Filomat 27:2 (2013), 341–351
DOI 10.2298/FIL1302341K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

k-tuple total domination in inflated graphs

Adel P. Kazemi

Department of Mathematics
University of Mohaghegh Ardabili
P.O.Box 5619911367, Ardabil, Iran

Abstract. The inflation or inflated graph G_I of a graph G with n vertices is obtained from G by replacing every vertex x_i of degree $d(x_i)$ of G by a clique X_i, which is isomorphic to the complete graph $K_{d(x_i)}$, and each edge (x_i, x_j) of G is replaced by an edge (u, v) in such a way that $u \in X_i$, $v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I. For integer $k \geq 1$, the k-tuple total domination number $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a k-tuple total dominating set of G, which is a set of vertices in G such that every vertex of G is adjacent to at least k vertices in it. For existing this number, must the minimum degree of G be at least k. Henning and Kazemi in [Total domination in inflated graphs, Discrete Applied Mathematics 160 (2012) 164-169] have studied the k-tuple total domination number of inflated graphs, when $k = 1$. Here, we continue their studying when $k \geq 2$. First we prove $nk \leq \gamma_{\times k,t}(G_I) \leq n(k+1) - 1$ when $\delta(G) \geq k+1$, and then we characterize graphs G that the k-tuple total domination number number of G_I is either nk or $nk+1$. Also we find some bounds for this number in the inflated graph G_I, when G has a cut-edge e or a cut-vertex v, in terms of the k-tuple total domination number of the inflation of the components of $G - e$ or of the v-components of $G - v$, respectively. Finally, we calculate this number for the inflation of some graphs.

1. Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [2]. Let $G = (V, E)$ be a graph with the *vertex set* V of *order* $n(G)$ and the *edge set* E of *size* $m(G)$. The *open neighborhood* and the *closed neighborhood* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The *degree* of a vertex v is also $deg_G(v) =\mid N_G(v) \mid$. The *minimum* and the *maximum degree* of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We say that a graph is *connected* if there is a path between every two vertices of the graph, and otherwise is called *disconnected*. In a connected graph G, a vertex (resp. edge) v is called a *cut-vertex* or (resp. *cut-edge*) if $G - v$ is disconnected. Every maximal connected subgraph of $G - v$ is called a (*connectedness*) *component* of it. Let v be a cut-vertex of a graph G and let S be the vertex set of a component of $G - v$. The induced subgraph by $S \cup \{v\}$ of G is called a *v-component* of G. We also write K_n, C_n and P_n for the *complete graph*, the *cycle* and the *path* of order n, respectively, while $G[S]$ and $K_{n_1,n_2,\ldots,n_p}$ denote the *subgraph induced* of G by a vertex set S of G and the *complete p-partite graph*, respectively.

An edge subset M in G is called a *matching* in G if any two edges of M has no vertex in common. If $e = vw \in M$, then we say either M *saturates* two vertices v and w or v and w are *M-saturated* (by e). A

2010 *Mathematics Subject Classification*. 05C69
Keywords. k-tuple total domination number, inflated graph, k-Hamiltonian-like decomposable graph
Received: 10 September 2012; Accepted: 10 October 2012
Communicated by Dragan Stevanovic
Email address: adelpkazemi@yahoo.com,adelpkazemi@uma.ac.ir (Adel P. Kazemi)

matching M is a *perfect matching* or a *maximum matching* if all vertices of G are M-saturated or there is no other matching M' with $|M'| > |M|$, respectively.

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3]. A set $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S, while the minimum cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G.

In [4] Henning and Kazemi generalized the definition of the total domination number to the k-tuple total domination number as follows: for each integer $k \geq 1$, a subset S of V is a *k-tuple total dominating set* of G, abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \geq k$, that is, S is a kTDS of G if every vertex has at least k neighbors in S. The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ of G is the minimum cardinality of a kTDS of G. We remark that $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a k-tuple total dominating set, its minimum degree must be at least k. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a *$\gamma_{\times k,t}(G)$-set*. When $k = 2$, a k-tuple total dominating set is called a *double total dominating set*, abbreviated DTDS, and the k-tuple total domination number is called the *double total domination number*. The redundancy involved in k-tuple total domination makes it useful in many applications. For more information see [5–7].

For the notation for inflated graphs, we follow that of [9]. The *inflation* or *inflated graph* G_I of the graph G without isolated vertices is obtained as follows: each vertex x_i of degree $d(x_i)$ of G is replaced by a clique $X_i \cong K_{d(x_i)}$ (that is, X_i is isomorphic to the complete graph $K_{d(x_i)}$) and each edge (x_i, x_j) of G is replaced by an edge (u, v) in such a way that $u \in X_i$, $v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I. An obvious consequence of the definition is that $n(G_I) = \sum_{x_i \in V(G)} d_G(x_i) = 2m(G)$, $\delta(G_I) = \delta(G)$ and $\Delta(G_I) = \Delta(G)$. There are two different kinds of edges in G_I. The edges of the clique X_i are colored red and the X_i's are called the *red cliques* (a red clique X_i is reduced to a point if x_i is a pendant vertex of G). The other ones, which correspond to the edges of G, are colored *blue* and they form a perfect matching of G_I. Every vertex of G_I belongs to exactly one red clique and one blue edge. Two adjacent vertices of G_I are said to *red-adjacent* if they belong to a same red clique, *blue-adjacent* otherwise. In general, we adopt the following notation: if x_i and x_j are two adjacent vertices of G, the end vertices of the blue edge of G_I replacing the edge (x_i, x_j) of G are called $x_i x_j$ in X_i and $x_j x_i$ in X_j, and this blue edge is $(x_i x_j, x_j x_i)$. Clearly an inflation is claw-free. More precisely, G_I is the line-graph $L(S(G))$ where the subdivision $S(G)$ of G is obtained by replacing each edge of G by a path of length 2. The study of various domination parameters in inflated graphs was originated by Dunbar and Haynes in [8]. Results related to the domination parameters in inflated graphs can be found in [9–11].

Henning and Kazemi in [6] have studied the k-tuple total domination number of inflated graphs, when $k = 1$. Here, we continue their studying when $k \geq 2$.

This paper is organized as follows. In Section 2, we prove that if $k \geq 2$ is an integer and G is a graph of order n with $\delta \geq k + 1$, then $nk \leq \gamma_{\times k,t}(G_I) \leq n(k + 1) - 1$, and then we characterize graphs G that $\gamma_{\times k,t}(G_I)$ is either nk or $nk + 1$. In Section 3, we find some upper and lower bounds for the k-tuple total domination number of the inflation of a graph G with a cut-edge e, in terms of the k-tuple total domination number of the inflation of the components of $G - e$. In a similar manner, we find some upper and lower bounds for the k-tuple total domination number of the inflation of a graph G with a cut-vertex v, in terms of the k-tuple total domination number of the inflation of the v-components of $G - v$. Finding the k-tuple total domination number of the inflation of the complete graphs is our next work. Finally, in Section 4, we calculate the k-tuple total domination number of the inflation of the generalized Petersen graphs, the Harary graphs and the complete bipartite graphs. Also we get an upper bound for this number in the inflation of the complete multipartite graphs.

2. Some bounds

First we give two general upper and lower bounds for the k-tuple total domination number of inflated graphs, where $\delta \geq k \geq 2$.

Theorem 2.1. *Let $k \geq 2$ be an integer, and let G be a graph of order n.*
i. If $\delta \geq k$, then $\gamma_{\times k,t}(G_I) \geq nk$,
ii. if $\delta \geq k+1$, then $\gamma_{\times k,t}(G_I) \leq n(k+1)-1$.

Proof. **Case i**. Let S be an arbitrary kTDS of G_I. Since every vertex v of the red clique X_i is adjacent to only one vertex of another red clique and is adjacent to $deg(v)-1 \geq \delta-1 \geq k-1$ vertices in X_i, we have $| S \cap X_i | \geq | N_{X_i}(v) | +1 \geq k$, for each vertex $v \in S \cap X_i$. Hence $\gamma_{\times k,t}(G_I) \geq nk$.

Case ii. Let $V(G) = \{x_i \mid 1 \leq i \leq n\}$. Set $S = S_1 \cup ... \cup S_n$ such that $S_1 = \{x_1x_j \mid 2 \leq j \leq k+1\}$ is a k-subset of X_1, and if $2 \leq j \leq k+1$, then $S_j = \{x_jx_1\} \cup S'_j$ that S_j' is a k-subset of $X_j - \{x_jx_1\}$, and if $k+1 < j \leq n$, then S_j is a $(k+1)$-subset of X_j. Since S is a kTDS of G_I of cardinality $n(k+1)-1$, we get $\gamma_{\times k,t}(G_I) \leq n(k+1)-1$. □

Now, we characterize graphs G of order n that the k-tuple total domination number of their inflations is either nk or $nk+1$. First we define the following three new concepts.

We know that a graph G is a *Hamiltonian graph* if it has a *Hamiltonian cycle*, that is, a cycle that contains all vertices of the graph. We extend this definition in such a way:

Definition 2.2. A graph G is a *Hamiltonian-like decomposable graph* if there exist t vertex-disjoint Hamiltonian subgraphs G_1, G_2, ..., G_t of G such that $V(G) = V(G_1) \cup V(G_2) \cup ... \cup V(G_t)$. A such partition we call a *Hamiltonian-like decomposition* of G and simply write $G = HLD(G_1, G_2, ..., G_t)$.

In generally, for each integer $k \geq 1$, we present the next definitions.

Definition 2.3. A graph G is a *k-Hamiltonian-like decomposable graph*, briefly kHLD-*graph*, if it has k Hamiltonian-like decomposition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs (where $1 \leq i \leq k$) such that every two distinct Hamiltonian subgraphs $G_{s_i}^{(i)}$ and $G_{s_j}^{(j)}$ have vertex-disjoint Hamiltonian cycles $C_{s_i}^{(i)}$ and $C_{s_j}^{(j)}$, respectively.

We note that 1-Hamiltonian-like decomposable graph is the same Hamiltonian-like decomposable graph.

Definition 2.4. A k-Hamiltonian-like decomposable graph G is a kHLPM-*graph* or a kHLMM-*graph* if G has a perfect matching or a maximum matching M, respectively, of cardinality $\lfloor n/2 \rfloor$ such that for each Hamiltonian-like decomposition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs (where $1 \leq i \leq k$) M satisfies the condition:

$$M \cap E(C_{\ell_i}^{(i)}) = \emptyset, \text{ for each } 1 \leq \ell_i \leq t_i, \tag{1}$$

where $C_{\ell_i}^{(i)}$ is a Hamiltonian cycle of $G_{\ell_i}^{(i)}$.

The next two theorems characterize graphs G with $\gamma_{\times k,t}(G_I) = nk$.

Theorem 2.5. *Let G be a graph of order n with $\delta(G) \geq 2k \geq 1$. Then $\gamma_{\times(2k),t}(G_I) = 2kn$ if and only if G is a kHLD-graph.*

Proof. Let $V(G) = \{x_i \mid 1 \leq i \leq n\}$. For $1 \leq i \leq k$ and some $t_i \geq 1$, let $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ be k Hamiltonian-like decompositions of G. For $1 \leq i \leq k$ and $1 \leq \ell_i \leq t_i$, let $C_{\ell_i}^{(i)} : x_1^{(i)}x_2^{(i)}...x_{c_{i,\ell_i}}^{(i)}$ be a Hamiltonian cycle of $G_{\ell_i}^{(i)}$. Set

$$S_{i,\ell_i} = \{x_m^{(i)}x_{m-1}^{(i)}, x_m^{(i)}x_{m+1}^{(i)} \mid 1 \leq m \leq c_{i,\ell_i}\}.$$

Then $S^{(i)} = S_{i,1} \cup S_{i,2} \cup ... \cup S_{i,t_i}$ is a DTDS of G_I of cardinality $2n$. Since G is k-Hamiltonian-like decomposable, we conclude that every two distinct $S^{(i)}$ and $S^{(\ell)}$ are disjoint. Hence $S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}$ is a 2kTDS of G_I of cardinality $2kn$. Thus $\gamma_{\times(2k),t}(G_I) \leq 2kn$, and Theorem 2.1 implies $\gamma_{\times(2k),t}(G_I) = 2kn$.

Conversely, let $\gamma_{\times(2k),t}(G_I) = 2kn$ and let S be a $\gamma_{\times(2k),t}(G_I)$-set. Then we may partition every $S \cap X_i$ to k 2-subsets $D_j^{(i)}$ (when $1 \leq j \leq k$) such that $D_j^{(1)} \cup D_j^{(2)} \cup ... \cup D_j^{(n)}$ is a union of some vertex-disjoint cycles. Since for each $1 \leq i \leq n$, $| S \cap X_i |= 2k$. Without loss of generality, we may assume that $D_j^{(1)} \cup D_j^{(2)} \cup ... \cup D_j^{(n)}$ is the cycle

$$C_j : x_1x_n, x_1x_2; x_2x_1, x_2x_3; x_3x_2, x_3x_4; ...; x_nx_{n-1}, x_nx_1.$$

Then G has the corresponding cycle $C'_j : x_1x_2x_3x_4...x_n$. Therefore, for every partition $D_j^{(1)} \cup D_j^{(2)} \cup ... \cup D_j^{(n)}$ there is a corresponding partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs $G_1^{(i)}$, $G_2^{(i)}$, ... and $G_{t_i}^{(i)}$. Hence G is a kHLD-graph. □

Theorem 2.6. *Let G be a graph of order n with $\delta(G) \geq 2k+1 \geq 1$. Then $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n$ if and only if G is a kHLPM-graph.*

Proof. Let $V(G) = \{x_i \mid 1 \leq i \leq n\}$. Let G be a kHLPM-graph with perfect matching M. We follow exactly the notation and terminology introduced in the first and second paragraphs of the proof of Theorem 2.5. Then similarly $S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}$ is a 2kTDS of G_I of cardinality $2kn$. Set $M_I = \{(x_ix_j, x_jx_i) \mid x_ix_j \in M\}$. Since for every partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs, M satisfies the condition (1), we have $V(M_I) \cap (S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}) = \emptyset$. One can verify that $V(M_I) \cup S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}$ is a (2k+1)TDS of G_I of cardinality $(2k+1)n$. Thus $\gamma_{\times(2k+1),t}(G_I) \leq (2k+1)n$ and Theorem 2.1 implies $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n$.

Conversely, let $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n$ and let S be a $\gamma_{\times(2k+1),t}(G_I)$-set. Since $| S \cap X_i |= 2k+1$ for each $1 \leq i \leq n$, similar to the proof of Theorem 2.5, we may partition every $S \cap X_i$ to k 2-subsets $D_j^{(i)}$ (when $1 \leq j \leq k$) such that $D_j^{(1)} \cup D_j^{(2)} \cup ... \cup D_j^{(n)}$ is a union of some disjoint cycles. Hence there is a corresponding partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs for it, and also $\bigcup_{1 \leq i \leq n}(S - (\bigcup_{1 \leq j \leq k} D_j^{(i)}))$ makes a blue matching M_I in G_I with size $\lfloor \frac{n}{2} \rfloor$. It can be easily verified that $M = \{x_ix_j \mid (x_ix_j, x_jx_i) \in M_I\}$ is a perfect matching in G that satisfies the condition (1). Hence G is a kHLPM-graph. □

Theorems 2.5 and 2.6 imply the next result.

Theorem 2.7. *Let G be a graph of order n with $\delta(G) \geq k \geq 1$. Then $\gamma_{\times k,t}(G_I) \geq nk+1$ if and only if k and n are odd or k is even and G is not a kHLD-graph or k is odd and G is not a kHLPM-graph.*

By closer look at the proofs of Theorems 2.5 and 2.6 we obtain the following observation.

Observation 2.8. *Let k be an integer and let G be a graph of order n with $\gamma_{\times k,t}(G) = nk$. Then for every $\gamma_{\times k,t}(G_I)$-set S, the induced subgraph $G_I[S]$ by S in G_I contains a union of vertex-disjoint Hamiltonian cycles (of some of the its subgraphs) and probably a perfect matching. Therefore, if we reduce the number of vertices of S in a red clique of G_I to less than k vertices, then there exist another unique red clique X of G_I and an unique vertex w of $X \cap S$ such that $| N(w) \cap S |< k$.*

The next theorem states a necessary and sufficient condition for $\gamma_{\times k,t}(G_I) = nk+1$, when k and n are both odd.

Theorem 2.9. *Let G be a graph of odd order n and let $1 \leq 2k+1 \leq \delta$. Then $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n+1$ if and only if G is a kHLMM-graph.*

Proof. Let $V(G) = \{x_i \mid 1 \le i \le n\}$ and let G be a kHLMM-graph with maximum matching M. Without loss of generality, we may assume that M does not saturate x_n. For $1 \le i \le k$ and some $t_i \ge 1$, let $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ be k Hamiltonian-like decompositions of G. For $1 \le \ell_i \le t_i$, let $C_{\ell_i}^{(i)} : x_1^{(i)} x_2^{(i)} ... x_{c_{i,\ell_i}}^{(i)}$ be a Hamiltonian cycle in $G_{\ell_i}^{(i)}$. Set

$$S_{i,\ell_i} = \{x_m^{(i)} x_{m-1}^{(i)}, x_m^{(i)} x_{m+1}^{(i)} \mid 1 \le m \le c_{i,\ell_i}\}.$$

Then $S^{(i)} = S_{i,1} \cup S_{i,2} \cup ... \cup S_{i,t_i}$ is a DTDS of G_I of cardinality $2n$. Also every two distinct $S^{(i)}$ and $S^{(\ell)}$ are disjoint. Hence $S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}$ is a 2kTDS of G_I of cardinality $2kn$. Since G is k-Hamiltonian-like decomposable. Set $M_I = \{(x_i x_j, x_j x_i) \mid x_i x_j \in M\}$. Since for each partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs M satisfies the condition (1), we have $V(M_I) \cap (S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}) = \emptyset$. One can verify that for every two arbitrary vertices $\alpha, \beta \in X_n - (S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)})$, the set $V(M_I) \cup S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)} \cup \{\alpha, \beta\}$ is a $(2k+1)$TDS of G_I of cardinality $(2k+1)n + 1$. Thus $\gamma_{\times(2k+1),t}(G_I) \le (2k+1)n + 1$ and Theorem 2.7 implies $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n + 1$.

Conversely, let $\gamma_{\times(2k+1),t}(G_I) = (2k+1)n + 1$ and let S be a $\gamma_{\times(2k+1),t}(G_I)$-set. Without loss of generality, we may assume that for each $1 \le i \le n-1$, $\mid S \cap X_i \mid = 2k+1$ and $\mid S \cap X_n \mid = 2k+2$. Similar to the proofs of the previous theorems, we may partition every $S \cap X_i$ to k 2-subsets $D_j^{(i)}$ (when $1 \le j \le k$) such that $D_j^{(1)} \cup D_j^{(2)} \cup ... \cup D_j^{(n)}$ is a union of some disjoint cycles. Hence there is a corresponding partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs for it, and also $\bigcup_{1 \le i \le n-1} (S - (\bigcup_{1 \le j \le k} D_j^{(i)}))$ makes a blue matching M_I in G_I with size $\lfloor \frac{n}{2} \rfloor$. It can be easily verified that $M = \{x_i x_j \mid (x_i x_j, x_j x_i) \in M_I\}$ is a maximum matching in G with size $\lfloor \frac{n}{2} \rfloor$ such that does not saturate x_n and for every partition $G = HLD(G_1^{(i)}, G_2^{(i)}, ..., G_{t_i}^{(i)})$ of Hamiltonian subgraphs it satisfies the condition (1). Hence G is a kHLMM-graph. □

3. The inflation of a connected graph which has a cut-edge or a cut-vertex

In the next theorem we present some upper and lower bounds for the k-tuple total domination number of the inflation of a graph F which contains a cut-edge e, in terms of the k-tuple total domination numbers of the inflation of the components of $F - e$.

Theorem 3.1. *Let F be a graph with a cut-edge e such that G and H are the components of $F - e$. If $2 \le k \le \min\{\delta(G), \delta(H)\}$, then*

$$\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) - k \le \gamma_{\times k,t}(F_I) \le \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I).$$

Proof. Let $V(G) = \{x_i \mid 1 \le i \le n\}$ and let $V(H) = \{y_i \mid 1 \le i \le m\}$. Without loss of generality, we may assume that $e = x_1 y_1$. Then $V(F_I) = V(G_I) \cup V(H_I) \cup \{x_1 y_1, y_1 x_1\}$ and

$$\begin{aligned} E(F_I) &= E(G_I) \cup E(H_I) \cup \{(x_1 x_j, x_1 y_1) \mid x_1 x_j \in X_1\} \cup \\ &\quad \{(y_1 y_j, y_1 x_1) \mid y_1 y_j \in Y_1\} \cup \{(x_1 y_1, y_1 x_1)\}. \end{aligned}$$

Let $X_1' = X_1 \cup \{x_1 y_1\}$ and let $Y_1' = Y_1 \cup \{y_1 x_1\}$. Let S_G and S_H be $\gamma_{\times k,t}(G_I)$-set and $\gamma_{\times k,t}(H_I)$-set, respectively. Since $S_G \cup S_H$ is a kTDS of F_I of cardinality $\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I)$, we obtain $\gamma_{\times k,t}(F_I) \le \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I)$.

Now let S_F be a $\gamma_{\times k,t}(F_I)$-set. If $S_F \cap \{x_1 y_1, y_1 x_1\} = \emptyset$, then $S_F \cap V(G_I)$ and $S_F \cap V(H_I)$ are k-tuple total dominating sets of G_I and H_I, respectively. Hence

$$\begin{aligned} \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) &\le \mid S_F \cap V(G_I) \mid + \mid S_F \cap V(H_I) \mid \\ &= \mid S_F \mid \\ &= \gamma_{\times k,t}(F_I). \end{aligned}$$

Therefore, we assume that $S_F \cap \{x_1 y_1, y_1 x_1\} \ne \emptyset$, and in the next two cases we will complete our proof.

Case i. $\mid S_F \cap \{x_1 y_1, y_1 x_1\} \mid = 1$.

Let $S_F \cap \{x_1y_1, y_1x_1\} = \{x_1y_1\}$. Then $S_F \cap V(H_I)$ is a kTDS of H_I and $|\, S_F \cap X_1 \,| \geq k$. Since $k \geq 2$ and each clique of every inflated graph contains at least k vertices of every kTDS and also $|\, S_F \cap X_1 \,| > k$ implies $|\, S_F \cap Y'_1 \,| = k-1$, we get $|\, S_F \cap X_1 \,| = k$. If $deg_G(x_1) = k$, then $(S_F \cap V(G_I)) \cup \{x_ix_1 \mid x_1x_i \in X_1\}$ is a kTDS of G_I of cardinality at most $|\, S_F \cap V(G_I) \,| + k$. Hence

$$\begin{aligned}\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) &\leq |\, S_F \cap V(G_I) \,| + k + |\, S_F \cap V(H_I) \,| \\ &= \gamma_{\times k,t}(F_I) + k - 1.\end{aligned}$$

If $deg_G(x_1) \neq k$, then for every $x_1x_j \in X_1 - S_F$ the set $(S_F \cap V(G_I)) \cup \{x_1x_j\}$ is a kTDS of G_I. Hence

$$\begin{aligned}\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) &\leq |\, (S_F \cap V(G_I)) \cup \{x_1x_j\} \,| + |\, S_F \cap V(H_I) \,| \\ &= \gamma_{\times k,t}(F_I).\end{aligned}$$

Case ii. $|\, S_F \cap \{x_1y_1, y_1x_1\} \,| = 2$.
Since $|\, S_F \cap X'_1 \,| \geq k$, $|\, S_F \cap Y'_1 \,| \geq k$ and $\{x_1y_1, y_1x_1\} \subseteq S_F$, we have $|\, S_F \cap X_1 \,| = k-1$ or $|\, S_F \cap Y_1 \,| = k-1$. Let $|\, S_F \cap X_1 \,| \geq |\, S_F \cap Y_1 \,| = k-1$. If $deg_H(y_1) = k$, then there exists a vertex $y_1y_j \in Y_1 - S_F$ such that $(S_F \cap V(H_I)) \cup \{y_1y_j, y_jy_1\}$ is a kTDS of H_I. If $deg_H(y_1) \geq k+1$, then there are two disjoint vertices $y_1y_j, y_1y_i \in Y_1 - S_F$ such that $(S_F \cap V(H_I)) \cup \{y_1y_j, y_1y_i\}$ is a kTDS of H_I.

Now, in each possible case, we will present a k-tuple total dominating set of G_I. If $|\, S_F \cap X'_1 \,| \geq k+1$, then $S_F \cap V(G_I)$ is a kTDS of G_I. Let $|\, S_F \cap X_1 \,| = k$ and let $deg_G(x_1) = k$. Then $(S_F \cap V(G_I)) \cup \{x_ix_1 \mid x_1x_i \in X_1\}$ is a kTDS of G_I of cardinality at most $|\, S_F \cap V(G_I) \,| + k$. If either $|\, S_F \cap X_1 \,| = k$ and $deg_G(x_1) = k+1$ or $|\, S_F \cap X_1 \,| = k-1$ and $deg_G(x_1) = k$, then for each $x_1x_j \in X_1 - S_F$ the set $(S_F \cap V(G_I)) \cup \{x_1x_j, x_jx_1\}$ is a kTDS of G_I. Finally, if either $|\, S_F \cap X_1 \,| = k$ and $deg_G(x_1) \geq k+2$ or $|\, S_F \cap X_1 \,| = k-1$ and $deg_G(x_1) \geq k+1$, then for every two distinct vertices $x_1x_j, x_1x_i \in X_1 - S_F$, the set $(S_F \cap V(G_I)) \cup \{x_1x_j, x_1x_i\}$ is a kTDS of G_I. Thus in Case (ii) we have proved that $\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) - k \leq \gamma_{\times k,t}(F_I)$.

With comparing the obtained bounds in Case (i) and Case (ii), we obtain $\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) - k \leq \gamma_{\times k,t}(F_I)$, and this completes our proof. □

By closer look at the proof of Theorem 3.1 we have the next theorem.

Theorem 3.2. *Let F be a graph with a cut-edge e such that G and H are the components of $F - e$. If $2 \leq k \leq \min\{\delta(G), \delta(H)\} - 1$, then*

$$\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) - 2 \leq \gamma_{\times k,t}(F_I) \leq \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I).$$

We now calculate the k-tuple total domination number of the inflation of the complete graphs and then continue our discussion.

Proposition 3.3. *Let $n > k \geq 2$ be integers. Then every complete graph K_n is $\lfloor \frac{n-1}{2} \rfloor$-Hamiltonian-like decomposable graph and*

$$\gamma_{\times k,t}((K_n)_I) = \begin{cases} nk+1 & \text{if } k \text{ and } n \text{ are odd}, \\ nk & \text{otherwise}. \end{cases}$$

Proof. Let $V(K_n) = \{i \mid 1 \leq i \leq n\}$. Since for each $1 \leq i \leq \lfloor (n-1)/2 \rfloor$ the edge set $E_i = \{(j, j+i) \mid 1 \leq j \leq n\}$ is the union of some disjoint cycles and $\bigcup_{1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor} E_i$ is a partition of $V(K_n)$, we conclude that K_n is $\lfloor \frac{n-1}{2} \rfloor$-Hamiltonian-like decomposable graph. We also see that $M = \{(i, i + \lfloor \frac{n}{2} \rfloor) \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ is a perfect matching or a maximum matching in K_n with size $\lfloor \frac{n}{2} \rfloor$, when n is even or odd, respectively. Then Theorems 2.6 and 2.9 complete our proof. □

Proposition 3.4. *Let $2 \leq k < n \leq m$ and let F be a graph with a cut-edge e such that $G \cong K_n$ and $H \cong K_m$ are the components of $F - e$. Then*

$$\gamma_{\times k,t}(F_I) = \begin{cases} k(n+m)+1 & \text{if } k \text{ is odd and } n \equiv m+1 \ (mod\ 2), \\ k(n+m) & \text{otherwise}. \end{cases}$$

Proof. Let $V(G) = \{x_i \mid 1 \le i \le n\}$, $V(H) = \{y_i \mid 1 \le i \le m\}$ and let $e = x_n y_m$. Since every complete graph K_t is $\lfloor \frac{t-1}{2} \rfloor$-Hamiltonian-like decomposable graph and $n \le m$, we conclude that F is $\lfloor \frac{n-1}{2} \rfloor$-Hamiltonian-like decomposable graph. We now continue our discussion in the next two cases.

Case i. $n \equiv m + 1 \pmod 2$.
If k is odd, then Theorem 2.7 implies that $\gamma_{\times k,t}(F_I) \ge k(n + m) + 1$. Without loss of generality, we may assume that n is odd and m is even. Then $\gamma_{\times k,t}(G_I) = kn + 1$ and $\gamma_{\times k,t}(H_I) = km$, by Proposition 3.3. If S_G and S_H are $\gamma_{\times k,t}(G_I)$-set and $\gamma_{\times k,t}(H_I)$-set, respectively, then $S_G \cup S_H$ is a kTDS of F_I of cardinality $k(n + m) + 1$. Hence $\gamma_{\times k,t}(F_I) = k(n + m) + 1$. For even k it can be similarly verified that $\gamma_{\times k,t}(F_I) = k(n + m)$.

Case ii. $n \equiv m \pmod 2$.
In this case, Theorem 2.1 implies that $\gamma_{\times k,t}(F_I) \ge k(n + m)$. If either $n \equiv m \equiv 0 \pmod 2$ or $n \equiv m \equiv 1 \pmod 2$ and k is even, then $\gamma_{\times k,t}(G_I) = kn$, and $\gamma_{\times k,t}(H_I) = km$, by Proposition 3.3. If S_G and S_H are $\gamma_{\times k,t}(G_I)$-set and $\gamma_{\times k,t}(H_I)$-set, respectively, then obviously $S_G \cup S_H$ is a kTDS of F_I of cardinality $k(n + m)$ and so $\gamma_{\times k,t}(F_I) = k(n + m)$.

Now let $n \equiv m \equiv 1 \pmod 2$ and let k be odd. Then $\gamma_{\times k,t}(G_I) = kn + 1$ and $\gamma_{\times k,t}(H_I) = km + 1$, by Proposition 3.3. Let $S_G = S_1 \cup \{\alpha, \beta\}$ be the given $\gamma_{\times k,t}(G_I)$-set in the second paragraph of the proof of Theorem 2.9 such that $S_1 = V(M_I) \cup S^{(1)} \cup S^{(2)} \cup ... \cup S^{(k)}$ and $\alpha, \beta \in X_n - (S_1 - V(M_I))$. With applying Theorem 2.9 for H, let also similarly $S_H = S'_1 \cup \{\alpha', \beta'\}$ be the given $\gamma_{\times k,t}(H_I)$-set in the second paragraph of the proof of Theorem 2.9 such that $S'_1 = V(M'_I) \cup S'^{(1)} \cup S'^{(2)} \cup ... \cup S'^{(k)}$ and $\alpha', \beta' \in Y_m - (S'_1 - V(M'_I))$. Then obviously $S = S_G \cup S_H \cup \{x_n y_m, y_m x_n\} - \{\alpha, \beta, \alpha', \beta'\}$ is a kTDS of F_I of cardinality $k(n+m)$. Hence $\gamma_{\times k,t}(F_I) = k(n+m)$. □

Proposition 3.3 implies that if $G = K_n$ and $H = K_m$, then

$$\gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) = \begin{cases} k(n+m) & \text{if } k \text{ is odd and } m \text{ and } n \text{ are both even,} \\ k(n+m)+1 & \text{if } k \text{ is odd and } n \equiv m+1 \pmod 2, \\ k(n+m)+2 & \text{if } k, m \text{ and } n \text{ are odd.} \end{cases}$$

Thus Proposition 3.4 implies the next result which states the given bounds in Theorem 3.1 are sharp.

Corollary 3.5. *Let $2 \le k < n \le m$ and let F be a graph with a cut-edge e such that $G \cong K_n$ and $H \cong K_m$ are the components of $F - e$. Then*

$$\gamma_{\times k,t}(F_I) = \begin{cases} \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) - 2 & \text{if } k, m \text{ and } n \text{ are all odd,} \\ \gamma_{\times k,t}(G_I) + \gamma_{\times k,t}(H_I) & \text{otherwise.} \end{cases}$$

Now in the next theorem we give some upper and lower bounds for the k-tuple total domination number of the inflation of a graph F, which contains a cut-vertex v, in terms of the k-tuple total domination numbers of the inflation of the v-components of $F - v$.

Theorem 3.6. *Let F be a graph with a cut-vertex v such that $G^1, G^2, ..., G^m$ are all v-components of $F - v$. If $2 \le k \le \min\{\delta(G^i) \mid 1 \le i \le m\} - 1$ and G^i_I is the inflation of G^i, then*

$$\sum_{1 \le i \le m} \gamma_{\times k,t}(G^i_I) - m(k+1) + k \le \gamma_{\times k,t}(F_I) \le \sum_{1 \le i \le m} \gamma_{\times k,t}(G^i_I),$$

and the given upper bound is sharp.

Proof. Let $V(G^i) = \{x^i_j \mid 1 \le j \le n_i\}$ for $i = 1, 2, ..., m$. Without loss of generality, we may assume that $x^1_1 = x^2_1 = ... = x^m_1 = v$. Then $V(F_I) = \bigcup_{1 \le i \le m} V(G^i_I)$ and

$$E(F_I) = (\bigcup_{1 \le i \le m} E(G^i_I) \bigcup \{(x^i_1 x^i_l, x^j_1 x^j_t) \mid x^i_1 x^i_l \in X^i, \text{ and } x^j_1 x^j_t \in X^j, \text{ for } 1 \le i < j \le m\}.$$

Let S^i be a $\gamma_{\times k,t}(G^i_I)$-set for $i = 1, 2, ..., m$. Since $\bigcup_{1 \le i \le m} S^i$ is a kTDS of F_I of cardinality $\sum_{1 \le i \le m} \gamma_{\times k,t}(G^i_I)$, we have $\gamma_{\times k,t}(F_I) \le \sum_{1 \le i \le m} \gamma_{\times k,t}(G^i_I)$.

Now let S be a $\gamma_{\times k,t}(F_I)$-set. Let $S^i = S \cap V(G_I^i)$, where $1 \le i \le m$. Then each S^i is a kTDS of $G_I^i - X_1^i$, where X_1^i is the corresponding clique of the vertex x_1^i. Let $| S \cap X_1^i |= t_i$, where $1 \le i \le m$. Then $\sum_{1\le i\le m} t_i \ge k$. The condition $\delta(G^i) > k$ allows us that by adding at most $k+1-t_i$ vertices of X_1^i to S, we obtain a kTDS S' of F_I such that every $S' \cap V(G_I^i)$ is a kTDS of G_I^i. Then

$$\begin{array}{lll} \sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i) & \le & \sum_{1\le i\le m} | S'_F \cap V(G_I^i) | \\ & \le & | S_F | + m(k+1) - \sum_{1\le i\le m} t_i \\ & \le & \gamma_{\times k,t}(F_I) + m(k+1) - k. \end{array}$$

Hence

$$\sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i) - m(k+1) + k \le \gamma_{\times k,t}(F_I) \le \sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i).$$

Now we show that the upper bound $\sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i)$ is sharp. Let F be a graph with a cut-vertex v such that G^1, G^2, ..., G^m are the all v-components of $F - v$ and $\gamma_{\times k,t}(G_I^i) = n_i k$, where $n_i = n(G^i)$. Let $V(G^i) = \{x_j^i \mid 1 \le j \le n_i\}$ and $x_1^1 = ... = x_1^m = v$. Let Y_v^F be the corresponding red clique with the vertex v in F. Let S^i be a $\gamma_{\times k,t}(G_I^i)$-set, where $1 \le i \le m$. Then every clique in G_I^i contains exactly k vertices of S^i, and so $S = \bigcup_{1\le i\le m} S^i$ is a kTDS of F_I of cardinality $\sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i) = \sum_{1\le i\le m} n_i k$ such that Y_v^F contains mk vertices of S.

We claim that S has minimum cardinality among all k-tuple total dominating sets of F_I. Observation 2.8 implies that every red clique other than Y_v^F must contains at least k vertices of every kTDS of F_I. Thus we cannot reduce the number of vertices of S in cliques (except probably Y_v^F). Since also reducing the number of the vertices of $S \cap Y_v^F$ reduce the cardinality of the k-tuple total domination number of G_I^i, we cannot do it, by Observation 2.8. Therefore S is a minimal kTDS of F_I. Now let S' be an arbitrary $\gamma_{\times k,t}(F_I)$-set of cardinality less than $\sum_{1\le i\le m} n_i k$. Then, by the previous discussion, there exists a v-component G^i of $F - v$ and a clique X of it other than $X_1^i = Y_v^F \cap V(G_I^i)$ such that $| S' \cap X |< k$. But this is not possible, by Observation 2.8. Therefore S is a $\gamma_{\times k,t}(F_I)$-set, and so $\gamma_{\times k,t}(F_I) = \sum_{1\le i\le m} \gamma_{\times k,t}(G_I^i) = \sum_{1\le i\le m} n_i k$. □

We see that if $G^1, G^2, ..., G^m$ and F are the given graphs in the second part of the proof of Theorem 3.6, then $n = n(F) = \sum_{1\le i\le m} n_i - m + 1$ and

$$\begin{array}{lll} \gamma_{\times k,t}(F_I) & = & \sum_{1\le i\le m} n_i k \\ & = & nk + (m-1)k \\ & \le & n(k+1) - 1. \end{array}$$

Thus this family of graphs are examples of the graphs G of order n with $\gamma_{\times k,t}(G_I) = nk + \alpha k \le n(k+1) - 1$, where α is an arbitrary positive integer.

4. The inflation of some graphs

In Section 3, we calculated the k-tuple total domination number of the inflation of the complete graphs. Now we find this number in the inflation of the generalized Petersen graphs, the Harary graphs and the complete bipartite graphs. Also we give an upper bound for this number of the inflation of the complete multipartite graph.

In [12], Watkins introduced the notion of generalized Petersen graph (GPG for short) as follows: for any integer $n \ge 3$ let Z_n be additive group on $\{1, 2, ..., n\}$ and $m \in Z_n - \{0\}$. The *generalized Petersen graph* $P(n,m)$ is defined on the set $\{a_i, b_i \mid i \in Z_n\}$ of $2n$ vertices with edges $a_i a_{i+1}$, $a_i b_i$, $b_i b_{i+m}$ for all i. If $m = \frac{n}{2}$, then every vertex b_i has degree 2 and every vertex a_i has degree 3, and otherwise $P(n,m)$ is 3-regular. Thus $\gamma_{\times 3,t}((P(n,m)_I) = n(G_I) = 6n$, when $m \neq \frac{n}{2}$. Since $M = \{a_i b_i \mid i \in Z_n\}$ is a perfect matching in $P(n,m)$, we get $S = \{a_i b_i, b_i a_i \mid i \in Z_n\}$ as a $\gamma_t((P(n,m))_I)$-set and so $\gamma_t((P(n,m)_I) = 2n$. In the next proposition we calculate $\gamma_{\times 2,t}((P(n,m)_I)$.

Proposition 4.1. *Let $n \geq 3$ and $m \geq 1$ be integers. Then*

$$\gamma_{\times 2,t}((P(n,m))_I) = \begin{cases} 4n+2 & \text{if } m = \frac{n}{2} \text{ is odd,} \\ 4n & \text{otherwise.} \end{cases}$$

Proof. Let $G = P(n,m)$. We first assume that $m \neq \frac{n}{2}$ and d is the greatest common divisor of m and n. Then the induced subgraph by $\{b_i \mid i \in Z_n\}$ of G has a partition to d disjoint cycles $C_i : b_i b_{i+m} b_{i+2m} ... b_{i+\alpha-m}$, where $1 \leq i \leq d$ and $\alpha = \min\{tm \mid tm \equiv 0 \pmod n\}$. Since the induced subgraph by $\{a_i \mid i \in Z_n\}$ of G is the cycle $C_a : a_1 a_2 a_3 ... a_n$, we conclude that G is a Hamiltonian-like decomposable graph and Theorem 2.5 implies $\gamma_{\times 2,t}(G_I) = 4n$.

Now let $m = \frac{n}{2}$. In this case, $b_i b_j \in E(G)$ if and only if $j \equiv i + m \pmod n$. Hence every vertex b_i has degree 2 and every vertex a_i has degree 3. Then there exist the $\lfloor \frac{m}{2} \rfloor$ disjoint cycles $b_i a_i a_{i+1} b_{i+1} b_{i+1+m} a_{i+1+m} a_{i+m} b_{i+m}$ with eight vertices. If m is even, then these cycles form a partition of $V(G)$. Hence G is a Hamiltonian-like decomposable graph and Theorem 2.5 implies $\gamma_{\times 2,t}(G_I) = 4n$. Otherwise, these cycles form a partition of $V(G) - \{a_m, b_m, b_n, a_n\}$. We notice that the induced subgraph of G by $\{a_m, b_m, b_n, a_n\}$ is the path $P_4 : a_m b_m b_n a_n$. Set

$$\begin{aligned} S &= S_1 \cup S_2 \cup ... \cup S_{\lfloor \frac{m}{2} \rfloor} \cup \\ &\quad \{a_m a_{m+1}, a_m a_{m-1}, a_m a_m; b_m a_m, b_m b_n; b_n b_m, b_n a_n; a_n b_n, a_n a_1, a_n a_{n-1}\}, \end{aligned}$$

where

$$\begin{aligned} S_i &= \{b_i b_{i+m}, b_i a_i; a_i b_i, a_i a_{i+1}; a_{i+1} a_i, a_{i+1} b_{i+1}; b_{i+1} a_{i+1}\} \cup \\ &\quad \{b_{i+1} b_{i+1+m}; b_{i+1+m} b_{i+1}, b_{i+m+1} a_{i+m+1}; a_{i+m+1} b_{i+m+1}\} \cup \\ &\quad \{a_{i+m+1} a_{i+m}; a_{i+m} a_{i+m+1}, a_{i+m} b_{i+m}; b_{i+m} a_{i+m}, b_{i+m} b_i\}, \end{aligned}$$

for each $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. One can verify that S is a minimum DTDS of G_I and so $\gamma_{\times 2,t}(G_I) = 4n + 2$. □

We now consider the Harary graphs which make an interesting family of graphs. Given $m < n$, place n vertices $1, 2, ..., n$ around a circle, equally spaced. If m is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m}{2}$ vertices in each direction around the circle. If m is odd and n is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m-1}{2}$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is m-regular. When m and n are both odd, index the vertices by the integers modulo n. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $(i, i + \frac{n-1}{2})$, for $0 \leq i \leq \frac{n-1}{2}$ (see [13]).

Proposition 4.2. *Let $2 \leq k \leq m < n$ be integers. Then the Harary graph $H_{m,n}$ is $\lfloor \frac{m}{2} \rfloor$-Hamiltonian-like decomposable graph and*

$$\gamma_{\times k,t}((H_{m,n})_I) = \begin{cases} nk+1 & \text{if } k \text{ and } n \text{ are both odd,} \\ nk & \text{otherwise.} \end{cases}$$

Proof. Since for each $1 \leq i \leq m$ the edge subset $E_i = \{(j, j+i) \mid 1 \leq j \leq n\}$ is the union of some disjoint cycles and $\bigcup_{1 \leq i \leq m} E_i$ is a partition of $V(H_{m,n})$, we conclude that $H_{m,n}$ is a m-Hamiltonian-like decomposable graph. Let m be odd. Then $M = \{(i, i + \lfloor \frac{n}{2} \rfloor) \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ is a perfect matching or a maximum matching of $H_{m,n}$ with size $\lfloor \frac{n}{2} \rfloor$ if n is even or odd, respectively. Now Theorems 2.6 and 2.9 complete our proof. □

In the following two theorems we consider the complete bipartite graphs $K_{p,q}$. First let $p = q$.

Proposition 4.3. *For integers $p \geq k \geq 2$, let G be the complete bipartite graph $K_{p,p}$. Then G is a $(\lfloor \frac{p}{2} \rfloor - 1)$HLPM-graph if p is even, and is a $\lfloor \frac{p}{2} \rfloor$-Hamiltonian-like decomposable graph, otherwise. Hence $\gamma_{\times k,t}(G_I) = 2pk$.*

Proof. We consider the partition $X \cup Y$ for $V(G)$, where $X = \{x_i \mid 1 \leq i \leq p\}$ and $Y = \{y_i \mid 1 \leq i \leq p\}$. For $0 \leq j \leq \lfloor \frac{p}{2} \rfloor - 1$, we choose $\lfloor \frac{p}{2} \rfloor$ sequences on $X \cup Y$ with length $2p$ that are alternatively from X and Y

with starting of vertex x_1 such that every three consequence numbers of them are x_i, y_{i+j}, and $x_{i+(2j+1)}$. Let $0 \leq j \leq \lfloor \frac{p}{2} \rfloor - 2$. If p does not divided by $2j+1$, then j-th sequence makes the cycle

$$C_j : x_1 y_{j+1} x_{2j+2} y_{3j+2} ... x_{p-2j} y_{p-j}$$

but if $p = (2j+1)t$, for some positive integer t, then it makes $2j+1$ disjoint cycles

$$C_i^j : x_i y_{i+j} x_{i+(2j+1)} y_{i+(3j+1)} ... x_{i+(t-1)(2j+1)} y_{i+(t-1)(2j+1)+j}$$

with length t, where $1 \leq i \leq 2j+1$. We notice that for odd p and $j = \lfloor \frac{p}{2} \rfloor - 1$ there exists another cycle with length $2p$ which is vertex-disjoint with the other cycles. If p is even and $j = \lfloor \frac{p}{2} \rfloor - 1$, the corresponding sequence makes a perfect matching M which is disjoint of the cycles. Then Theorems 2.5 and 2.6 imply $\gamma_{\times k,t}(G_I) = 2pk$. □

Proposition 4.4. *For integers $q \geq p > k \geq 2$, let G be the complete bipartite graph $K_{p,q}$. Then*

$$\gamma_{\times k,t}(G_I) = 2pk + (q-p)(k+1).$$

Proof. We consider the partition $X \cup Y$ for $V(G)$, where $X = \{x_i \mid 1 \leq i \leq p\}$ and $Y = \{y_i \mid 1 \leq i \leq q\}$. Let S be an arbitrary $\gamma_{\times k,t}(G_I)$-set such that α red cliques of G_I contain k vertices and $p+q-\alpha$ red cliques of G_I contain $k+1$ vertices of S. Since G is bipartite, then $\frac{\alpha}{2}$ cliques must be selected among the q red cliques Y_i, where $1 \leq i \leq q$, and the second $\frac{\alpha}{2}$ cliques must be selected among the p red cliques X_i, where $1 \leq i \leq p$. We notice that this choosing is possible. Because, by Proposition 4.3, $K_{p,p}$ is $(\lfloor \frac{p}{2} \rfloor - 1)$HLPM-graph or $\lfloor \frac{p}{2} \rfloor$-Hamiltonian-like decomposable graph, when p is even or odd, respectively. Thus $\alpha \leq 2p$ and so

$$\begin{aligned}
\gamma_{\times k,t}(G_I) &= \min\{| S | \mid S \text{ is a kTDS of } G_I\} \\
&= \min\{\alpha k + (q+p-\alpha)(k+1) \mid 0 \leq \alpha \leq 2p\} \\
&= \min\{(q+p)(k+1) - \alpha \mid 0 \leq \alpha \leq 2p\} \\
&= (q+p)(k+1) - 2p \\
&= 2pk + (q-p)(k+1).
\end{aligned}$$

□

Theorem 2.1 implies that if G is a graph of order n with $\delta(G) \geq k \geq 2$, then $n(k+1) - n \leq \gamma_{\times k,t}(G_I) \leq n(k+1) - 1$. In the next result, we show that for each $n(k+1) - n \leq m = n(k+1) - 2\ell \leq n(k+1) - 1$ there exist an integer k and a graph G that its k-tuple total domination number is m.

Theorem 4.5. *For each integers n, k and ℓ with the condition $2 \leq k < \ell \leq \lfloor \frac{n}{2} \rfloor$, there exists a graph G of order n such that $\gamma_{\times k,t}(G_I) = n(k+1) - 2\ell$.*

Proof. Let $G = K_{\ell,n-\ell}$. Then Proposition 4.4 implies

$$\begin{aligned}
\gamma_{\times k,t}(G_I) &= 2\ell k + (n-2\ell)(k+1) \\
&= n(k+1) - 2\ell.
\end{aligned}$$

□

The next theorem gives an upper bound for the k-tuple total domination number of the complete multipartite graphs.

Proposition 4.6. *Let G be the complete multipartite graph $K_{n_1,n_2,...,n_m}$ of order n. If $2 \leq k < n' = \max\{\sum_{i\in J} n_i \mid J \subseteq \{1,2,..m\}$ and $\sum_{i\in J} n_i \leq \frac{n}{2}\}$, then $\gamma_{\times k,t}(G_I) \leq n(k+1) - 2n'$.*

Proof. We assume that $V(G) = X^{(1)} \cup X^{(2)} \cup ... \cup X^{(m)}$ is the partition of the vertices of G, where $X^{(i)} = \{x_j^{(i)} \mid 1 \le j \le n_i\}$. Let $n' = \sum_{i \in J} n_i \le \frac{n}{2}$, for some $J \subseteq \{1, 2, ..m\}$. Let $X = \bigcup_{i \in J} X^{(i)}$ and $Y = \bigcup_{i \notin J} X^{(i)}$. Then every vertex of X is adjacent to all vertices of Y. If H is the complete bipartite with the vertex set $X \cup Y$, then it is a subgraph of G and so $\gamma_{\times k,t}(G_I) \le \gamma_{\times k,t}(H_I) = n(k+1) - 2n'$, by Proposition 4.4. □

At the end of our paper we state the following problems.

Problem 4.7. *Can the upper bound $n(k+1) - 1$ in Theorem 2.1 be improved?*

Problem 4.8. *Is the lower bound $\sum_{1 \le i \le m} \gamma_{\times k,t}(G_I^i) - m(k+1) + k$ in Theorem 3.6 sharp?*

Problem 4.9. *Characterize all graphs G that satisfy $\gamma_{\times k,t}(G_I) = nk + 1$.*

References

[1] F. Harary, T. W. Haynes, Double domination in graphs, Ars Combin. 55 (2000) 201–213.
[2] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), Fundamentals of Domination in Graphs, Marcel Dekker, Inc. New York, 1998.
[3] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc. New York, 1998.
[4] M. A. Henning, A. P. Kazemi, *k*-tuple total domination in graphs, Discrete Applied Mathematics 158 (2010) 1006–1011.
[5] M. A. Henning, A. P. Kazemi, *k*-tuple total domination in cross product of graphs, J. Comb. Optim. 24 (2012) 339-346.
[6] M. A. Henning, A. P. Kazemi, Total domination in inflated graphs, Discrete Applied Mathematics 160 (2012) 164–169 .
[7] A. P. Kazemi, *k*-tuple total domination in complementary prisms, ISRN Discrete Mathematics DOI:10.5402/2011/681274.
[8] J. E. Dunbar, T. W. Haynes, Domination in inflated graphs, Congr. Numer. 118 (1996) 143–154.
[9] O. Favaron, Irredundance in inflated graphs, J. Graph Theory 28 (1998) 97–104.
[10] O. Favaron, Infated graphs with equal independent number and upper irredundance number, Discrete Mathematics 236 (2001) 81–94.
[11] J. Puech, The lower irredundance and domination parameters are equal for inflated trees, J. Combin. Math. Combin. Comput. 33 (2000) 117–127.
[12] M. E. Watkins, A Theorem on Tait coloring with an application to the generalized Petersen graphs, J. Combin. Theory 6 (1969) 152–164.
[13] D. B. West, Introduction to Graph Theorey, (2nd edition), Prentice Hall, USA, 2001.