For the square lattice which also has degree four, the asymptotic growth constant is $z_{sq} = 0.40749510126068800045...$ [14] that is larger than our result here.

As mentioned previously, the number of dimer-monomers on the graph $H(n)$ illustrated in Fig. 8 is the same as the number of independent sets on the two-dimensional Sierpinski gasket $SG_2(n)$. Similar to Eq. (1), one can define a constant for the exponential growth of the number of dimer-monomers:

$$z'_G = \lim_{v(G)\to\infty} \frac{\ln N_{DM}(G)}{v(G)} \,, \tag{30}$$

where $N_{DM}(G)$ is the number of dimer-monomers on a graph G. As the number of vertices of $H(n)$ is 3^n+3, we have the following corollary.

Corollary 3.6. *The asymptotic growth constant for the number of dimer-monomers on the graph $H(n)$ in the large n limit is* $z'_H = 0.57646430165052837528....$

This result can be obtained from the asymptotic formula given in [42].

4. The number of independent sets on $SG_{2,3}(n)$

The method given in the previous section can be applied to the number of independent sets on $SG_{d,b}(n)$ with larger values of d and b. The number of configurations to be considered increases as d and b increase, and the recursion relations must be derived individually for each d and b. In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with the number of layers b equal to three. For $SG_{2,3}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3 \times 6^n , \tag{31}$$

$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5} , \tag{32}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. The initial values for the number of independent sets with various conditions are the same as those for SG_2: $f_{2,3}(0) = 1$, $g_{2,3}(0) = 1$, $h_{2,3}(0) = 0$ and $p_{2,3}(0) = 0$. The recursion relations for $SG_{2,3}(n)$ are lengthy and given in the appendix. Some values of $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$, $p_{2,3}(n)$, $m_{2,3}(n)$ are listed in Table 3. These numbers grow exponentially, and do not have simple integer factorizations.

Table 3: The first few values of $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$, $p_{2,3}(n)$, $m_{2,3}(n)$.

n	0	1	2	3
$f_{2,3}(n)$	1	19	172,371,175	93,818,345,014,803,648,739,612,995,034,820,933,103,277,876,214,071
$g_{2,3}(n)$	1	9	80,291,169	43,700,938,182,461,202,772,695,141,988,444,331,720,442,482,282,619
$h_{2,3}(n)$	0	4	37,399,906	20,356,061,468,851,869,739,344,457,713,631,919,274,541,443,648,604
$p_{2,3}(n)$	0	2	17,420,990	9,481,930,039,890,479,716,613,035,420,873,292,623,048,215,623,126
$m_{2,3}(n)$	4	60	542,865,390	295,471,274,008,633,345,992,344,829,561,922,978,711,277,869,630,866

The values of the ratios $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$, $\gamma_{2,3}(n)$ defined in Eq. (9) for small n are listed in Table 4. The sequence of $\alpha_{2,3}(n)$ decreases monotonically as n increases, while $\beta_{2,3}(n)$ increases monotonically. Except the first term $\gamma_{2,3}(1)$, $\gamma_{2,3}(n)$ also increases monotonically for $n \geq 2$. We again have $\alpha_{2,3}(n), \beta_{2,3}(n), \gamma_{2,3}(n) \in (0, 1]$ but $\gamma_{2,3}(n) \leq \beta_{2,3}(n) \leq \alpha_{2,3}(n)$ for $n \geq 2$, in contrast to Lemma 3.3.

Table 4: The first few values of $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$, $\gamma_{2,3}(n)$. The last digits given are rounded off.

n	1	2	3
$\alpha_{2,3}(n)$	0.47368421052631578947	0.46580391994195085112	0.46580376338514186621
$\beta_{2,3}(n)$	0.44444444444444444444	0.46580348082863259844	0.46580376338514186620
$\gamma_{2,3}(n)$	0.5	0.46580304239267339335	0.46580376338514186619

By the same argument given in Lemma 3.4, we have the upper and lower bounds of the asymptotic growth constant for the number of independent sets on $SG_{2,3}(n)$:

$$\frac{1}{7 \times 6^m}\Big\{5 \ln f_{2,3}(m) + \ln\Big[1 + \gamma_{2,3}^3(m)\Big] + 6 \ln\Big[1 + \gamma_{2,3}^2(m)\Big]\Big\} \leq z_{SG_{2,3}}$$
$$\leq \frac{1}{7 \times 6^m}\Big\{5 \ln f_{2,3}(m) + \ln\Big[1 + \alpha_{2,3}^3(m)\Big] + 6 \ln\Big[1 + \alpha_{2,3}^2(m)\Big]\Big\} , \tag{33}$$

with m a positive integer. The convergence of the upper and lower bounds remains rapid. More than a hundred significant figures for $z_{SG_{2,3}}$ can be obtained when m is equal to five. We have the following proposition.

Proposition 4.1. *The asymptotic growth constant for the number of independent sets on the two-dimensional Sierpinski gasket $SG_{2,3}(n)$ in the large n limit is* $z_{SG_{2,3}} = 0.38135033366164857274....$

5. The number of independent sets on $SG_3(n)$

In this section, we derive the asymptotic growth constant of independent sets on the three-dimensional Sierpinski gasket $SG_3(n)$. We use the following definitions.

Definition 5.1. *Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n. (i) Define $m_3(n) \equiv N_{IS}(SG_3(n))$ as the number of independent sets. (ii) Define $f_3(n)$ as the number of independent sets such that all four outmost vertices are not in the vertex subset. (iii) Define $g_3(n)$ as the number of independent sets such that only one certain outmost vertex is in the vertex subset. (iv) Define $h_3(n)$ as the number of independent sets such that exact two certain outmost vertices are in the vertex subset. (v) Define $p_3(n)$ as the number of independent sets such that exact three certain outmost vertices are in the vertex subset. (vi) Define $q_3(n)$ as the number of independent sets such that all four outmost vertices are in the vertex subset.*

The quantities $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$ are illustrated in Fig. 9, where only the outmost vertices are shown. There are $\binom{4}{1} = 4$ equivalent configurations for $g_3(n)$, $\binom{4}{2} = 6$ equivalent configurations for $h_3(n)$, and $\binom{4}{1} = 4$ equivalent configurations for $p_3(n)$. By definition,

$$m_3(n) = f_3(n) + 4g_3(n) + 6h_3(n) + 4p_3(n) + q_3(n) . \tag{34}$$

The initial values at stage zero are $f_3(0) = 1$, $g_3(0) = 1$, $h_3(0) = 0$, $p_3(0) = 0$, $q_3(0) = 0$ and $m_3(0) = 5$.

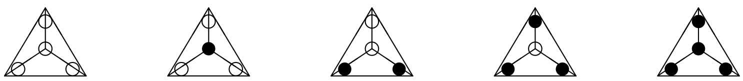


Figure 9: Illustration for the configurations $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$. Only the four outmost vertices are shown explicitly, where a solid circle is in the vertex subset and an open circle is not.

The recursion relations are lengthy and given in the appendix. Some values of $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$, $q_3(n)$, $m_3(n)$ are listed in Table 5. These numbers grow exponentially, and do not have simple integer factorizations.

Table 5: The first few values of $f_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$, $q_3(n)$, $m_3(n)$.

n	0	1	2	3
$f_3(n)$	1	10	25,817	1,292,964,293,737,151,090
$g_3(n)$	1	4	11,387	571,820,791,550,665,532
$h_3(n)$	0	2	5,050	252,892,039,471,313,074
$p_3(n)$	0	1	2,252	111,843,868,747,687,217
$q_3(n)$	0	1	1,010	49,464,202,269,253,193
$m_3(n)$	5	43	111,683	5,594,439,374,027,693,723

Define ratios

$$\alpha_3(n) = \frac{g_3(n)}{f_3(n)} , \qquad \beta_3(n) = \frac{h_3(n)}{g_3(n)} , \qquad \gamma_3(n) = \frac{p_3(n)}{h_3(n)} , \qquad \delta_3(n) = \frac{q_3(n)}{p_3(n)} \tag{35}$$

for a positive integer n as in Eq. (9). As n increases, we find $\alpha_3(n)$ increases monotonically while $\beta_3(n)$, $\gamma_3(n)$, $\delta_3(n)$ decrease monotonically with the relation $\alpha_3(n) \le \beta_3(n) \le \gamma_3(n) \le \delta_3(n)$. The values of these ratios for small n are listed in Table 6. Numerically, we find

$$\lim_{n\to\infty} \alpha_3(n) = \lim_{n\to\infty} \beta_3(n) = \lim_{n\to\infty} \gamma_3(n) = \lim_{n\to\infty} \delta_3(n) = 0.442256573677178603386... \tag{36}$$

Table 6: The first few values of $\alpha_3(n)$, $\beta_3(n)$, $\gamma_3(n)$, $\delta_3(n)$. The last digits given are rounded off.

n	1	2	3	4
$\alpha_3(n)$	0.4	0.441065964287098	0.442255671189410	0.442256573676665
$\beta_3(n)$	0.5	0.443488188284886	0.442257510059261	0.442256573677711
$\gamma_3(n)$	0.5	0.445940594059406	0.442259349015113	0.442256573678758
$\delta_3(n)$	1	0.448490230905861	0.442261188057088	0.442256573679804

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of independent sets on $SG_3(n)$ is bounded:

$$\frac{1}{2\times 4^m}\ln[f_3(m)] + \frac{1}{4^m}\ln[1+\alpha_3^2(m)] \le z_{SG_3} \le \frac{1}{2\times 4^m}\ln[f_3(m)] + \frac{1}{4^m}\ln[1+\delta_3^2(m)] \, , \tag{37}$$

where m is a positive integer. More than a hundred significant figures for z_{SG_3} can be obtained when m is equal to seven. We have the following proposition.

Proposition 5.2. *The asymptotic growth constant for the number of independent sets on the three-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit is $z_{SG_3} = 0.328599605721479557 61....$*

6. The number of independent sets on $SG_4(n)$

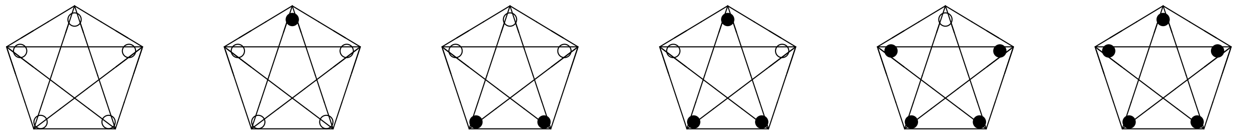
In this section, we derive the asymptotic growth constant of independent sets on the four-dimensional Sierpinski gasket $SG_4(n)$. We use the following definitions.

Definition 6.1. *Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n. (i) Define $m_4(n) \equiv N_{IS}(SG_4(n))$ as the number of independent sets. (ii) Define $f_4(n)$ as the number of independent sets such that all five outmost vertices are not in the vertex subset. (iii) Define $g_4(n)$ as the number of independent sets such that only one certain outmost vertex is in the vertex subset. (iv) Define $h_4(n)$ as the number of independent sets such that exact two certain outmost vertices are in the vertex subset. (v) Define $p_4(n)$ as the number of independent sets such that exact three certain outmost vertices are in the vertex subset. (vi) Define $q_4(n)$ as the number of independent sets such that exact four certain outmost vertices are in the vertex subset. (vii) Define $r_4(n)$ as the number of independent sets such that all five outmost vertices are in the vertex subset.*

The quantities $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$ and $r_4(n)$ are illustrated in Fig. 10, where only the outmost vertices are shown. There are $\binom{5}{1} = 5$ equivalent $g_4(n)$, $\binom{5}{2} = 10$ equivalent $h_4(n)$, $\binom{5}{3} = 10$ equivalent $p_3(n)$, and $\binom{5}{4} = 5$ equivalent $q_3(n)$. By definition,

$$m_4(n) = f_4(n) + 5g_4(n) + 10h_4(n) + 10p_4(n) + 5q_4(n) + r_4(n) \, . \tag{38}$$

The initial values at stage zero are $f_4(0) = 1$, $g_4(0) = 1$, $h_4(0) = 0$, $p_4(0) = 0$, $q_4(0) = 0$, $r_4(0) = 0$ and $m_4(0) = 6$.



$f_4(n)$ $g_4(n)$ $h_4(n)$ $p_4(n)$ $q_4(n)$ $r_4(n)$

Figure 10: Illustration for the configurations $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$ and $r_4(n)$. Only the five outmost vertices are shown explicitly, where a solid circle is in the vertex subset and an open circle is not.

Table 7: The first few values of $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$, $m_4(n)$.

n	0	1	2	3
$f_4(n)$	1	26	48,645,865	1,209,689,823,065,753,613,801,849,265,389,348,210,254
$g_4(n)$	1	10	19,499,025	485,275,031,765,121,996,003,377,748,244,728,141,942
$h_4(n)$	0	4	7,827,058	194,671,321,306,020,419,533,199,834,929,606,628,798
$p_4(n)$	0	2	3,146,558	78,093,721,039,746,646,163,976,217,053,630,607,240
$q_4(n)$	0	1	1,266,948	31,327,833,873,772,900,771,790,623,812,192,536,505
$r_4(n)$	0	1	510,980	12,567,379,442,065,248,794,102,222,711,306,394,841
$m_4(n)$	6	142	262,722,870	6,532,921,954,159,964,003,443,553,868,217,630,357,710

The recursion relations are lengthy and given in the appendix. Some values of $f_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$, $m_4(n)$ are listed in Table 7. These numbers grow exponentially, and do not have simple integer factorizations.

Define ratios

$$\alpha_4(n) = \frac{g_4(n)}{f_4(n)} , \quad \beta_4(n) = \frac{h_4(n)}{g_4(n)} , \quad \gamma_4(n) = \frac{p_4(n)}{h_4(n)} , \quad \delta_4(n) = \frac{q_4(n)}{p_4(n)} , \quad \eta_4(n) = \frac{r_4(n)}{q_4(n)} \tag{39}$$

for a positive integer n as in Eq. (9). As $n \geq 2$ increases, we find $\alpha_4(n)$ increases monotonically while $\beta_4(n)$, $\gamma_4(n)$, $\delta_4(n)$, $\eta_4(n)$ decrease monotonically with the relation $\alpha_4(n) \leq \beta_4(n) \leq \gamma_4(n) \leq \delta_4(n) \leq \eta_4(n)$. The values of these ratios for small n are listed in Table 8. Numerically, we find

$$\lim_{n\to\infty} \alpha_4(n) = \lim_{n\to\infty} \beta_4(n) = \lim_{n\to\infty} \gamma_4(n) = \lim_{n\to\infty} \delta_4(n) = \lim_{n\to\infty} \eta_4(n) = 0.401156636030339443965... \tag{40}$$

Table 8: The first few values of $\alpha_4(n)$, $\beta_4(n)$, $\gamma_4(n)$, $\delta_4(n)$, $\eta_4(n)$. The last digits given are rounded off.

n	1	2	3	4
$\alpha_4(n)$	0.384615384615385	0.400836227292906	0.401156579572832	0.401156636030338
$\beta_4(n)$	0.4	0.401407660126596	0.401156681393497	0.401156636030341
$\gamma_4(n)$	0.5	0.402010308343186	0.401156783217105	0.401156636030344
$\delta_4(n)$	0.5	0.402645684586141	0.401156885043655	0.401156636030347
$\eta_4(n)$	1	0.403315684621626	0.401156986873147	0.401156636030350

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of independent sets on $SG_4(n)$ is bounded:

$$\frac{2}{5^{m+1}} \ln[f_4(m)] + \frac{1}{5^m} \ln[1 + \alpha_4^2(m)] \leq z_{SG_4} \leq \frac{2}{5^{m+1}} \ln[f_4(m)] + \frac{1}{5^m} \ln[1 + \eta_4^2(m)] , \tag{41}$$

where m is a positive integer. More than a hundred significant figures for z_{SG_4} can be obtained when m is equal to seven. We have the following proposition.

Proposition 6.2. *The asymptotic growth constant for the number of independent sets on the four-dimensional Sierpinski gasket $SG_4(n)$ in the large n limit is* $z_{SG_4} = 0.28916553234872775551....$

7. Bounds of the asymptotic growth constants

For the d-dimensional Sierpinski gasket $SG_d(n)$, we conjecture that similar upper and lower bounds for the asymptotic growth constant as in Lemma 3.4 hold,

$$\frac{2}{(d+1)^{m+1}} \ln[f_d(m)] + \frac{1}{(d+1)^m} \ln[1+\alpha_d^2(m)] \le z_{SG_d} \le \frac{2}{(d+1)^{m+1}} \ln[f_d(m)] + \frac{1}{(d+1)^m} \ln[1+\zeta_d^2(m)] \quad (42)$$

with a positive integer m, where the ratios are defined as

$$\alpha_d(n) = \frac{g_d(n)}{f_d(n)}, \qquad \zeta_d(n) = \frac{t_d(n)}{s_d(n)}, \quad (43)$$

for a positive integer n. $f_d(n)$ again is the number of independent sets such that all $d+1$ outmost vertices are not in the vertex subset, $g_d(n)$ is the number of independent sets such that one certain outmost vertex is in the vertex subset, $s_d(n)$ is the number of independent sets such that all but one certain outmost vertex are in the vertex subset, and $t_4(n)$ is the number of independent sets such that all $d+1$ outmost vertices are in the vertex subset.

Although the quantities in Eq. (42) for general m are difficult to obtain, one can consider the simplest case $m = 1$. Denote the upper and lowers bounds at $m = 1$ as $\bar{z}_{SG_d}$ and $\underline{z}_{SG_d}$, respectively. Because $s_d(1) = t_d(1) = 1$ and $g_d(1) = f_{d-1}(1)$, we have

$$\begin{aligned} \bar{z}_{SG_d} &= \frac{2}{(d+1)^2} \ln[f_d(1)] + \frac{1}{d+1} \ln(2), \\ \underline{z}_{SG_d} &= \frac{2}{(d+1)^2} \ln[f_d(1)] + \frac{1}{d+1} \ln\Big[1 + \Big(\frac{f_{d-1}(1)}{f_d(1)}\Big)^2\Big], \end{aligned} \quad (44)$$

and the task reduces to the determination of $f_d(1)$. It is easy to see that $f_1(1) = 2$ and we formally set $f_0(1) = 1$, then $f_d(1)$ satisfies the recursion relation

$$f_d(1) = f_{d-1}(1) + d f_{d-2}(1) \quad (45)$$

for $d \ge 2$. This relation can be understood as follows. The d-dimensional Sierpinski gasket $SG_d(1)$ at stage one is the juxtaposition of $d+1$ complete graphs K_{d+1}. For the enumeration of $f_d(1)$, consider one of the complete graphs. In the case that all d interior vertices of the complete graph are not in the vertex subset, the number is the same as $g_d(1) = f_{d-1}(1)$, which is given as the first term on the right-hand-side of Eq. (45). In the case that one of the d interior vertices of the complete graph is in the vertex subset, the number is given by $f_{d-2}(1)$, which gives the second term on the right-hand-side of Eq. (45). It follows that $f_d(1)$ is equal to the number of permutation involutions on $d+1$ elements, which is given by

$$f_d(1) = \sum_{n=0}^{[(d+1)/2]} \frac{(d+1)!}{2^n n!(d+1-2n)!} \quad (46)$$

as sequence A000085 in Ref. [43]. The values of $f_d(1)$, $\underline{z}_{SG_d}$, $\bar{z}_{SG_d}$ for small d are listed in Table 9. We notice that $\underline{z}_{SG_d}$ is closer to z_{SG_d} compared with $\bar{z}_{SG_d}$, and serves as an approximation for z_{SG_d}. Furthermore, it is easy to see that $f_{d-1}(1) \ll f_d(1)$ when d is large using Eq. (45), such that the second term of $\underline{z}_{SG_d}$ in Eq. (44) approaches zero in the infinite d limit. While the term $\ln(2)/(d+1)$ of $\bar{z}_{SG_d}$ also approaches zero in the infinite d limit, $\frac{2}{(d+1)^2} \ln[f_d(1)]$ decreases as d increases. The asymptotic behavior of $f_d(1)$ and the ratio $f_d(1)/f_{d-1}(1)$ has been discussed in [44] and improved in [45]. Using the results in [45], we have the following conjecture.

Conjecture 7.1. *The asymptotic growth constant for the number of independent sets on the d-dimensional Sierpinski gasket SG_d with large d can be approximated as*

$$z_{SG_d} \sim \frac{\ln(d+1)-1}{d+1} + \frac{2}{(d+1)^{3/2}} + \frac{1/2-\ln 2}{(d+1)^2} - \frac{5}{12(d+1)^{5/2}} + \frac{17}{48(d+1)^3} \,. \tag{47}$$

Table 9: Numerical values of $\underline{z}_{SG_d}$, $\bar{z}_{SG_d}$, and some ratios of them to z_{SG_d}. The last digits given are rounded off.

d	$f_d(1)$	$\underline{z}_{SG_d}$	$\bar{z}_{SG_d}$	z_{SG_d}	$\underline{z}_{SG_d}/z_{SG_d}$	$\bar{z}_{SG_d}/z_{SG_d}$
2	4	0.3824465974	0.5391144738	0.3843095344	0.9951525088	1.402813164
3	10	0.3249281379	0.4611099318	0.3285996057	0.9888269257	1.403257715
4	26	0.2882396119	0.3992771592	0.2891655323	0.9967979570	1.380790981
5	76	0.2590427565	0.3561208268	-	-	-
6	232	0.2368781125	0.3213368369	-	-	-
7	764	0.2184809121	0.2940986410	-	-	-
8	2620	0.2034116955	0.2713602941	-	-	-
9	9496	0.1905090814	0.2524872368	-	-	-
10	35696	0.1794854089	0.2362827010	-	-	-

Appendix A. Recursion relations for $SG_{2,3}(n)$

We give the recursion relations for the generalized two-dimensional Sierpinski gasket $SG_{2,3}(n)$ here. Since the subscript is $d = 2, b = 3$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_{2,3}(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} f_{n+1} = {} & f_n^6 + 6f_n^4g_n^2 + 9f_n^2g_n^4 + 6f_n^3g_n^2h_n + 2g_n^6 + 12f_ng_n^4h_n + 6f_n^2g_n^2h_n^2 + 9g_n^4h_n^2 + 6f_ng_n^2h_n^3 + 6g_n^2h_n^4 + h_n^6 + f_n^3g_n^3 \\ & +6f_n^2g_n^3h_n + 9f_ng_n^3h_n^2 + 3f_n^2g_nh_n^3 + 3f_ng_n^4p_n + 2g_n^3h_n^3 + 6f_ng_n^2h_n^2p_n + 6g_n^4h_np_n + 6f_ng_nh_n^4 + 3f_nh_n^4p_n \\ & +9g_n^2h_n^3p_n + 3g_n^3h_np_n^2 + 6g_nh_n^3p_n^2 + h_n^3p_n^3 \,, \end{aligned} \tag{A.1}$$

$$\begin{aligned} g_{n+1} = {} & f_n^5g_n + 2f_n^4g_nh_n + 4f_n^3g_n^3 + 3f_ng_n^5 + 9f_n^2g_n^3h_n + 2f_n^3g_nh_n^2 + f_n^3g_n^2p_n + 4g_n^5h_n + 10f_ng_n^3h_n^2 + 2f_n^2g_nh_n^3 \\ & +2f_ng_n^4p_n + 2f_n^2g_n^2h_np_n + 7g_n^3h_n^3 + 3f_ng_n^2h_n^2p_n + 3g_n^4h_np_n + 2f_ng_nh_n^4 + 2g_nh_n^5 + 4g_n^2h_n^3p_n + h_n^5p_n + f_n^2g_n^4 \\ & +4f_ng_n^4h_n + 2f_n^2g_n^2h_n^2 + 3g_n^4h_n^2 + 8f_ng_n^2h_n^3 + 2f_ng_n^3h_np_n + f_n^2g_nh_n^2p_n + g_n^5p_n + 4g_n^2h_n^4 + 4f_ng_nh_n^3p_n \\ & +8g_n^3h_n^2p_n + 2f_ng_n^2h_np_n^2 + 2f_nh_n^5 + 7g_nh_n^4p_n + 5g_n^2h_n^2p_n^2 + 2f_nh_n^3p_n^2 + g_n^3p_n^3 + 2h_n^4p_n^2 + 4g_nh_n^2p_n^3 + h_n^2p_n^4 \,, \end{aligned} \tag{A.2}$$

$$\begin{aligned} h_{n+1} = {} & f_n^4g_n^2 + 2f_n^2g_n^4 + 4f_n^3g_n^2h_n + 7f_ng_n^4h_n + 5f_n^2g_n^2h_n^2 + 2f_n^2g_n^3p_n + f_n^3h_n^3 + 4g_n^4h_n^2 + 8f_ng_n^2h_n^3 + 4f_ng_n^3h_np_n \\ & +2f_n^2g_nh_n^2p_n + 2g_n^5p_n + 3g_n^2h_n^4 + 2f_ng_nh_n^3p_n + 8g_n^3h_n^2p_n + f_ng_n^2h_np_n^2 + f_nh_n^5 + 4g_nh_n^4p_n + 2g_n^2h_n^2p_n^2 + h_n^4p_n^2 \\ & +f_ng_n^5 + 2g_n^5h_n + 4f_ng_n^3h_n^2 + 7g_n^3h_n^3 + 3f_ng_n^2h_n^2p_n + 2g_n^4h_np_n + 3f_ng_nh_n^4 + 4g_nh_n^5 + 2f_nh_n^4p_n + 10g_n^2h_n^3p_n \\ & +2g_n^3h_np_n^2 + 2f_ng_nh_n^2p_n^2 + 9g_nh_n^3p_n^2 + 3h_n^5p_n + 2g_n^2h_np_n^3 + f_nh_n^2p_n^3 + 4h_n^3p_n^3 + 2g_nh_np_n^4 + h_np_n^5 \,, \end{aligned} \tag{A.3}$$

$$\begin{aligned} p_{n+1} = {} & f_n^3g_n^3 + 6f_n^2g_n^3h_n + 9f_ng_n^3h_n^2 + 3f_n^2g_nh_n^3 + 3f_ng_n^4p_n + 2g_n^3h_n^3 + 6f_ng_n^2h_n^2p_n + 6g_n^4h_np_n + 6f_ng_nh_n^4 + 3f_nh_n^4p_n \\ & +9g_n^2h_n^3p_n + 3g_n^3h_np_n^2 + 6g_nh_n^3p_n^2 + h_n^3p_n^3 + g_n^6 + 6g_n^4h_n^2 + 9g_n^2h_n^4 + 6g_n^3h_n^2p_n + 2h_n^6 + 12g_nh_n^4p_n + 6g_n^2h_n^2p_n^2 \\ & +9h_n^4p_n^2 + 6g_nh_n^2p_n^3 + 6h_n^2p_n^4 + p_n^6 \,. \end{aligned} \tag{A.4}$$

There are always $128 = 2^7$ terms (counting multiplicity) in these equations.

Appendix B. Recursion relations for $SG_3(n)$

We give the recursion relations for the three-dimensional Sierpinski gasket $SG_3(n)$ here. Since the subscript is $d = 3$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_3(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$f_{n+1} = f_n^4 + 6f_n^2g_n^2 + 12f_ng_n^2h_n + 3g_n^4 + 4f_nh_n^3 + 12g_n^2h_n^2 + 4g_n^3p_n + 3h_n^4 + 12g_nh_n^2p_n + 6h_n^2p_n^2 + p_n^4 \,, \tag{B.1}$$

$$\begin{aligned} g_{n+1} = {} & f_n^3g_n + 3f_n^2g_nh_n + 3f_ng_n^3 + 6f_ng_nh_n^2 + 6g_n^3h_n + 3f_ng_n^2p_n + 7g_nh_n^3 + 3f_nh_n^2p_n + 9g_n^2h_np_n + g_n^3q_n + 6h_n^3p_n \\ & +6g_nh_np_n^2 + 3g_nh_n^2q_n + 3h_np_n^3 + 3h_n^2p_nq_n + p_n^3q_n \,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} h_{n+1} = {} & f_n^2g_n^2 + 4f_ng_n^2h_n + f_n^2h_n^2 + g_n^4 + 2f_nh_n^3 + 7g_n^2h_n^2 + 2g_n^3p_n + 4f_ng_nh_np_n + 2h_n^4 + 12g_nh_n^2p_n + 2g_n^2p_n^2 \\ & +2f_nh_np_n^2 + 2g_n^2h_nq_n + 7h_n^2p_n^2 + 2g_np_n^3 + 2h_n^3q_n + 4g_nh_np_nq_n + p_n^4 + 4h_np_n^2q_n + h_n^2q_n^2 + p_n^2q_n^2 \,, \end{aligned} \tag{B.3}$$

$$\begin{aligned} p_{n+1} = {} & f_ng_n^3 + 3f_ng_nh_n^2 + 3g_n^3h_n + 6g_nh_n^3 + 3f_nh_n^2p_n + 6g_n^2h_np_n + 7h_n^3p_n + 9g_nh_np_n^2 + f_np_n^3 + 3g_nh_n^2q_n + 6h_np_n^3 \\ & +6h_n^2p_nq_n + 3g_np_n^2q_n + 3p_n^3q_n + 3h_np_nq_n^2 + p_nq_n^3 \,, \end{aligned} \tag{B.4}$$

$$q_{n+1} = g_n^4 + 6g_n^2h_n^2 + 3h_n^4 + 12g_nh_n^2p_n + 12h_n^2p_n^2 + 4g_np_n^3 + 4h_n^3q_n + 3p_n^4 + 12h_np_n^2q_n + 6p_n^2q_n^2 + q_n^4 \,. \tag{B.5}$$

There are always $64 = 2^6$ terms (counting multiplicity) in these equations.

Appendix C. Recursion relations for $SG_4(n)$

We give the recursion relations for the four-dimensional Sierpinski gasket $SG_4(n)$ here. Since the subscript is $d = 4$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_4(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} f_{n+1} = {} & f_n^5 + 10f_n^3g_n^2 + 30f_n^2g_n^2h_n + 15f_ng_n^4 + 10f_n^2h_n^3 + 60f_ng_n^2h_n^2 + 20f_ng_n^3p_n + 30g_n^4h_n + 15f_nh_n^4 + 60f_ng_nh_n^2p_n \\ & +70g_n^2h_n^3 + 60g_n^3h_np_n + 5g_n^4q_n + 30f_nh_n^2p_n^2 + 12h_n^5 + 120g_nh_n^3p_n + 60g_n^2h_np_n^2 + 30g_n^2h_n^2q_n + 5f_np_n^4 \\ & +70h_n^3p_n^2 + 60g_nh_np_n^3 + 15h_n^4q_n + 60g_nh_n^2p_nq_n + 30h_np_n^4 + 60h_n^2p_n^2q_n + 10h_n^3q_n^2 + 20g_np_n^3q_n + 30h_np_n^2q_n^2 \\ & +15p_n^4q_n + 10p_n^2q_n^3 + q_n^5 \,, \end{aligned} \tag{C.1}$$

$$\begin{aligned} g_{n+1} = {} & f_n^4g_n + 4f_n^3g_nh_n + 6f_n^2g_n^3 + 12f_n^2g_nh_n^2 + 24f_ng_n^3h_n + 6f_n^2g_n^2p_n + 3g_n^5 + 28f_ng_nh_n^3 + 6f_n^2h_n^2p_n + 36f_ng_n^2h_np_n \\ & +4f_ng_n^3q_n + 36g_n^3h_n^2 + 10g_n^4p_n + 24f_nh_n^3p_n + 24f_ng_nh_np_n^2 + 12f_ng_nh_n^2q_n + 31g_nh_n^4 + 90g_n^2h_n^2p_n + 12g_n^3p_n^2 \\ & +16g_n^3h_nq_n + g_n^4r_n + 12f_nh_np_n^3 + 12f_nh_n^2p_nq_n + 36h_n^4p_n + 102g_nh_n^2p_n^2 + 12g_n^2p_n^3 + 36g_n^2h_np_nq_n + 36g_nh_n^3q_n \\ & +6g_n^2h_n^2r_n + 4f_np_n^3q_n + 54h_n^2p_n^3 + 52h_n^3p_nq_n + 60g_nh_np_n^2q_n + 3h_n^4r_n + 12g_nh_n^2p_nr_n + 13g_np_n^4 + 12g_nh_n^2q_n^2 \\ & +52h_np_n^3q_n + 30h_n^2p_nq_n^2 + 12h_n^2p_n^2r_n + 6p_n^5 + 4h_n^3q_nr_n + 12g_np_n^2q_n^2 + 4g_np_n^3r_n + 12h_np_nq_n^3 + 12h_np_n^2q_nr_n \\ & +18p_n^3q_n^2 + 3p_n^4r_n + 4p_nq_n^4 + 6p_n^2q_n^2r_n + q_n^4r_n \,, \end{aligned} \tag{C.2}$$

$$\begin{aligned} h_{n+1} = {} & f_n^3g_n^2 + 6f_n^2g_n^2h_n + f_n^3h_n^2 + 3f_ng_n^4 + 3f_n^2h_n^3 + 21f_ng_n^2h_n^2 + 6f_ng_n^3p_n + 6f_n^2g_nh_np_n + 9g_n^4h_n + 6f_nh_n^4 \\ & +36f_ng_nh_n^2p_n + 6f_ng_n^2p_n^2 + 3f_n^2h_np_n^2 + 31g_n^2h_n^3 + 6f_ng_n^2h_nq_n + 30g_n^3h_np_n + 2g_n^4q_n + 21f_nh_n^2p_n^2 + 6f_ng_np_n^3 \\ & +6f_nh_n^3q_n + 12f_ng_nh_np_nq_n + 7h_n^5 + 72g_nh_n^3p_n + 51g_n^2h_np_n^2 + 27g_n^2h_n^2q_n + 6g_n^3p_nq_n + 2g_n^3h_nr_n + 3f_np_n^4 \\ & +12f_nh_np_n^2q_n + 3f_nh_n^2q_n^2 + 54h_n^3p_n^2 + 54g_nh_np_n^3 + 15h_n^4q_n + 78g_nh_n^2p_nq_n + 6g_n^2h_nq_n^2 + 15g_n^2p_n^2q_n + 6g_nh_n^3r_n \\ & +6g_n^2h_np_nr_n + 3f_np_n^2q_n^2 + 27h_np_n^4 + 81h_n^2p_n^2q_n + 30g_nh_np_nq_n^2 + 13h_n^3q_n^2 + 26g_np_n^3q_n + 12h_n^3p_nr_n \\ & +12g_nh_np_n^2r_n + 6g_nh_n^2q_nr_n + 51h_np_n^2q_n^2 + 18p_n^4q_n + 6h_n^2q_n^3 + 6g_np_nq_n^3 + 14h_np_n^3r_n + 18h_n^2p_nq_nr_n + h_n^3r_n^2 \\ & +6g_np_n^2q_nr_n + 3h_nq_n^4 + 15p_n^2q_n^3 + 12h_np_nq_n^2r_n + 3h_np_n^2r_n^2 + 12p_n^3q_nr_n + q_n^5 + 6p_nq_n^3r_n + 3p_n^2q_nr_n^2 + q_n^3r_n^2 \,, \end{aligned} \tag{C.3}$$

$$\begin{aligned} p_{n+1} = {} & f_n^2g_n^3 + 3f_n^2g_nh_n^2 + 6f_ng_n^3h_n + g_n^5 + 12f_ng_nh_n^3 + 3f_n^2h_n^2p_n + 12f_ng_n^2h_np_n + 15g_n^3h_n^2 + 3g_n^4p_n + 14f_nh_n^3p_n \\ & +18f_ng_nh_np_n^2 + f_n^2p_n^3 + 6f_ng_nh_n^2q_n + 18g_nh_n^4 + 51g_n^2h_n^2p_n + 6g_n^3p_n^2 + 6g_n^3h_nq_n + 12f_nh_np_n^3 + 12f_nh_n^2p_nq_n \\ & +6f_ng_np_n^2q_n + 27h_n^4p_n + 81g_nh_n^2p_n^2 + 13g_n^2p_n^3 + 30g_n^2h_np_nq_n + 26g_nh_n^3q_n + 3g_n^2h_n^2r_n + 6f_np_n^3q_n \\ & +6f_nh_np_nq_n^2 + 54h_n^2p_n^3 + 54h_n^3p_nq_n + 78g_nh_np_n^2q_n + 6g_n^2p_nq_n^2 + 3h_n^4r_n + 12g_nh_n^2p_nr_n + 15g_np_n^4 + 15g_nh_n^2q_n^2 \end{aligned}$$

$$
\begin{aligned}
&+3g_n^2p_n^2r_n + 2f_np_nq_n^3 + 72h_np_n^3q_n + 51h_n^2p_nq_n^2 + 21h_n^2p_n^2r_n + 7p_n^5 + 6h_n^3q_nr_n + 27g_np_n^2q_n^2 + 6g_nh_nq_n^3 \\
&+12g_nh_np_nq_nr_n + 6g_np_n^3r_n + 30h_np_nq_n^3 + 36h_np_n^2q_nr_n + 31p_n^3q_n^2 + 6h_n^2q_n^2r_n + 6p_n^4r_n + 6g_np_nq_n^2r_n + 2g_nq_n^4 \\
&+3h_n^2p_nr_n^2 + 6h_nq_n^3r_n + 9p_nq_n^4 + 21p_n^2q_n^2r_n + 6h_np_nq_nr_n^2 + 3p_n^3r_n^2 + 3q_n^4r_n + 6p_nq_n^2r_n^2 + p_n^2r_n^3 + q_n^2r_n^3 \,, \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
q_{n+1} \;=\;& f_ng_n^4 + 6f_ng_n^2h_n^2 + 4g_n^4h_n + 3f_nh_n^4 + 12f_ng_nh_n^2p_n + 18g_n^2h_n^3 + 12g_n^3h_np_n + 12f_nh_n^2p_n^2 + 4f_ng_np_n^3 + 4f_nh_n^3q_n \\
&+6h_n^5 + 52g_nh_n^3p_n + 30g_n^2h_np_n^2 + 12g_n^2h_n^2q_n + 3f_np_n^4 + 12f_nh_np_n^2q_n + 54h_n^3p_n^2 + 52g_nh_np_n^3 + 13h_n^4q_n \\
&+60g_nh_n^2p_nq_n + 12g_n^2p_n^2q_n + 4g_nh_n^3r_n + 6f_np_n^2q_n^2 + 36h_np_n^4 + 102h_n^2p_n^2q_n + 36g_nh_np_nq_n^2 + 12h_n^3q_n^2 \\
&+36g_np_n^3q_n + 12h_n^3p_nr_n + 12g_nh_np_n^2r_n + f_nq_n^4 + 90h_np_n^2q_n^2 + 31p_n^4q_n + 12h_n^2q_n^3 + 16g_np_nq_n^3 + 24h_np_n^3r_n \\
&+24h_n^2p_nq_nr_n + 12g_np_n^2q_nr_n + 10h_nq_n^4 + 36p_n^2q_n^3 + 36h_np_nq_n^2r_n + 6h_np_n^2r_n^2 + 28p_n^3q_nr_n + 4g_nq_n^3r_n + 3q_n^5 \\
&+6h_nq_n^2r_n^2 + 24p_nq_n^3r_n + 12p_n^2q_nr_n^2 + 6q_n^3r_n^2 + 4p_nq_nr_n^3 + q_nr_n^4 \,, \qquad \text{(C.5)}
\end{aligned}
$$

$$
\begin{aligned}
r_{n+1} \;=\;& g_n^5 + 10g_n^3h_n^2 + 15g_nh_n^4 + 30g_n^2h_n^2p_n + 30h_n^4p_n + 60g_nh_n^2p_n^2 + 10g_n^2p_n^3 + 20g_nh_n^3q_n + 70h_n^2p_n^3 + 60h_n^3p_nq_n \\
&+60g_nh_np_n^2q_n + 5h_n^4r_n + 15g_np_n^4 + 120h_np_n^3q_n + 60h_n^2p_nq_n^2 + 30h_n^2p_n^2r_n + 12p_n^5 + 30g_np_n^2q_n^2 + 60h_np_nq_n^3 \\
&+60h_np_n^2q_nr_n + 70p_n^3q_n^2 + 15p_n^4r_n + 5g_nq_n^4 + 20h_nq_n^3r_n + 30p_nq_n^4 + 60p_n^2q_n^2r_n + 10p_n^3r_n^2 + 15q_n^4r_n \\
&+30p_nq_n^2r_n^2 + 10q_n^2r_n^3 + r_n^5 \,. \qquad \text{(C.6)}
\end{aligned}
$$

There are always $1024 = 2^{10}$ terms (counting multiplicity) in these equations.

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