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Improved results on the Drazin inverse of a 2×2 block matrix in terms of Banachiewicz-Schur forms

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Abstract. In this paper, we derive a representation for the Drazin inverse of a block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under the assumptions $AA^{\pi}B = 0$, $CA^{\pi}B = 0$, $AA^DBSS^{\pi} = 0$, $SS^DCWAA^D(AW)^{\pi} = 0$ and $(AW)^{\pi}AA^DBSS^D = 0$, where $S = D - CA^DB$ is the generalized Schur complement. And the representation can be regarded as an unified form of M^D because it covers the case either S is nonsingular or zero. Moreover, some alternative representations for the Drazin inverse are presented. Several situations are analyzed and recent results are generalized.

1. Introduction

Let A be a square complex matrix. As we know, the Drazin inverse [1] of A, denoted by A^D, is the unique matrix satisfying the following three equations

$$A^kA^DA = A^k, \quad A^DAA^D = A^D, \quad AA^D = A^DA,$$

where $k = ind(A)$ is the index of A. If $ind(A) = 1$, then the Drazin inverse of A is reduced to the group inverse, denote by $A^{\#}$. If $ind(A) = 0$, then $A^D = A^{-1}$. In addition, we denote $A^{\pi} = I - AA^D$, and define $A^0 = I$, where I is the identity matrix with proper sizes.

The Drazin inverse is very useful, and the applications in singular differential or difference equations, Markov chains, cryptography, iterative method and numerical analysis can be found in [1, 2, 11], respectively.

In this paper, we consider the Drazin inverse of a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where A and D are square complex matrices but need not to be the same size. This problem was first proposed by Campbell and Meyer [2], and is quite complicated. To the best of our knowledge, there was no

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explicit formula for the Drazin inverse of M with arbitrary blocks. However, representations for the Drazin inverse of the block matrices were presented in the literature under some conditions [3-11, 13-21].

As is well known that, if A and the Schur complement $S = D - CA^{-1}B$ are nonsingular, then M is also nonsingular, and the inverse of M can be expressed as

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}. \tag{2}$$

The generalized Schur complement of A in M denoted by $S = D - CA^DB$ plays an important role in the representations for M^D. When M and A are nonsingular, we focus attention on the Drazin inverse of M, it is natural to see whether M^D has a form (2) with A^{-1} and S^{-1} replaced by A^D and S^D, respectively. In recent papers [14, 16], the necessary and sufficient conditions for this problem were established.

When S is either nonsingular or zero, Hartwig et al. [11] extended the results in [17] by replacing the assumptions $A^\pi B = 0$ and $CA^\pi = 0$ with $CA^\pi B = 0$ and $AA^\pi B = 0$. In the case S is nonsingular, Martínez-Serrano and Castro-González [10] gave some new results under weaker conditions $A^2A^\pi B = 0$, $CAA^\pi B = 0$ and $CA^\pi B = 0$; in the case S is zero, they derived the expressions of M^D under the assumptions $A^2A^\pi B = 0$, $CAA^\pi B = 0$ and $BCA^\pi B = 0$ (or $ABCA^\pi = 0$, BCA^π is nilpotent). Recently, when $S = 0$, Yang and Liu [15] presented a formula of M^D under the conditions $AA^\pi BC = 0$ and $CA^\pi BC = 0$. Deng and Wei [18] presented some new results on the formulae of M^D and $M^\#$. Recently, Castro-González and Martínez-Serrano [13] developed the formula of M^D under the conditions $ind(S) \leqslant 1$ and $W = I + A^DBS^\pi CA^D$ is nonsingular. Li [19] developed a representation under certain conditions when S is group invertible.

To the best of our knowledge, it is still an open problem to find an explicit formula for M^D when the generalized Schur complement S is an arbitrary matrix. And most of the existing results mentioned above were obtained when S is zero or $ind(S) \leqslant 1$. In this paper, we give several representations for the Drazin inverse of M without the restrictions $ind(S) \leqslant 1$ and $W = I + A^DBS^\pi CA^D$ is nonsingular.

It is known that, if matrix A and B are similar, i.e., there exists a nonsingular matrix P such that $A = PBP^{-1}$, then $A^D = PB^DP^{-1}$. Based on this fact, Castro-González and Martínez-Serrano [13] used it to determine the expression of M^D. Here, we adopt this technique again, but the choices of P are different.

Before proceeding to the next section, we first introduce the following results which will come in handy in the proofs of our theorems.

Lemma 1.1. ([12]) *Let $P, Q \in C^{n\times n}$, and $ind(P) = r$, $ind(Q) = s$, such that $PQ = 0$, then*

$$(P+Q)^D = Q^\pi \sum_{i=0}^{s-1} Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi.$$

Lemma 1.2. ([14]) *Let M be a given matrix of form (1) and $S = D - CA^DB$ be the generalized Schur complement of A in M. Then*

$$M^D = \begin{pmatrix} A^D + A^DBS^DCA^D & -A^DBS^D \\ -S^DCA^D & S^D \end{pmatrix}$$

if and only if

$$A^\pi BS^D = A^DBS^\pi, \quad S^\pi CA^D = S^DCA^\pi$$

and

$$\begin{pmatrix} AA^\pi & A^\pi B \\ S^\pi CA^\pi & SS^\pi \end{pmatrix}$$

is a nilpotent operator (or matrix).

In the following, we always denote $S = D - CA^DB$ and $W = I + A^DBS^\pi CA^D$.

Lemma 1.3. *Let $A \in C^{n\times n}$ with $ind(A) = r$. Then*

$$(A^2A^DW)^D = AA^D(AW)^D = (AW)^DAA^D = (AW)^D, \ \ (A^2A^DW)^\pi = (AW)^\pi; \tag{3}$$

$$(WA^2A^D)^D = AA^D(WA)^D = (WA)^DAA^D = (WA)^D, \ \ (WA^2A^D)^\pi = (WA)^\pi. \tag{4}$$

Moreover, if W is nonsingular, then

$$(A^2A^DW)^D = W^{-1}A^D, \ \ (A^2A^DW)^\pi = A^\pi;$$
$$(WA^2A^D)^D = A^DW^{-1}, \ \ (WA^2A^D)^\pi = A^\pi.$$

And the following statements are equivalent:
(1) *W is nonsingular;*
(2) *$AA^D(AW)^\pi = (AW)^\pi AA^D = 0$;*
(3) *$AA^D(WA)^\pi = (WA)^\pi AA^D = 0$.*

Proof. Without loss of generality, we assume that matrix A and $BS^\pi C$ can be represented as

$$A = \begin{pmatrix} \Sigma & 0 \\ 0 & N \end{pmatrix}, \ \ BS^\pi C = \begin{pmatrix} (BS^\pi C)_{11} & (BS^\pi C)_{12} \\ (BS^\pi C)_{21} & (BS^\pi C)_{22} \end{pmatrix},$$

where Σ is nonsingular and N is nilpotent with index r, Σ and $(BS^\pi C)_{11}$ have the same size. Hence,

$$A^2A^DW = \begin{pmatrix} \Sigma(I + \Sigma^{-1}(BS^\pi C)_{11}\Sigma^{-1}) & 0 \\ 0 & 0 \end{pmatrix}, \ \ AW = \begin{pmatrix} \Sigma(I + \Sigma^{-1}(BS^\pi C)_{11}\Sigma^{-1}) & 0 \\ 0 & N \end{pmatrix}.$$

Therefore, the conclusion (3) is evident. Similarly, we can prove (4). In the case W is nonsingular, the results have been shown in [13]. And the equivalence of statements among (1)-(3) can be easily verified. □

2. Main results

Castro-González and Martínez-Serrano [13] gave some representations for the Drazin inverse of block matrix M when $ind(S) \leqslant 1$ and W is nonsingular together with some conditions. In this section, we present several representations for M^D under some weaker conditions than those in [13]. Furthermore, some analogous results are given.

According to Lemma 1.3, the nonsingularity of W is equivalent to $AA^D(AW)^\pi = (AW)^\pi AA^D = 0$, next, we consider the representation of M^D extended the case where $AA^D(AW)^\pi = (AW)^\pi AA^D = 0$ are substituted with $SS^DCWAA^D(AW)^\pi = 0$ and $(AW)^\pi AA^DBSS^D = 0$.

Theorem 2.1. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $AA^\pi B = 0$, $CA^\pi B = 0$, $AA^DBSS^\pi = 0$, $SS^DCWAA^D(AW)^\pi = 0$ and $(AW)^\pi AA^DBSS^D = 0$, then*

$$\begin{aligned} M^D = &\left[\begin{pmatrix} I & 0 \\ S^\pi CA^D & I \end{pmatrix} + \begin{pmatrix} A^\pi BS^\pi CA^D & A^\pi B \\ 0 & 0 \end{pmatrix} R + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^j + \sum_{j=1}^{s-1} \begin{pmatrix} A^\pi BS^jS^\pi CA^D & 0 \\ 0 & 0 \end{pmatrix} R^{j+1}\right] R \\ &\times\left[\begin{pmatrix} I & 0 \\ -S^\pi CA^D & I \end{pmatrix} + R\begin{pmatrix} -BS^\pi CA^D & BS^\pi \\ C(I-W) & CA^DBS^\pi \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^\pi & 0 \end{pmatrix}\right. \\ &\left. + \sum_{i=0}^{r-1} R^{i+2} \begin{pmatrix} BS^\pi CA^iA^\pi & 0 \\ CA^DBS^\pi CA^iA^\pi & 0 \end{pmatrix}\right], \end{aligned} \tag{5}$$

where

$$R = \begin{pmatrix} (AW)^D + (AW)^DBS^DCW(AW)^D & -(AW)^DBS^D \\ -S^DCW(AW)^D & S^D \end{pmatrix}.$$

Proof. Consider the following splitting of M

$$M = \begin{pmatrix} I & 0 \\ S^{\pi}CA^{D} & I \end{pmatrix} \tilde{M} \begin{pmatrix} I & 0 \\ -S^{\pi}CA^{D} & I \end{pmatrix},$$

where

$$\tilde{M} = \begin{pmatrix} A + BS^{\pi}CA^{D} & B \\ SS^{\pi}CA^{D} + SS^{D}CA^{D}BS^{\pi}CA^{D} + CA^{\pi} + SS^{D}CAA^{D} & S + SS^{D}CA^{D}B \end{pmatrix}.$$

Hence

$$M^{D} = \begin{pmatrix} I & 0 \\ S^{\pi}CA^{D} & I \end{pmatrix} \tilde{M}^{D} \begin{pmatrix} I & 0 \\ -S^{\pi}CA^{D} & I \end{pmatrix}. \tag{6}$$

And write $\tilde{M}$ as

$$\tilde{M} = X + Y + Z,$$

where

$$\begin{aligned} X &= \begin{pmatrix} A^{2}A^{D}W & AA^{D}B \\ SS^{\pi}CA^{D} + SS^{D}CWAA^{D} & S + SS^{D}CA^{D}B \end{pmatrix}, \\ Y &= \begin{pmatrix} A^{\pi}BS^{\pi}CA^{D} & A^{\pi}B \\ 0 & 0 \end{pmatrix}, \\ Z &= \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}. \end{aligned}$$

Since $AA^{\pi}B = 0$ and $CA^{\pi}B = 0$, we get $Z(X + Y) = 0$, $XY = 0$, $Y^{2} = 0$ and Z is nilpotent with index $r + 1$. Applying Lemma 1.1 yields

$$\tilde{M}^{D} = \sum_{i=0}^{r} \left((X + Y)^{D}\right)^{i+1} Z^{i}, \tag{7}$$

$$(X + Y)^{D} = X^{D} + Y(X^{D})^{2},$$

and, for $i \geqslant 0$,

$$\left((X + Y)^{D}\right)^{i+1} = (X^{D})^{i+1} + Y(X^{D})^{i+2}.$$

Therefore,

$$\tilde{M}^{D} = \sum_{i=0}^{r} \left((X^{D})^{i+1} + Y(X^{D})^{i+2}\right) Z^{i}. \tag{8}$$

Next, we compute X^{D}. Rewrite X as

$$\begin{aligned} X &= \begin{pmatrix} A^{2}A^{D}W & AA^{D}BSS^{D} \\ SS^{D}CWAA^{D} & S^{2}S^{D} + SS^{D}CA^{D}BSS^{D} \end{pmatrix} + \begin{pmatrix} 0 & AA^{D}BS^{\pi} \\ 0 & SS^{D}CA^{D}BS^{\pi} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ SS^{\pi}CA^{D} & SS^{\pi} \end{pmatrix} \\ &\triangleq X_{1} + X_{2} + X_{3}. \end{aligned} \tag{9}$$

Since $AA^{D}BSS^{\pi} = 0$, we have $(X_{1} + X_{2})X_{3} = 0$, X_{3} is nilpotent with index s, and $X_{2}X_{1} = 0$, $X_{2}^{2} = 0$. Therefore, using Lemma 1.1 gives

$$X^{D} = \sum_{j=0}^{s-1} X_{3}^{j} \left((X_{1} + X_{2})^{D}\right)^{j+1}, \tag{10}$$

and for $j \geqslant 0$,

$$\left((X_1 + X_2)^D\right)^{j+1} = (X_1^D)^{j+1} + (X_1^D)^{j+2} X_2. \tag{11}$$

It follows from $(AW)^\pi AA^D BSS^D = 0$ that

$$\begin{aligned}
& S^2S^D + SS^DCA^DBSS^D - SS^DCWAA^D(A^2A^DW)^DAA^DBSS^D \\
= \ & S^2S^D + SS^DCA^DBSS^D - SS^DCW(AW)^DAA^DBSS^D \\
= \ & S^2S^D + SS^DCA^DBSS^D - SS^DCA^DAW(AW)^DAA^DBSS^D \\
= \ & S^2S^D + SS^DCA^D(AW)^\pi AA^DBSS^D \\
= \ & S^2S^D
\end{aligned}$$

i.e., the generalized Schur complement of A^2A^DW in X_1 is S^2S^D. By assumptions, we can verify that X_1 satisfies the conditions of Lemma 1.2, thus,

$$X_1^D = \begin{pmatrix} (AW)^D + (AW)^DBS^DCW(AW)^D & -(AW)^DBS^D \\ -S^DCW(AW)^D & S^D \end{pmatrix} \triangleq R. \tag{12}$$

Substituting (12) into (11) produces

$$\left((X_1 + X_2)^D\right)^{j+1} = R^{j+1} + R^{j+2} \begin{pmatrix} 0 & BS^\pi \\ 0 & CA^DBS^\pi \end{pmatrix}. \tag{13}$$

By substituting (13) into (10) follows that

$$X^D = \left[I + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^j\right] R \left[I + R \begin{pmatrix} 0 & BS^\pi \\ 0 & CA^DBS^\pi \end{pmatrix}\right].$$

Further, for all $i \geqslant 0$,

$$(X^D)^{i+1} = \left[I + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^j\right] R^{i+1} \left[I + R \begin{pmatrix} 0 & BS^\pi \\ 0 & CA^DBS^\pi \end{pmatrix}\right]. \tag{14}$$

In terms of (8) and (14) gives

$$\begin{aligned}
\tilde{M}^D &= \left[I + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^j + YR + Y\sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^{j+1}\right] R \sum_{i=0}^{r} \left[R^iZ^i + R^{i+1} \begin{pmatrix} 0 & BS^\pi \\ 0 & CA^DBS^\pi \end{pmatrix} Z^i\right] \\
&= \left[I + \begin{pmatrix} A^\pi BS^\pi CA^D & A^\pi B \\ 0 & 0 \end{pmatrix} R + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^\pi CA^D & 0 \end{pmatrix} R^j + \sum_{j=1}^{s-1} \begin{pmatrix} A^\pi BS^jS^\pi CA^D & 0 \\ 0 & 0 \end{pmatrix} R^{j+1}\right] \\
&\quad \times R\left[I + R \begin{pmatrix} 0 & BS^\pi \\ 0 & CA^DBS^\pi \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^\pi & 0 \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+2} \begin{pmatrix} BS^\pi CA^iA^\pi & 0 \\ CA^DBS^\pi CA^iA^\pi & 0 \end{pmatrix}\right].
\end{aligned} \tag{15}$$

Substituting (15) into (6) gives formula (5). □

As an application of Theorem 2.1, we can deduce following result.

Corollary 2.2. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $AA^{\pi}B = 0$, $CA^{\pi}B = 0$, $BSS^{\pi} = 0$ and $BS^{\pi}C = 0$, then*

$$\begin{aligned} M^D &= \left[\begin{pmatrix} I & 0 \\ S^{\pi}CA^D & I \end{pmatrix} + \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix} R + \sum_{j=1}^{s-1} \begin{pmatrix} 0 & 0 \\ S^jS^{\pi}CA^D & 0 \end{pmatrix} R^j \right] \\ &\quad \times R\left[\begin{pmatrix} I & A^DBS^{\pi} \\ -S^{\pi}CA^D & I \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^{\pi} & 0 \end{pmatrix}\right], \end{aligned}$$

where R is defined as in Theorem 2.1 with $W = I$.

Similarly, we can prove the following result.

Theorem 2.3. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $AA^{\pi}B = 0$, $CA^{\pi}B = 0$, $SS^{\pi}CAA^D = 0$, $SS^DCWAA^D(AW)^{\pi} = 0$ and $(AW)^{\pi}AA^DBSS^D = 0$, then*

$$\begin{aligned} M^D &= \left[\begin{pmatrix} I & 0 \\ S^{\pi}CA^D & I \end{pmatrix} + \begin{pmatrix} A^{\pi}BS^{\pi}CA^D & A^{\pi}B \\ 0 & 0 \end{pmatrix} R \right] R \\ &\quad \times \left[\begin{pmatrix} I & 0 \\ -S^{\pi}CA^D & I \end{pmatrix} + R \begin{pmatrix} -BS^{\pi}CA^D & 0 \\ C(I-W) & 0 \end{pmatrix} + \sum_{j=0}^{s-1} R^{j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix} \right. \\ &\quad \left. + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^{\pi} & 0 \end{pmatrix} + \sum_{i=0}^{r-1}\sum_{j=0}^{s-1} R^{i+j+2} \begin{pmatrix} BS^jS^{\pi}CA^iA^{\pi} & 0 \\ CA^DBS^jS^{\pi}CA^iA^{\pi} & 0 \end{pmatrix}\right], \end{aligned}$$

where R is defined as in Theorem 2.1.

Proof. If $SS^{\pi}CAA^D = 0$, then (9) reduces to

$$X = X_1 + X_2,$$

where

$$\begin{aligned} X_1 &= \begin{pmatrix} A^2A^DW & AA^DBSS^D \\ SS^DCWAA^D & S^2S^D + SS^DCA^DBSS^D \end{pmatrix}, \\ X_2 &= \begin{pmatrix} 0 & AA^DBS^{\pi} \\ 0 & SS^{\pi} + SS^DCA^DBS^{\pi} \end{pmatrix}. \end{aligned}$$

Note that $X_2X_1 = 0$, and X_2 is nilpotent with index $s + 1$. By Lemma 1.1 and (12), we can compute that

$$(X^D)^{i+1} = R^{i+1}\left[I + \sum_{j=0}^{s-1} R^{j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix}\right].$$

By (8), we get

$$\begin{aligned} \tilde{M}^D &= \left[I + \begin{pmatrix} A^{\pi}BS^{\pi}CA^D & A^{\pi}B \\ 0 & 0 \end{pmatrix} R\right] \sum_{i=0}^{r} R^{i+1} \left[I + \sum_{j=0}^{s-1} R^{j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix}\right] Z^i \\ &= \left[I + \begin{pmatrix} A^{\pi}BS^{\pi}CA^D & A^{\pi}B \\ 0 & 0 \end{pmatrix} R\right] R \left[I + \sum_{j=0}^{s-1} R^{j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix} + \sum_{i=1}^{r} R^i \begin{pmatrix} A^iA^{\pi} & 0 \\ CA^{i-1}A^{\pi} & 0 \end{pmatrix} \right. \\ &\quad \left. + \sum_{i=1}^{r}\sum_{j=0}^{s-1} R^{i+j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix} \begin{pmatrix} A^iA^{\pi} & 0 \\ CA^{i-1}A^{\pi} & 0 \end{pmatrix}\right]. \end{aligned}$$

Hence, this Theorem can be easily obtained. □

Using Theorem 2.3, we can derive the following result.

Corollary 2.4. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $AA^{\pi}B = 0$, $CA^{\pi}B = 0$, $SS^{\pi}C = 0$ and $BS^{\pi}C = 0$, then*

$$M^D = \left[\begin{pmatrix} I & 0 \\ S^{\pi}CA^D & I \end{pmatrix} + \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix} R\right] R \left[\begin{pmatrix} I & 0 \\ -S^{\pi}CA^D & I \end{pmatrix} + \sum_{j=0}^{s-1} R^{j+1} \begin{pmatrix} 0 & BS^jS^{\pi} \\ 0 & CA^DBS^jS^{\pi} \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^{\pi} & 0 \end{pmatrix}\right],$$

where R is defined as in Theorem 2.1 with $W = I$.

If $ind(S) \leqslant 1$, by Lemma 1.3, then Theorem 2.1 and Theorem 2.3 reduce to the following corollary.

Corollary 2.5. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) \leqslant 1$. If $AA^{\pi}B = 0$, $CA^{\pi}B = 0$, and W is nonsingular, then*

$$\begin{aligned} M^D = & \left[\begin{pmatrix} I & 0 \\ S^{\pi}CA^D & I \end{pmatrix} + \begin{pmatrix} A^{\pi}BS^{\pi}CA^D & A^{\pi}B \\ 0 & 0 \end{pmatrix} R\right] R \left[\begin{pmatrix} I & 0 \\ -S^{\pi}CA^D & I \end{pmatrix} + R \begin{pmatrix} -BS^{\pi}CA^D & BS^{\pi} \\ C(I-W) & CA^DBS^{\pi} \end{pmatrix}\right. \\ & \left. + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^iA^{\pi} & 0 \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+2} \begin{pmatrix} BS^{\pi}CA^iA^{\pi} & 0 \\ CA^DBS^{\pi}CA^iA^{\pi} & 0 \end{pmatrix}\right], \end{aligned}$$

where

$$R = \begin{pmatrix} W^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^D + A^DBS^DCA^D & -A^DBS^D \\ -S^DCA^D & S^D \end{pmatrix}.$$

If the condition $AA^{\pi}B = 0$ in Theorem 2.1 and Theorem 2.3 is replaced by $CAA^{\pi} = 0$, then the following results can be deduced by a similar approach.

Theorem 2.6. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $CAA^{\pi} = 0$, $CA^{\pi}B = 0$, $SS^{\pi}CAA^D = 0$, $SS^DC(WA)^{\pi}AA^D = 0$ and $(WA)^{\pi}AA^DWBSS^D = 0$, then*

$$\begin{aligned} M^D = & \left[\begin{pmatrix} I & -A^DBS^{\pi} \\ 0 & I \end{pmatrix} + \begin{pmatrix} -A^DBS^{\pi}C & (I-W)B \\ S^{\pi}C & S^{\pi}CA^DB \end{pmatrix} \tilde{R} + \sum_{i=0}^{r-1} \begin{pmatrix} 0 & A^iA^{\pi}B \\ 0 & 0 \end{pmatrix} \tilde{R}^{i+1}\right. \\ & \left. + \sum_{i=0}^{r-1} \begin{pmatrix} A^iA^{\pi}BS^{\pi}C & A^iA^{\pi}BS^{\pi}CA^DB \\ 0 & 0 \end{pmatrix} \tilde{R}^{i+2}\right] \tilde{R} \left[\begin{pmatrix} I & A^DBS^{\pi} \\ 0 & I \end{pmatrix} + \tilde{R} \begin{pmatrix} A^DBS^{\pi}CA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}\right. \\ & \left. + \sum_{j=1}^{s-1} \tilde{R}^j \begin{pmatrix} 0 & A^DBS^jS^{\pi} \\ 0 & 0 \end{pmatrix} + \sum_{j=1}^{s-1} \tilde{R}^{j+1} \begin{pmatrix} A^DBS^jS^{\pi}CA^{\pi} & 0 \\ 0 & 0 \end{pmatrix}\right], \end{aligned}$$

where

$$\tilde{R} = \begin{pmatrix} (WA)^D + (WA)^DWBS^DC(WA)^D & -(WA)^DWBS^D \\ -S^DC(WA)^D & S^D \end{pmatrix}.$$

Proof. Consider the splitting of M

$$M = \begin{pmatrix} I & -A^DBS^{\pi} \\ 0 & I \end{pmatrix} \tilde{M} \begin{pmatrix} I & A^DBS^{\pi} \\ 0 & I \end{pmatrix},$$

where

$$\tilde{M} = \begin{pmatrix} A + A^DBS^{\pi}C & A^{\pi}B + AA^DBSS^D + A^DBSS^{\pi} + A^DBS^{\pi}CA^DBSS^D \\ C & S + CA^DBSS^D \end{pmatrix}.$$

Hence

$$M^D = \begin{pmatrix} I & -A^DBS^\pi \\ 0 & I \end{pmatrix} \tilde{M}^D \begin{pmatrix} I & A^DBS^\pi \\ 0 & I \end{pmatrix}.$$

And write $\tilde{M}$ as

$$\tilde{M} = X + Y + Z,$$

where

$$\begin{aligned} X &= \begin{pmatrix} WA^2A^D & AA^DWBSS^D + A^DBSS^\pi \\ CAA^D & S + CA^DBSS^D \end{pmatrix}, \\ Y &= \begin{pmatrix} A^DBS^\pi CA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}, \\ Z &= \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The following derivative process is similar to Theorem 2.1, we omit the details. □

The following result can be easily obtained from Theorem 2.6.

Corollary 2.7. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $CAA^\pi = 0$, $CA^\pi B = 0$, $SS^\pi C = 0$ and $BS^\pi C = 0$, then*

$$M^D = \left[\begin{pmatrix} I & -A^DBS^\pi \\ S^\pi CA^D & I \end{pmatrix} + \sum_{i=0}^{r-1}\begin{pmatrix} 0 & A^iA^\pi B \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+1}\right]\tilde{R}\left[\begin{pmatrix} I - A^DBS^DCA^\pi & A^DBS^\pi \\ S^DCA^\pi & I \end{pmatrix} + \sum_{j=1}^{s-1}\tilde{R}^j\begin{pmatrix} 0 & A^DBS^jS^\pi \\ 0 & 0 \end{pmatrix}\right],$$

where $\tilde{R}$ is defined as in Theorem 2.6 with $W = I$.

Similarly, we can deduce the following consequence.

Theorem 2.8. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $CAA^\pi = 0$, $CA^\pi B = 0$, $AA^DBSS^\pi = 0$, $SS^DC(WA)^\pi AA^D = 0$ and $(WA)^\pi AA^DWBSS^D = 0$, then*

$$\begin{aligned} M^D = &\left[\begin{pmatrix} I & -A^DBS^\pi \\ 0 & I \end{pmatrix} + \begin{pmatrix} -A^DBS^\pi C & (I-W)B \\ 0 & 0 \end{pmatrix}\tilde{R} + \sum_{j=0}^{s-1}\begin{pmatrix} 0 & 0 \\ S^jS^\pi C & S^jS^\pi CA^DB \end{pmatrix}\tilde{R}^{j+1}\right. \\ &\left. + \sum_{i=0}^{r-1}\begin{pmatrix} 0 & A^iA^\pi B \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+1} + \sum_{i=0}^{r-1}\sum_{j=0}^{s-1}\begin{pmatrix} A^iA^\pi BS^jS^\pi C & A^iA^\pi BS^jS^\pi CA^DB \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+j+2}\right] \\ &\times\tilde{R}\left[\begin{pmatrix} I & A^DBS^\pi \\ 0 & I \end{pmatrix} + \tilde{R}\begin{pmatrix} A^DBS^\pi CA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}\right], \end{aligned}$$

where $\tilde{R}$ is defined as in Theorem 2.6.

By Theorem 2.8, the following corollary is evident.

Corollary 2.9. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) = s$. If $CAA^\pi = 0$, $CA^\pi B = 0$, $BSS^\pi = 0$ and $BS^\pi C = 0$, then*

$$\begin{aligned} M^D = &\left[\begin{pmatrix} I & -A^DBS^\pi \\ 0 & I \end{pmatrix} + \sum_{j=0}^{s-1}\begin{pmatrix} 0 & 0 \\ S^jS^\pi C & S^jS^\pi CA^DB \end{pmatrix}\tilde{R}^{j+1} + \sum_{i=0}^{r-1}\begin{pmatrix} 0 & A^iA^\pi B \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+1}\right] \\ &\times\tilde{R}\left[\begin{pmatrix} I & A^DBS^\pi \\ 0 & I \end{pmatrix} + \tilde{R}\begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}\right], \end{aligned}$$

where $\tilde{R}$ is defined as in Theorem 2.6 with $W = I$.

If $ind(S) \leqslant 1$, then Theorem 2.6 and Theorem 2.8 reduce to the following corollary.

Corollary 2.10. *Let M be a given matrix of form (1) with $ind(A) = r$, $ind(S) \leqslant 1$. If $CAA^\pi = 0$, $CA^\pi B = 0$, and W is nonsingular, then*

$$\begin{aligned} M^D \;=\; & \Bigg[\begin{pmatrix} I & -A^D BS^\pi \\ 0 & I \end{pmatrix} + \begin{pmatrix} -A^D BS^\pi C & (I-W)B \\ S^\pi C & S^\pi CA^D B \end{pmatrix}\tilde{R} + \sum_{i=0}^{r-1}\begin{pmatrix} 0 & A^i A^\pi B \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+1} \\ & + \sum_{i=0}^{r-1}\begin{pmatrix} A^i A^\pi BS^\pi C & A^i A^\pi BS^\pi CA^D B \\ 0 & 0 \end{pmatrix}\tilde{R}^{i+2}\Bigg]\tilde{R}\Bigg[\begin{pmatrix} I & A^D BS^\pi \\ 0 & I \end{pmatrix} + \tilde{R}\begin{pmatrix} A^D BS^\pi CA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}\Bigg], \end{aligned}$$

where

$$\tilde{R} = \begin{pmatrix} A^D + A^D BS^D CA^D & -A^D BS^D \\ -S^D CA^D & S^D \end{pmatrix}\begin{pmatrix} W^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Remark 2.11. *By the above theorems the other special cases can also be deduced, such as: Theorem 2.6 or Theorem 2.8 in [13]; Theorem 3.1 or Corollary 3.2 (S is nonsingular), Theorem 4.1 or Corollary 4.2 (S is zero) in [11].*

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References

[1] A. Ben-Israel and T. N. E. Greville, Generalized Inverses: Theory and Applications, 2nd ed., Springer, New York, 2003.
[2] S. L. Campbell, C. D. Meyer, Generalized Inverse of Linear Transformations, Dover, New York, 1991.
[3] N. Castro-González, E. Dopazo, M. F. Martínez-Serrano, On the Drazin inverse of the sum of two operators and its application to operator matrices, J. Math. Anal. Appl. 350 (2008) 207-215.
[4] X. Li, Y. Wei, A note on the representations for the Drazin inverse of 2×2 block matrices, Linear Algebra Appl. 423 (2007) 332-338.
[5] Y. Wei, Expressions for the Drazin inverse of a 2×2 block matrix, Linear and Multilinear Algebra 45 (1998) 131-146.
[6] N. Castro-González, E. Dopazo, J. Robles, Formulas for the Drazin inverse of special block matrices, Appl. Math. Comput. 174 (2006) 252-270.
[7] C. D. Meyer, N. J. Rose, The index and the Drazin inverse of block triangular matrices, SIAM J. Appl. Math. 33 (1977) 1-7.
[8] D. S. Cvetković-Ilić, A note on the representation for the Drazin inverse of 2×2 block matrices, Linear Algebra Appl. 429 (2008) 242-248.
[9] L. Guo, X. Du, Representations for the Drazin inverses of 2×2 block matrices, Appl. Math. Comput. 217 (2010) 2833-2842.
[10] M. F. Martínez-Serrano, N. Castro-González, On the Drazin inverse of block matrices and generalized Schur complement, Appl. Math. Comput. 215 (2009) 2733-2740.
[11] R. E. Hartwig, X. Li, Y. Wei, Representations for the Drazin inverse of a 2×2 block matrix, SIAM J. Matrix Anal. Appl. 27 (2006) 757-771.
[12] R. E. Hartwig, G. Wang, Y. Wei, Some additive results on Drazin inverse, Linear Algebra Appl. 322 (2001) 207-217.
[13] N. Castro-González, M.F. Martínez-Serrano, Drazin inverse of partitioned matrices in terms of Banachiewicz-Schur forms, Linear Algebra Appl. 432 (2010) 1691-1702.
[14] C. Deng, A note on the Drazin inverses with Banachiewicz-Schur forms, Appl. Math. Comput. 213 (2009) 230-234.
[15] H. Yang, X. Liu, The Drazin inverse of the sum of two matrices and its applications, J. Comput. Appl. Math. 235 (2011) 1412-1417.
[16] D. S. Cvetković-Ilić, Expression of the Drazin and MP-inverse of partitioned matrix and quotient identity of generalized Schur complement, Appl. Math. Comput. 213 (2009) 18-24.
[17] J. Miao, Results of the Drazin inverse of block matrices, J. Shanghai Normal Univ. 18 (1989) 25-31. (in Chinese)
[18] C. Deng, Y. Wei, Representations for the Drazin inverse of 2×2 block-operator matrix with singular Schur complement, Linear Algebra Appl. 435 (2011) 2766-2783.
[19] X. Li, A representation for the Drazin inverse of block matrices with a singular generalized Schur complement, Appl. Math. Comput. 217 (2011) 7531-7536.
[20] C. Deng, D. S. Cvetković-Ilić, Y. Wei, Some results on the generalized Drazin inverse of operator matrices, Linear and Multilinear Algebra 58 (2010) 503-521.
[21] C. Deng, A comment on some recent results concerning the Drazin inverse of an anti-triangular block matrix, Filomat 26:2 (2012), 341–351.