Filomat 27:2 (2013), 333–339
DOI 10.2298/FIL1302333S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Paracompactness with respect to an ideal

N. Sathiyasundari[a], V. Renukadevi[b]

[a]*Department of Mathematics, ANJA College, Sivaksi 626124, Tamil Nadu, India.*
[b]*Department of Mathematics, ANJA College, Sivaksi 626124, Tamil Nadu, India.*

Abstract. In this paper, we study $\mathcal{I}$-paracompact spaces and discuss their properties. Also, we characterize $\mathcal{I}$-paracompact spaces. Some of the results in paracompact spaces have been generalized in terms of $\mathcal{I}-$paracompact spaces.

1. Introduction

The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [14]. An *ideal* $\mathcal{I}$ on a set X is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be a $\sigma-$ *ideal* [9] if it is countably additive. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $()^\star : \wp(X) \to \wp(X)$, called a *local function* [9] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star()$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called $\star-$*topology*, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ [9]. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. A subset A of a topological space (X, τ) is said to be a *generalized* $F_\sigma-$*subset* [13] if for each open subset U of X containing A, there exists an $F_\sigma-$subset B of X which is contained in U and contains A. A space X is said to be *totally normal* [12] if it is normal and every open subset G of X is expressible as a union of a locally finite (in G) family of open $F_\sigma-$subset of X. A space X is said to be *perfectly normal* [6] if it is normal and in which each open set is an $F_\sigma-$set. A subset A of a space (X, τ) is said to be *g-closed* [11] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U \in \tau$. By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the *closure* and *interior* of A in (X, τ).

Lemma 1.1. [1] *The union of a finite family of locally finite collection of sets in a space* (X, τ) *is again locally finite.*

Lemma 1.2. [1] *If* $\mathcal{V}$ *is a locally finite family of sets in a space* (X, τ)*, then* $\lambda = \{cl(Q) \mid Q \in \mathcal{V}\}$ *is locally finite in* X.

Lemma 1.3. [3] *If* $\{A_\alpha \mid \alpha \in \Delta\}$ *is a locally finite family of subsets in a space* (X, τ)*, and if* $B_\alpha \subset A_\alpha$ *for each* $\alpha \in \Delta$*, then the family* $\{B_\alpha \mid \alpha \in \Delta\}$ *is locally finite in* X.

2010 *Mathematics Subject Classification*. Primary 54D15; Secondary 54D20, 03E75
Keywords. Ideal, paracompact modulo $\mathcal{I}$ space, totally normal, perfectly normal, generalized $F_\sigma-$set
Received: 16 August 2012; Accepted: 31 December 2012
Communicated by Ljubiša D. R. Kočinac
Email addresses: sathyamat03@yahoo.co.in (N. Sathiyasundari), renu_siva2003@yahoo.com (V. Renukadevi)

2. $\mathcal{I}$−paracompact subsets

The concept of paracompactness with respect to an ideal was introduced by Zahid [15] and is further studied by T.R. Hamlett, D. Rose and D. Janković [8]. An ideal space $(X, \tau, \mathcal{I})$ is said to be *paracompact modulo* $\mathcal{I}$ or $\mathcal{I}$*−paracompact* [8] if and only if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ (not necessarily a cover) such that $X - \cup\{V \mid V \in \mathcal{V}\} \in \mathcal{I}$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$ *− paracompact relative to* X ($\mathcal{I}$ *− paracompact subset* [8]) if for any open cover $\mathcal{U}$ of A, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}$ of open sets such that $\mathcal{V}$ refines $\mathcal{U}$ and $A \subset \cup\{V \mid V \in \mathcal{V}\} \cup I$. A is said to be $\mathcal{I}$*−paracompact* ($\mathcal{I}$−paracompact *subspace* [8]) if $(A, \tau_A, \mathcal{I}_A)$ is $\mathcal{I}_A$−paracompact as a subspace, where τ_A is the usual subspace topology. Theorem 2.1 below shows that a space (X, τ) is paracompact if and only if it is paracompact modulo $\{\emptyset\}$, the easy proof of which is omitted. A space X is said to be *hereditarily* $\mathcal{I}$*−paracompact* if every subset of X is $\mathcal{I}$−paracompact. In this section, we characterize $\mathcal{I}$− paracompact spaces.

Theorem 2.1. *Let (X, τ) be a space with an ideal $\mathcal{I} = \{\emptyset\}$. Then (X, τ) is paracompact if and only if (X, τ) is paracompact modulo $\mathcal{I}$.*

The following Theorem 2.2 gives a property of subsets of X which are $\mathcal{I}$−paracompact.

Theorem 2.2. *If every open subset of $(X, \tau, \mathcal{I})$ is $\mathcal{I}$−paracompact, then every subset of X is $\mathcal{I}$−paracompact.*

Proof. Let B be a subset of X and $\mathcal{U}_B = \{U_\alpha \cap B \mid \alpha \in\Delta\}$ be a τ_B−open cover of B, where each U_α is open in X. Then $\mathcal{U} = \{U_\alpha \mid \alpha \in\Delta\}$ is a τ−open cover of V where $V = \cup U_\alpha$. By hypothesis, there exist $I \in \mathcal{I}$ and τ−locally finite family $\mathcal{V} = \{V_\beta \mid \beta \in \nabla\}$ which refines $\mathcal{U}$ such that $V = \cup\{V_\beta \mid \beta \in \nabla\} \cup I$. Then $V \cap B = \Big(\cup \{V_\beta \mid \beta \in \nabla\} \cup I\Big) \cap B$ which implies that $B = \cup\{V_\beta \cap B \mid \beta \in \nabla\} \cup (I \cap B)$ which implies that $B = \cup\{V_\beta \cap B \mid \beta \in \nabla\} \cup I_B$ where $I_B = I \cap B \in \mathcal{I}_B$. Let $x \in B$. Since $\mathcal{V}$ is τ−locally finite, there exists $U \in \tau(x)$ such that $V_\beta \cap U = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$ and so $(V_\beta \cap U) \cap B = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$. Hence $(V_\beta \cap B) \cap (U \cap B) = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$. Therefore, $\mathcal{V}_B = \{V_\beta \cap B \mid \beta \in \nabla\}$ is τ_B−locally finite. Let $V_\beta \cap B \in \mathcal{V}_B$. Then $V_\beta \in \mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $U_\alpha \in \mathcal{U}$ such that $V_\beta \subset U_\alpha$ which implies that $V_\beta \cap B \subset U_\alpha \cap B$. Therefore, $\mathcal{V}_B$ refines $\mathcal{U}_B$. Hence every subset of X is an $\mathcal{I}$−paracompact subspace. □

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.2, we have the following Corollary 2.3.

Corollary 2.3. [4, 7] *If every open subset of a space (X, τ) is paracompact, then every subset of X is paracompact.*

Hamlett, Rose and Janković [8] established that every closed subset of an $\mathcal{I}$−paracompact space is $\mathcal{I}$−paracompact. The following Theorem 2.4 is a generalization of the above result. If $\mathcal{I} = \{\emptyset\}$ in the Theorem 2.4, we have Corollary 2.6.

Theorem 2.4. *Every F_σ−set (countable union of closed sets) of an $\mathcal{I}$−paracompact space $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$−paracompact subspace of X.*

Proof. Let A be an F_σ−subset of X. Then $A = \cup\{A_i \mid i \in \mathbf{N}\}$ where each A_i is closed. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in\Delta\}$ be a τ_A−open cover of A where $U_\alpha = V_\alpha \cap A$ such that V_α is open in X. Then $\mathcal{U}_1 = \{V_\alpha \mid \alpha \in\Delta\} \cup \{X - A_i \mid i \in \mathbf{N}\}$ is an open cover of X. By hypothesis, there exist $I \in \mathcal{I}$ and open locally finite family $\mathcal{V}_1 = \{V_\beta \mid \beta \in\Delta_0\}$ which refines $\mathcal{U}_1$ such that $X = \cup\{V_\beta \mid \beta \in\Delta_0\} \cup I$. Let $\mathcal{B} = \{V_\beta \mid V_\beta \in \mathcal{V}_1$ and $V_\beta \cap A_i \neq \emptyset$ for every $i\}$. Then $\mathcal{B}$ is locally finite. Let $V_\beta \in \mathcal{B}$. Then $V_\beta \in \mathcal{V}_1$ and since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there exists some U in $\mathcal{U}_1$ such that $V_\beta \subset U$. This U must be some V_α. Suppose, if $U = X - A_i$ for some i, then $V_\beta \subset X - A_i$ for some i which implies that $V_\beta \cap A_i = \emptyset$. Then $V_\beta \notin \mathcal{B}$, which is a contradiction. Therefore, U must be some V_α. Since $X = \cup\{V_\beta \mid \beta \in\Delta_0\} \cup I$, $A = (\cup\{V_\beta \mid \beta \in\Delta_0\} \cup I) \cap A = \cup\{(V_\beta \cap A) \mid \beta \in\Delta_0\} \cup (I \cap A)$ which implies that $A \subset \cup\{(V_\beta \cap A) \mid \beta \in\Delta_0\} \cup I$. Let $\mathcal{B}_A = \{V_\beta \cap A \mid V_\beta \in \mathcal{B}$ and $\beta \in\Delta_0\}$. Let $x \in A$. Since $\mathcal{B}$ is locally finite, there exists $W \in \tau(x)$ such that $V_\beta \cap W = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ...\beta_n$. Now $(V_\beta \cap W) \cap A = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ...\beta_n$ implies that $(V_\beta \cap A) \cap (W \cap A) = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ...\beta_n$. Hence $\mathcal{B}_A = \{V_\beta \cap A \mid V_\beta \in \mathcal{B}$ and $\beta \in\Delta_0\}$ is τ_A−locally finite. Let $V_\beta \cap A \in \mathcal{B}_A$ where $V_\beta \in \mathcal{B}$. Since every element of $\mathcal{B}$ is contained in some V_α, $V_\beta \subset V_\alpha$ for some α which implies that $V_\beta \cap A \subset V_\alpha \cap A$ and so $V_\beta \cap A \subset U_\alpha$. Therefore, $\mathcal{B}_A$ refines $\mathcal{U}$. Hence A is an $\mathcal{I}$−paracompact subspace. □

Corollary 2.5. [8] *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}-$paracompact space. If $A \subseteq X$ is closed, then A is $\mathcal{I}-$paracompact.*

Corollary 2.6. [7, P.218, Theorem 8] *Every $F_\sigma-$set of a paracompact space (X, τ) is paracompact.*

Theorem 2.7. *Let $(X, \tau, \mathcal{I})$ be a space and let A be a subset of X such that for each open set $U \supset A$, there is an $\mathcal{I}-$paracompact set B with $A \subset B \subset U$. Then A is $\mathcal{I}-$paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a τ_A-open cover of A where $U_\alpha = A \cap V_\alpha$ such that V_α is open in X. By the given condition, there exists an $\mathcal{I}-$paracompact subset B of X such that $A \subset B \subset \cup V_\alpha$. Then $\mathcal{U}_B = \{V_\alpha \cap B \mid \alpha \in \Delta\}$ is a τ_B-open cover of B. By hypothesis, there exist $I \cap B = I_B \in \mathcal{I}_B$ and τ_B-locally finite family $\mathcal{V}_B = \{V_\beta \cap B \mid \beta \in \nabla\}$ which refines $\mathcal{U}_B$ such that $B \subset \cup\{V_\beta \cap B \mid \beta \in \nabla\} \cup (I \cap B)$. Then $A = B \cap A \subset (\cup\{V_\beta \cap B \mid \beta \in \nabla\} \cup (I \cap B)) \cap A = \cup\{V_\beta \cap B \cap A \mid \beta \in \nabla\} \cup (I \cap A)$ which implies that $A \subset \cup\{V_\beta \cap A \mid \beta \in \nabla\} \cup I_A$. Let $x \in A$. Since $\mathcal{V}_B = \{V_\beta \cap B \mid \beta \in \nabla\}$ is τ_B-locally finite, there exists $W \in \tau(x)$ such that $(V_\beta \cap B) \cap W = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$ which implies that $((V_\beta \cap B) \cap (W \cap B)) \cap A = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$. Hence $(V_\beta \cap B \cap A) \cap (W \cap B \cap A) = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$ and so $(V_\beta \cap A) \cap (W \cap A) = \emptyset$ for all $\beta \neq \beta_1, \beta_2, ..., \beta_n$. Therefore, $\mathcal{V} = \{V_\beta \cap A \mid \beta \in \nabla\}$ is τ_A-locally finite. Let $V_\beta \cap A \in \mathcal{V}$. Then $V_\beta \cap B \in \mathcal{V}_B$. Since $\mathcal{V}_B$ refines $\mathcal{U}_B$, there is some $V_\alpha \cap B \in \mathcal{U}_B$ such that $V_\beta \cap B \subset V_\alpha \cap B$. Also, $A \subset B$ implies that $V_\beta \cap A \subset V_\beta \cap B$. Thus, $V_\beta \cap A \subset V_\alpha \cap A = U_\alpha$ so that $\mathcal{V}$ refines $\mathcal{U}$. Hence A is $\mathcal{I}-$paracompact. □

Corollary 2.8. *Every generalized $F_\sigma-$subset of an $\mathcal{I}-$paracompact space $(X, \tau, \mathcal{I})$ is $\mathcal{I}-$paracompact.*

Proof. Let X be an $\mathcal{I}-$paracompact space. Let A be a generalized $F_\sigma-$subset of X. Then for every open subset U of X containing A, there exists an $F_\sigma-$ subset B of X which is contained in U and contains A. By Theorem 2.4, B is $\mathcal{I}-$paracompact. Therefore, by Theorem 2.7, A is $\mathcal{I}-$paracompact. □

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.7, we have the following Corollary 2.9.

Corollary 2.9. [6] *Let (X, τ) be a space and let A be a subset of X such that for each open set $U \supset A$, there is a paracompact set B with $A \subset B \subset U$. Then A is paracompact.*

If $\mathcal{I} = \{\emptyset\}$ in the above Corollary 2.8, we have Corollary 2.10.

Corollary 2.10. *Every generalized $F_\sigma-$subset of a paracompact space (X, τ) is paracompact.*

Theorem 2.11. *Every subset of a perfectly normal $\mathcal{I}-$paracompact space $(X, \tau, \mathcal{I})$ is $\mathcal{I}-$paracompact.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is a perfectly normal $\mathcal{I}-$ paracompact space. Since X is perfectly normal, every open set is an F_σ set and so every open set is $\mathcal{I}-$paracompact, by Theorem 2.4. Therefore, by Theorem 2.2, every subset of X is $\mathcal{I}-$paracompact. □

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.11, we have Corollary 2.12.

Corollary 2.12. [5, 7] *Every subset of a perfectly normal, paracompact space (X, τ) is paracompact.*

Corollary 2.13. *Every perfectly normal $\mathcal{I}-$paracompact space $(X, \tau, \mathcal{I})$ is hereditarily $\mathcal{I}-$paracompact.*

If $\mathcal{I} = \{\emptyset\}$ in the above Corollary 2.13, we have Corollary 2.14.

Corollary 2.14. *Every perfectly normal paracompact space (X, τ) is hereditarily paracompact.*

Theorem 2.15. *Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a locally finite open covering of a space $(X, \tau, \mathcal{I})$ such that each $cl(V_\alpha)$ is $\mathcal{I}-$paracompact relative to X. Then X is $\mathcal{I}-$paracompact.*

Proof. Let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Delta_0\}$ be an open cover of X. Then for each α, $\mathcal{U}$ is a cover of $cl(V_\alpha)$ by $\tau-$open sets. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_1 = \{V_\beta \mid \beta \in \Delta_1\}$ of open sets which refines $\mathcal{U}$ such that $cl(V_\alpha) \subset \cup\{V_\beta \mid \beta \in \Delta_1\} \cup I$. Now $V_\alpha = cl(V_\alpha) \cap V_\alpha \subset \big(\cup\{V_\beta \mid \beta \in \Delta_1\} \cup I\big) \cap V_\alpha = \cup\{V_\beta \cap V_\alpha \mid \beta \in \Delta_1\} \cup (I \cap V_\alpha)$ which implies that $V_\alpha \subset \cup\{V_\beta \cap V_\alpha \mid \beta \in \Delta_1\} \cup I$. Since $\{V_\alpha \mid \alpha \in \Delta\}$ is an open covering of X, $X = \cup\{V_\beta \cap V_\alpha \mid \alpha \in \Delta,\ \beta \in \Delta_1\} \cup I$. Since $\{V_\alpha \mid \alpha \in \Delta\}$ and $\mathcal{V}_1 = \{V_\beta \mid \beta \in \Delta_1\}$ are locally finite, $\mathcal{V} = \{V_\beta \cap V_\alpha \mid \alpha \in \Delta,\ \beta \in \Delta_1\}$ is locally finite. If $V_\beta \cap V_\alpha \in \mathcal{V}$, then $V_\beta \in \mathcal{V}_1$ and since $\mathcal{V}_1$ refines $\mathcal{U}$, there is some $U_\gamma \in \mathcal{U}$ such that $V_\beta \subset U_\gamma$. Also, $V_\beta \cap V_\alpha \subset V_\beta \subset U_\gamma$ implies that $V_\beta \cap V_\alpha \subset U_\gamma$. Therefore, $\mathcal{V}$ refines $\mathcal{U}$. Hence X is $\mathcal{I}-$paracompact. □

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.15, we have the following Corollary 2.16.

Corollary 2.16. *Let $\{V_\alpha \mid \alpha \in \Delta\}$ be a locally finite open covering of a space (X, τ) such that each $cl(V_\alpha)$ is paracompact relative to X. Then X is paracompact.*

Theorem 2.17. *Every subset of a totally normal $\mathcal{I}-$paracompact space $(X, \tau, \mathcal{I})$ is $\mathcal{I}-$paracompact.*

Proof. Let X be a totally normal $\mathcal{I}-$paracompact space. Let G be an open subset of X. Since X is totally normal, $G = \cup G_i$ where $G_i' s$ are open F_σ-subset of X and locally finite in G. Therefore, $\{G_i\}$ is a locally finite open covering of G. Also, for each i, $cl(G_i)$ is a closed subsets of X and so by Theorem IV.3[8], $cl(G_i)$ is $\mathcal{I}-$paracompact relative to X for each i. Then $cl(G_i)$ is $\mathcal{I}-$paracompact relative to G for each i. Therefore, G is $\mathcal{I}-$paracompact, by Theorem 2.15. Since G is an open subset of X, by Theorem 2.2, every subset of X is $\mathcal{I}-$paracompact. □

Corollary 2.18. *Every totally normal $\mathcal{I}-$paracompact space is hereditarily $\mathcal{I}-$paracompact.*

If $\mathcal{I} = \{\emptyset\}$ in the above Theorem 2.17, we have the following Corollary 2.19.

Corollary 2.19. *Every subset of a totally normal paracompact space is paracompact.*

A collection $\mathcal{V}$ of subsets of X is said to be an $\mathcal{I}$ − *cover* [15] of X if $X - \cup\{V_\alpha \mid V_\alpha \in \mathcal{V}\} \in \mathcal{I}$. A collection $\mathcal{A}$ of subsets of a space (X, τ) is said to be σ − *locally finite* [8] if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where each collection $\mathcal{A}_n$ is a locally finite. The following Theorem 2.20 gives a property of $\mathcal{I}-$paracompact spaces.

Theorem 2.20. *Let $(X, \tau, \mathcal{I})$ be a regular ideal space. If X is $\mathcal{I}-$paracompact, then every open cover of X has a closed locally finite $\mathcal{I}-$cover refinement.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, let $U_x \in \mathcal{U}$ such that $x \in U_x$. Since (X, τ) is regular, for each $x \in X$, there exists a neighborhood V_x of x such that $cl(V_x) \subset U_x$. Now $\mathcal{U}_1 = \{V_x \mid x \in X\}$ is an open cover of X and so there exist an $I \in \mathcal{I}$ and a locally finite family $\mathcal{W}_1 = \{W_\beta \mid \beta \in \Delta\}$ of open sets which refines $\mathcal{U}_1$ such that $X = \cup\{W_\beta \mid \beta \in \Delta\} \cup I$ which implies that $X = \cup\{cl(W_\beta) \mid \beta \in \Delta\} \cup I$. Since the family $\mathcal{W}_1 = \{W_\beta \mid \beta \in \Delta\}$ is locally finite, the family $\mathcal{W} = \{cl(W_\beta) \mid W_\beta \in \mathcal{W}_1\}$ is locally finite, by Lemma 1.2. Let $cl(W_\beta) \in \mathcal{W}$. Then $W_\beta \in \mathcal{W}_1$. Since $\mathcal{W}_1$ refines $\mathcal{U}_1$, there is some $V_x \in \mathcal{U}_1$ such that $W_\beta \subset V_x$ and so $cl(W_\beta) \subset cl(V_x)$. Also, $cl(V_x) \subset U_x$ implies that $cl(W_\beta) \subset U_x$. Hence $\mathcal{W}$ refines $\mathcal{U}$. Thus, $\mathcal{W} = \{cl(W_\beta) \mid \beta \in \Delta\}$ is a closed locally finite family which refines $\mathcal{U}$ which completes the proof. □

Corollary 2.21. [7, P.210, Lemma 2] *If every covering of a regular space X has a locally finite refinement, then every open covering of that space also has closed locally finite refinement.*

Theorem 2.22. *Let $(X, \tau, \mathcal{I})$ be a regular ideal space. Then X is $\mathcal{I}-$paracompact if and only if every open cover of X has an open $\sigma-$locally finite $\mathcal{I}-$cover refinement.*

Proof. Since every locally finite refinement is $\sigma-$locally finite refinement, it is enough to prove the sufficiency. Let $\mathcal{U}$ be an open cover of X. Then there exists $I \in \mathcal{I}$ and open $\sigma-$locally finite refinement $\mathcal{V}$ of $\mathcal{U}$ such that $X \subset \cup\{V \mid V \in \mathcal{V}\} \cup I$. Also, $\mathcal{V} = \bigcup_{n \in N} \mathcal{V}_n$ where each $\mathcal{V}_n$ is locally finite. For each $n \in N$, let

$W_n = \cup\{V \mid V \in \mathcal{V}_n\}$. Then $X \subset \cup\{W_n \mid n \in N\} \cup I$. For each $n \in N$, let $W_n' = W_n - \overset{n-1}{\underset{i=1}{\cup}} W_i$. Then W_n' refines W_n. Let $x \in X$ and n be the smallest member of $\{n \in N \mid x \in W_n\}$. Then $x \in W_n'$. Hence $X \subset \cup\{W_n' \mid n \in N\} \cup I$. Also, W_{nx} is a neighborhood of x that intersect only finite number of members of W_n' so that $\{W_n' \mid n \in N\}$ is locally finite. Let $\mathcal{W} = \{W_n' \cap V \mid n \in N \text{ and } V \in \mathcal{V}_n\}$. Let $x \in X$. Since $\{W_n' \mid n \in N\}$ is locally finite, there exists a neighborhood P containing x that intersects only a finite number of members of $\{W_n' \mid n \in N\}$. Also, for each $i = 1, 2, ...k$, there exists a neighborhood $O_{x(i)}$ containing x that intersects only a finite number of members of $\mathcal{V}_{n,i}$. Then $P \cap O_{x(i)}$ is a neighborhood of x that intersects only a finite number of members of $\mathcal{W}$. Hence $\mathcal{W}$ is locally finite. Let $W_n' \cap V \in \mathcal{W}$. Then $V \in \mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $U \in \mathcal{U}$ such that $V \subset U$. Then $W_n' \cap V \subset V \subset U$. Thus, $\mathcal{W}$ refines $\mathcal{U}$. Since $X \subset \cup\{V \mid V \in \mathcal{V}\} \cup I$ and $X \subset \cup\{W_n' \mid n \in N\} \cup I$, $X \subset \cup\{(W_n' \cap V) \mid n \in N \text{ and } V \in \mathcal{V}\} \cup I$. Therefore, $(X, \tau, \mathcal{I})$ is $\mathcal{I}-$paracompact. □

Corollary 2.23. [7, P.210, Theorem 4] *Let (X, τ) be a regular space. Then (X, τ) is paracompact if and only if every open cover of X has an open $\sigma-$locally finite refinement.*

3. Relative $\mathcal{I}-$paracompact subsets

In this section, we discuss some of the properties of subsets of $\mathcal{I}-$ paracompact spaces.

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If B is an open subset of X, $A \subset B$ and A is $\mathcal{I}-$ paracompact relative to X, then A is $\mathcal{I}-$paracompact subset of B.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a cover of A by sets open in B. Then $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ is a open cover of A, since B is open in X. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V} = \{V_\beta \mid \beta \in \Delta_0\}$ by sets open in X which refines $\mathcal{U}$ such that $A \subset \cup\{V_\beta \mid \beta \in \Delta_0\} \cup I$ which implies $A \subset \cup\{V_\beta \cap B \mid \beta \in \Delta_0\} \cup I$. Let $x \in B$. Since $\mathcal{V} = \{V_\beta \mid \beta \in \Delta_0\}$ is locally finite in X, there exists $W \in \tau(x)$ such that $W \cap V_\beta = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$ which implies $(W \cap V_\beta) \cap B = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$ which implies $(W \cap B) \cap (V_\beta \cap B) = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$. Therefore, the family $\mathcal{V}_1 = \{V_\beta \cap B \mid \beta \in \Delta_0\}$ is $B-$locally finite. Let $V_\beta \cap B \in \mathcal{V}_1$. Then $V_\beta \in \mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $U_\alpha \in \mathcal{U}$ such that $V_\beta \subset U_\alpha$ which implies $V_\beta \cap B \subset U_\alpha \cap B \subset U_\alpha$. Hence $\mathcal{V}_1$ refines $\mathcal{U}$. Therefore, A is $\mathcal{I}-$paracompact relative to B. □

Theorem 3.2. *Let S be a closed subspace of an ideal space $(X, \tau, \mathcal{I})$. If $F \subseteq S$ is $\mathcal{I}-$paracompact relative to S and if there exists an open set G in X such that $F \subset G \subset S$, then F is $\mathcal{I}-$paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be an open cover of F by sets open in X. Then $\mathcal{U}_1 = \{U_\alpha \cap G \mid \alpha \in \Delta\}$ is an open cover of F by sets open in G so that $\mathcal{U}_1 = \{U_\alpha \cap G \mid \alpha \in \Delta\}$ is an open cover of F by sets open in S. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_1 = \{W_\beta \mid \beta \in \Delta_0\}$ in S where each $W_\beta = V_\beta \cap S$ is open in S which refines $\mathcal{U}_1$ such that $F \subset \cup\{V_\beta \cap S \mid \beta \in \Delta_0\} \cup I$. Then $F \cap G \subset \Big(\cup\{V_\beta \cap S \mid \beta \in \Delta_0\} \cup I\Big) \cap G$ which implies that $F \subset \cup\{V_\beta \cap G \mid \beta \in \Delta_0\} \cup (I \cap G)$ implies that $F \subset \cup\{V_\beta \cap G \mid \beta \in \Delta_0\} \cup I$. Let $x \in X$. If $x \in S$, there exists $W \in \tau(x)$ such that $(V_\beta \cap S) \cap W = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$ which implies $((V_\beta \cap S) \cap W) \cap G = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$ which implies $(V_\beta \cap G) \cap W = \emptyset$ for $\beta \neq \beta_1, \beta_2, ...\beta_n$. If $x \in X - S$, then $X - S$ is an open set containing x such that $(V_\beta \cap G) \cap (X - S) = \emptyset$. Thus, the family $\mathcal{V} = \{V_\beta \cap G \mid \beta \in \Delta_0\}$ is locally finite in X. Let $V_\beta \cap G \in \mathcal{V}$. Then $V_\beta \cap S \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there is some $U_\alpha \cap G \in \mathcal{U}_1$ such that $V_\beta \cap S \subset U_\alpha \cap G$ which implies $V_\beta \cap G \subset U_\alpha$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, F is $\mathcal{I}-$paracompact relative to X. □

Theorem 3.3. *If A is $\mathcal{I}-$ paracompact relative to X and B is a closed subset of X, then $A \cap B$ is $\mathcal{I}-$ paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Delta_0\}$ be an open cover of $A \cap B$. Then $\mathcal{U}_A = \{U_\gamma \mid \gamma \in \Delta_0\} \cup (X - B)$ is an open cover of A. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_A = \{V_\alpha \cup (X - B) \mid \alpha \in \Delta\}$ which refines $\mathcal{U}_A$ such that $A \subset \cup\{V_\alpha \cup (X - B) \mid \alpha \in \Delta\} \cup I$. Then $A \cap B \subset \cup\{(V_\alpha \cup (X - B)) \cap B \mid \alpha \in \Delta\} \cup (I \cap B)$ which implies that $A \cap B \subset \cup\{V_\alpha \cap B \mid \alpha \in \Delta\} \cup I$. Let $x \in X$. Since $\mathcal{V}_A = \{V_\alpha \cup (X - B) \mid \alpha \in \Delta\}$ is locally finite, there exists $W \in \tau(x)$ such that $(V_\alpha \cup (X - B)) \cap W = \emptyset$ for $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$ which implies $(V_\alpha \cap W) \cup ((X - B) \cap W) = \emptyset$

for $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$ which implies $((V_\alpha \cap W) \cup ((X - B) \cap W)) \cap B = \emptyset$ for $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$ which implies $((V_\alpha \cap W) \cap B) \cup ((X - B) \cap W \cap B) = \emptyset$ for $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$ which implies $(V_\alpha \cap B) \cap W = \emptyset$ for $\alpha \neq \alpha_1, \alpha_2, ...\alpha_n$. Therefore, the family $\mathcal{V} = \{V_\alpha \cap B \mid \alpha \in \Delta\}$ is locally finite. Let $V_\alpha \cap B \in \mathcal{V}$. Then $V_\alpha \cup (X - B) \in \mathcal{V}_A$. Since $\mathcal{V}_A$ refines $\mathcal{U}_A$, there is some $U_\gamma \cup (X - B) \in \mathcal{U}_A$ such that $V_\alpha \cup (X - B) \subset U_\gamma \cup (X - B)$ which implies $(V_\alpha \cup (X - B)) \cap B \subset (U_\gamma \cup (X - B)) \cap B$ which implies $V_\alpha \cap B \subset U_\gamma \cap B \subset U_\gamma$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, $A \cap B$ is $\mathcal{I}-$paracompact relative to X. □

Corollary 3.4. *If A is $\mathcal{I}-$ paracompact relative to X and $B \subset A$ is a closed subset of X, then B is $\mathcal{I}-$ paracompact relative to X.*

Theorem 3.5. *Let A be $\mathcal{I}-$paracompact relative to X and B an open set contained in A. Then $A - B$ is $\mathcal{I}-$paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a cover of $A - B$ by sets open in X. Then $\mathcal{U}_1 = \{U_\alpha \mid \alpha \in \Delta\} \cup B$ is a cover of A by sets open in X. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_1 = \{V_\beta \mid \beta \in \Delta_0\} \cup B$ which refines $\mathcal{U}_1$ such that $A \subset \cup(\{V_\beta \mid \beta \in \Delta_0\} \cup B) \cup I$. Then $A - B \subset \cup\Big((\{V_\beta \mid \beta \in \Delta_0\} \cup B) \cup I\Big) - B$ which implies that $A - B \subset \cup\{V_\beta - B \mid \beta \in \Delta_0\} \cup I$. Since the family $\mathcal{V}_1 = \{V_\beta \cup B \mid \beta \in \Delta_0\}$ is locally finite, the family $\mathcal{V} = \{V_\beta - B \mid \beta \in \Delta_0\}$ is locally finite, by Lemma 1.3. Let $V_\beta - B \in \mathcal{V}$. Then $V_\beta \cup B \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there is some $U_\alpha \cup B \in \mathcal{U}_1$ such that $V_\beta \cup B \subset U_\alpha \cup B$ which implies $(V_\beta \cup B) - B \subset (U_\alpha \cup B) - B$ and so $V_\beta - B \subset U_\alpha - B \subset U_\alpha$. Therefore, $\mathcal{V}$ refines $\mathcal{U}$. Hence $A - B$ is $\mathcal{I}-$paracompact relative to X. □

Theorem 3.6. *In a space $(X, \tau, \mathcal{I})$, if A and B are $\mathcal{I}-$paracompact relative to X, then $A \cup B$ is $\mathcal{I}-$paracompact relative to X.*

Proof. Let $\mathcal{U} = \{U_\gamma \mid \gamma \in \Delta\}$ be a cover of $A \cup B$ by sets open in X. Then $\mathcal{U} = \{U_\gamma \mid \gamma \in \Delta\}$ is an open cover of A and B. By hypothesis, there exist $I_A, I_B \in \mathcal{I}$ and locally finite families $\mathcal{V}_A = \{V_\alpha \mid \alpha \in \Delta_0\}$ of A and $\mathcal{V}_B = \{V_\beta \mid \beta \in \Delta_1\}$ of B which refines $\mathcal{U}$ such that $A \subset \cup\{V_\alpha \mid \alpha \in \Delta_0\} \cup I_A$ and $B \subset \cup\{V_\beta \mid \beta \in \Delta_1\} \cup I_B$. Then $A \cup B \subset \Big(\cup\{V_\alpha \mid \alpha \in \Delta_0\} \cup I_A\Big) \cup \Big(\cup\{V_\beta \mid \beta \in \Delta_1\} \cup I_B\Big)$ which implies that $A \cup B \subset \cup\{V_\alpha \cup V_\beta \mid \alpha \in \Delta_0, \beta \in \Delta_1\} \cup (I_A \cup I_B)$ which implies $A \cup B \subset \cup\{V_\alpha \cup V_\beta \mid \alpha \in \Delta_0, \beta \in \Delta_1\} \cup I$ where $I = I_A \cup I_B$. Since the family $\mathcal{V}_A$ and $\mathcal{V}_B$ are locally finite, the family $\mathcal{V} = \{V_\alpha \cup V_\beta \mid \alpha \in \Delta_0, \beta \in \Delta_1\}$ is locally finite, by Lemma 1.1, which refines $\mathcal{U}$. Therefore, $A \cup B$ is $\mathcal{I}-$paracompact relative to X. □

Theorem 3.7. *Every $g-$closed subset of an $\mathcal{I}-$paracompact space is $\mathcal{I}-$paracompact relative to X.*

Proof. Let A be a $g-$closed subset of $(X, \tau, \mathcal{I})$. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be an open cover of A. Then $A \subset \cup U_\alpha$. Since A is $g-$closed, $cl(A) \subset \cup U_\alpha$. Then $\mathcal{U}_1 = \{U_\alpha \mid \alpha \in \Delta\} \cup (X - cl(A))$ is an open cover of X. By hypothesis, there exist $I \in \mathcal{I}$ and locally finite family $\mathcal{V}_1 = \{V_\beta \cup V \mid \beta \in \Delta_0\}$ ($V_\beta \subset U\alpha$ and $V \subset X - cl(A)$) which refines $\mathcal{U}_1$ such that $X = \cup\{V_\beta \cup V \mid \beta \in \Delta_0\} \cup I$. Then $cl(A) - \bigcup_\beta V_\beta = cl(A) - (V \cup (\bigcup_\beta V_\beta)) \subset X - (V \cup (\bigcup_\beta V_\beta)) \in \mathcal{I}$. Thus, $cl(A) - \bigcup_\beta V_\beta \in \mathcal{I}$. Since $A - \bigcup_\beta V_\beta \subset cl(A) - \bigcup_\beta V_\beta$, $A - \bigcup_\beta V_\beta \in \mathcal{I}$, by hereditary. Since $\mathcal{V}_1 = \{V_\beta \cup V \mid \beta \in \Delta_0\}$ is locally finite, the family $\mathcal{V} = \{V_\beta \mid \beta \in \Delta_0\}$ is locally finite, by Lemma 1.3. Thus, the family $\mathcal{V}$ is locally finite which refines $\mathcal{U}$. Therefore, A is $\mathcal{I}-$paracompact relative to X. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be a perfectly normal ideal space with a $\sigma-$ideal $\mathcal{I}$ and G be a subset of X such that G is the union of countable number of open subsets G_n of X. Then each $G_n, n \in \mathbf{N}$ is $\mathcal{I}-$paracompact relative to X if and only if G is $\mathcal{I}-$paracompact relative to X.*

Proof. Suppose each $G_n, n \in \mathbf{N}$ is $\mathcal{I}-$paracompact relative to X. Let $\mathcal{U} = \{V_\alpha \mid \alpha \in \Delta\}$ be a cover of G by sets open in X. Then $\mathcal{U}_n = \{V_\alpha \cap G_n \mid \alpha \in \Delta\}$ is an open cover of G_n for each $n \in \mathbf{N}$. By hypothesis, there exist $I_n \in \mathcal{I}$ and locally finite family $\mathcal{V}_n = \{V_{\beta,n} \mid \beta \in \Delta_1\}$ which refines $\mathcal{U}_n$ such that $G_n \subset \cup\{V_{\beta,n} \mid \beta \in \Delta_1\} \cup I_n$. Then $\bigcup_n G_n \subset \bigcup_n(\cup\{V_{\beta,n} \mid \beta \in \Delta_1\} \cup I_n)$ which implies that $G \subset \bigcup_n\{W_n \mid n \in \mathbf{N}\} \cup I$ where $W_n = \cup\{V_{\beta,n} \mid \beta \in \Delta_1\}$ and $I = \cup\{I_n \mid n \in \mathbf{N}\}$. Let $x \in X$. Since $\mathcal{V}_n = \{V_{\beta,n} \mid \beta \in \Delta_1\}$ is locally finite, there exists a neighborhood U containing x such that $U \cap V_{\beta,n} \neq \emptyset$ for every $\beta \in \Delta_0$ where Δ_0 is a finite subset of Δ_1. Suppose $\mathcal{V} = \{W_n \mid n \in \mathbf{N}\}$

is not locally finite. Then there exists an element $x \in X$ such that for all neighborhood U of x, we have $U \cap W_i = \emptyset$ for all $i = 1, 2, ..., k$ which implies that $U \cap (\cup V_{\beta,i}) = \emptyset$ for all $i = 1, 2, ..., k$ which in turn implies that $\cup(U \cap V_{\beta,i}) = \emptyset$ for all $i = 1, 2, ..., k$ and so $U \cap V_{\beta,i} = \emptyset$ for all $i = 1, 2, ..., k$, which is a contradiction to the fact that $\mathcal{V}_n$ is locally finite. Therefore, $\mathcal{V} = \{W_n \mid n \in \mathbf{N}\}$ is locally finite. Let $W_n \in \mathcal{V}$ where $W_n = \underset{\beta}{\cup} V_{\beta,n}$. Then $V_{\beta,n} \in \mathcal{V}_n$. Since $\mathcal{V}_n$ refines $\mathcal{U}_n$, there is some $V_\alpha \cap G_n \in \mathcal{U}_n$ such that $V_{\beta,n} \subset V_\alpha \cap G_n$ which implies $V_{\beta,n} \subset V_\alpha$. Thus, $\underset{\beta}{\cup} V_{\beta,n} \subset V_\alpha$ and so $W_n \subset V_\alpha$ for some $V_\alpha \in \mathcal{U}$. Therefore, $\mathcal{V}$ refines $\mathcal{U}$. Hence G is $\mathcal{I}-$paracompact.

Conversely, suppose G is $\mathcal{I}-$paracompact. Since the subset of a perfectly normal space is perfectly normal, G is perfectly normal. Then each G_n is an F_σ-set. Therefore, by Theorem 2.4, each G_n is $\mathcal{I}-$paracompact. □

References

[1] A.V. Arkhangel'skii, V.I. Ponomarëv, Fundamentals of General Topology–Problems and Exercises, Hindustan Pub. Corporation, Delhi, 1966

[2] C.E. Aull, Paracompact and countably paracompact susets, In: General Topology and its Relations to Modern Analysis and Algebra, Proc. Kanpur Topol. Conf. 1966, Academic Press, New York, 1971, 49–53.

[3] N. Bourbaki, General Topology, Addison-Wesley, Mass., 1966.

[4] J. Dieudonn*é*, Une generalization des escapes compacts, J. Math. Pures Appl. 23 (1944) 65–76.

[5] C. H. Dowker, An imbedding theorem for paracompact metric spaces, Duke Math. J. 14 (1947) 639–645.

[6] J. Dugundji, Topology, Allyn and Bacon, Boston, 1996.

[7] R. Engelking, General Topology, Helderman Verlag, 1989.

[8] T.R. Hamlett, D. Rose, D. Janković, Paracompactness with respect to an ideal, Internat. J. Math. Math. Sci. 20 (1997) 433–442.

[9] D. Janković, T.R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97 (1990) 295–310.

[10] K. Kuratowski, Topolgie I, Warszawa, 1933.

[11] N. Levine, Generalised closed sets in topology, Rend, Circ. Mat. Palermo 19 (1970) 89–96.

[12] M.K. Singal, S.P. Arya, On the closure preserving sum theorem, Proc. Amer. Math. Soc. 53 (1975) 518–522.

[13] M.K. Singal, P. Jain, On subparacompact and countably subparacompact spaces, Bull. Austral. Math. Soc. 5 (1971) 289–304.

[14] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, New York, 1946.

[15] M.I. Zahid, Para *H*-closed spaces, locally para *H*-closed spaces and their minimal topologies, Ph.D dissertation, University of Pittsburgh, 1981.