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Using linear operators to approximate signals of $Lip(\alpha, p), (p \geq 1)$-class

Vishnu Narayan Mishra[a], Kejal Khatri[a], Lakshmi Narayan Mishra[b]

[a]*Department of Mathematics, S.V. National Institute of Technology, Ichchhanath Mahadev Road, Surat, Surat-395007 (Gujarat) India*
[b]*Dr. Ram Manohar Lohia Avadh University, Hawai Patti Allahabad Road, Faizabad, Faizabad 224 001 (Uttar Pradesh), India*

Abstract. In this paper, we investigate trigonometric polynomials associated with $f \in Lip(\alpha, p), (0 \leq \alpha \leq 1, p \geq 1)$ to approximate f in L_p norm to the degree of $O(n^{-\alpha})(0 < \alpha \leq 1)$ for a more general class of lower triangular regular matrices with non-negative entries and row sums t_n.

1. Introduction

Let f be a 2Π-periodic signal and let $f \in L_p[0, 2\Pi] = L_p$ for $p \geq 1$. Then the Fourier series of function (signal) f at any point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cosx + b_k sinx) \equiv \sum_{k=0}^{\infty} A_k(f;x). \tag{1}$$

Denote by $s_n(f;x), n = 0, 1, ...$ the nth partial sums of the series (1) at the point x, that is, $s_n(f;x) = \sum_{k=0}^{\infty} A_k(f;x)$, a trigonometric polynomial of degree (or order) n,
where
$A_0(f;x) = \frac{a_0}{2}, A_k(f;x) = a_k cosx + b_k sinx, k = 1, 2, ...$

We define

$$\tau_n(f;x) = \sum_{k=0}^{\infty} a_{n,k} s_k(f;x) \ \ \forall n \geq 0, \tag{2}$$

where $T \equiv (a_{n,k})$ is a linear operator represented by a lower triangular regular matrix with non-negative entries and row sums t_n. The forward difference operator Δ is defined by $\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$. Such a matrix T is said to have monotone rows if, for each n, $\{a_{n,k}\}$ is either non-increasing or non- decreasing in k, $0 \leq k \leq n$. The series (2) is said to be $T-$ summable to s, if $\tau_n(f;x) \to s$ as $n \to \infty$.
The $T-$ operator reduces to the Nörlund (N_p)-operator, if

$$a_{n,k} = \begin{cases} \frac{p_{n-k}}{P_n}, & o \leq k \leq n, \\ 0, & k > n, \end{cases}$$

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Email addresses: vishnunarayanmishra@gmail.com (Vishnu Narayan Mishra), kejal0909@gmail.com (Kejal Khatri), lakshminarayanmishra04@gmail.com (Lakshmi Narayan Mishra)

where $P_n = \sum_{k=0}^{\infty} p_k \neq 0$ and $p_{-1} = 0 = P_{-1}$. In this case, the transform $\tau_n(f;x)$ reduces to the Nörlund transform $N_n(f;x)$.
The $T-$ operator reduces to the weighted (Riesz) $(\bar{N}_p)$- operator, if

$$a_{n,k} = \begin{cases} \frac{p_k}{P_n}, & o \leq k \leq n, \\ 0, & k > n, \end{cases}$$

where $P_n = \sum_{k=0}^{\infty} p_k \neq 0$ and $p_{-1} = 0 = P_{-1}$. In this case, the transform $\tau_n(f;x)$ reduces to the Nörlund transform $\bar{N}_n(f;x)$ (or $R_n(f;x)$).
A function (signal) $f \in Lip(\alpha, p)$ for $p \geq 1$, $0 \leq \alpha \leq 1$, if

$$\left(\int_0^{2\Pi} \left| f(x+t) - f(x) \right|^p dx \right)^{\frac{1}{p}} = O(t^\alpha).$$

The integral modulus of continuity of function $f \in L_p[0, 2\Pi]$ is defined by

$$\omega_p(\delta; f) = \sup_{0<|h|\leq\delta} \left(\frac{1}{2\Pi} \int_0^{2\Pi} \left| f(x+h) - f(x) \right|^p dx \right)^{\frac{1}{p}}.$$

If, for $\alpha > 0$,

$$\omega_p(\delta; f) = O(\delta^\alpha),$$

then $f \in Lip(\alpha, p)$, $(p \geq 1)$.
The L_p -norm of f is defined by

$$||f||_p = \left(\frac{1}{2\Pi} \int_0^{2\Pi} |f(x)|^p dx \right)^{\frac{1}{p}} \quad (f \in L_p(p \geq 1)).$$

Also

$$s_n(f) = \frac{1}{\Pi} \int_0^{2\Pi} f(x+t) D_n(t) dt, \quad \sigma_n(f;x) = \frac{1}{n+1} \sum_{m=0}^{n} s_m(f;x),$$

$$D_n(t) = \frac{sin(n+1/2)t}{2sin(t/2)},$$

the Dirichlet Kernel of degree (or order) n.
A positive sequence $c := \{c_n\}$ is called almost monotone decreasing (or increasing) if there exists a constant $K := K(c)$, depending on the sequence c only, such that for all $n \leq m$,

$$c_n \leq K c_m \quad (K c_n \geq c_m).$$

Such sequences will be denoted by $c \in AMDS$ and $c \in AMIS$, respectively. A sequence which is either AMDS or AMIS is called monotone and will be denoted by $c \in AMS$.

$$N_n(f;x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k(f;x), \quad P_n = \sum_{r=0}^{n} p_r \neq 0, p_{-1} = 0 = P_{-1}.$$

$$R_n(f;x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k(f;x), \quad P_n = \sum_{r=0}^{n} p_r \neq 0, p_{-1} = 0 = P_{-1}.$$

$$A_{n,k} = \sum_{r=k}^{n} a_{n,r}, \quad t_n = \sum_{k=0}^{n} a_{n,k} = A_{n,0}, \quad b_{n,k} = \frac{A_{n,k} - A_{n,0}}{k} \ \forall 1 \leq k \leq n,$$

$$\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}, \quad \Delta f_k g_k = g_k \Delta f_k + f_{k+1} \Delta g_k,$$

$[x]$- denotes the greatest integer not exceeding x.
A signal (function) f is approximated by trigonometric polynomials τ of order (or degree) n and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \min_n \|f(x) - \tau_n(f;x)\|_p.$$

This method of approximation is called trigonometric Fourier Approximation (TFA).
Let $\sigma_n(f)$ denote the nth term of the $(C, 1)$ transform of the partial sums of the Fourier series of a 2Π -periodic function f. The approximation properties of the Cesàro means $\sigma_n(f)$ in Lipschitz classes $Lip(\alpha, p)$ $1 \le p < \infty$, $0 < \alpha \le 1$ were investigated by Quade in [10]. He proved:

Theorem 1.1. ([10]) *If $f \in Lip(\alpha, p)$ for $0 < \alpha \le 1$, then*

$$\|f(x) - \sigma_n(f;x)\|_p = O(n^{-\alpha}). \tag{3}$$

for either (i) $p > 1$ and $0 < \alpha \le 1$ or (ii) $p = 1$ and $0 < \alpha < 1$.
and if $p = \alpha = 1$, then

$$\|f(x) - \sigma_n(f;x)\|_1 = O(n^{-1} log(n + 1)). \tag{4}$$

In the paper Chandra [1] extended the work of Quade [10], gave some conditions on the sequence $\{p_n\}_0^\infty$ and obtained very satisfactory results about approximation by the means $N_n(f)$ and $R_n(f)$ in $Lip(\alpha, p)$, $1 \le p < \infty$, $0 < \alpha \le 1$, where $N_n(f)$ and $R_n(f)$ denote the nth terms of the Nörlund and weighted mean transforms of the sequences of partial sums, respectively. He proved:

Theorem 1.2. ([1]) *Let $f \in Lip(\alpha, p)$ and let $\{p_n\}$ be a positive sequence such that*

$$(n + 1)p_n = O(P_n). \tag{5}$$

If either
(i) $p > 1$, $0 < \alpha \le 1$ and
(ii) $\{p_n\}$ is monotonic,
or
(i) $p = 1$, $0 < \alpha < 1$ and
(ii) $\{p_n\}$ is non-decreasing sequence,
then

$$\|f(x) - N_n(f;x)\|_p = O(n^{-\alpha}). \tag{6}$$

Theorem 1.3. ([1]) *Let $f \in Lip(\alpha, p)$ and let $\{p_n\}$ be positive. Suppose that either*
(i) $p > 1$, $0 < \alpha \le 1$, and
(ii) $\sum_{k=0}^{n-1} \left|\Delta\left(\frac{P_k}{k+1}\right)\right| = O\left(\frac{P_n}{n+1}\right)$,
or
(i) $p = 1$, $0 < \alpha < 1$ and
(ii) $\{p_n\}$ with (5) is positive and non-decreasing.
Then

$$\|f(x) - R_n(f;x)\|_p = O(n^{-\alpha}). \tag{7}$$

Later, Leindler in [2] weakened the conditions given by Chandra [1] on the generating sequence $\{p_n\}_0^\infty$ and generalized his results. He proved:

Theorem 1.4. ([2]) *Let $f \in Lip(\alpha, p)$ and let $\{p_n\}$ be positive. If one of the conditions*
(i) $p > 1$, $0 < \alpha < 1$, $\{p_n\} \in AMDS$,
(ii) $p > 1$, $0 < \alpha < 1$ and $\{p_n\} \in AMIS$ and (5) holds,
(iii) $p > 1$, $\alpha = 1$ and $\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$,
(iv) $p > 1$, $\alpha = 1$, $\sum_{k=0}^{n-1} |\Delta p_k| = O(P_n/n)$ and (5) holds,
(v) $p = 1$, $0 < \alpha < 1$, $\sum_{k=-1}^{n-1} |\Delta p_k| = O(P_n/n)$
maintains, then (6) holds.

Theorem 1.5. ([2]) *Let $f \in Lip(\alpha, 1)$, $0 < \alpha < 1$. If the positive sequence $\{p_n\}$ satisfies (5) and the condition $\sum_{k=0}^{n-1} |\Delta p_k| = O(P_n/n)$ holds, then (7) holds.*

Recently Mittal et al. [6] have generalized two theorems 1.2 and 1.3 (Chandra [1],Theorems 1.1 and 1.2) to more general classes of triangular matrix methods. They proved:

Theorem 1.6. ([6]) *Let $f \in Lip(\alpha, p)$ and let T have monotone rows and satisfy*

$$|t_n - 1| = O(n^{-\alpha}). \tag{8}$$

(i) If $p > 1$, $0 < \alpha < 1$, and T also satisfies

$$(n + 1)\ max\{a_{n,o}, a_{n,r}\} = O(1), \tag{9}$$

where $r := [n/2]$, then

$$\|\tau_n(f; x) - f(x)\|_p = O(n^{-\alpha}). \tag{10}$$

(ii) If $p > 1$, $\alpha = 1$ and T also satisfies (9), then (10) is satisfied.
(iii) If $p = 1$, $0 < \alpha < 1$, and T also satisfies

$$(n + 1)\ max\{a_{n,o}, a_{n,r}\} = O(1), \tag{11}$$

then (10) is satisfied.

2. Main result

It is well known that the theory of approximation i.e. TFA, which originated from a theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 131 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis [8], in general and in Digital Signal Processing [9] in particular, in view of the classical Shannon sampling theorem. Broadly speaking, signals are treated as function of one variable and images are represented by functions of two variable.
This has motivated Mishra [3], Mittal et al. ([5, 7]) and Mishra and Mishra [4] to obtain many interesting results on TFA using different summability matrices without monotone rows.
In this paper, we extend Theorems 1.4 and 1.5 of Leindler ([2], Theorems 1.1 and 1.2) to more general classes of triangular matrix methods with non-negative entries and row sums t_n. Our Theorem 2.1 also generalize partially Theorem 1.6 of Mittal et al. [6], by dropping monotonicity on the elements of the matrix rows (that is, weakening the conditions on the filter T, we improve the quality of the digital filter). We prove:

Theorem 2.1. *Let $f \in Lip(\alpha, p)$, and let T have monotone rows and satisfy*

$$|t_n - 1| = O(n^{-\alpha}). \tag{12}$$

(i) $p > 1$, $0 < \alpha < 1$, $\{a_{n,k}\} \in AMS$ in k and satisfies

$$(n + 1)\ max\{a_{n,o}, a_{n,r}\} = O(1), \tag{13}$$

where $r := [n/2]$, *then (10) is satisfied.*

$$(ii)\ p > 1, \alpha = 1 \text{ and } \sum_{k=0}^{n-1}(n-k)|\Delta_k a_{n,k}| = O(1), \text{ or} \tag{14}$$

$$(iii)\ p > 1, \alpha = 1 \text{ and } \sum_{k=0}^{n}|\Delta_k a_{n,k}| = O(a_{n,0}), \text{ or} \tag{15}$$

$$(iv)\ p = 1, 0 < \alpha < 1,\ \sum_{k=0}^{n}|\Delta_k a_{n,k}| = O(a_{n,0}), \tag{16}$$

$$\text{and also } (n+1)\max\{a_{n,o}, a_{n,n}\} = O(1), \text{ hold then } (10) \text{ is satisfied}. \tag{17}$$

It is easy to examine that the conditions of Theorems 2.1 claim less than the requirements of our Theorems 1.6 for $A_{n,0} = t_n$.
For example, the condition on the sum in (14) is always satisfied if the sequence $\{a_{n,k}\}$ is non-decreasing in k, then using (17), we get

$$\begin{aligned}\sum_{k=0}^{n-1}(n-k)|\Delta_k a_{n,k}| &= \sum_{k=0}^{n-1}(n-k)|a_{n,k} - a_{n,k+1}| = \sum_{k=0}^{n-1}(n-k)(a_{n,k+1} - a_{n,k}) \\ &= A_{n,o} - (n+1)a_{n,o} = t_n + O(1) = O(n^{-1}) + O(1) = O(1).\end{aligned}$$

If $\{a_{n,k}\}$ is non-increasing in k and (17) holds then

$$\sum_{k=0}^{n}|\Delta_k a_{n,k}| = \sum_{k=0}^{n}|a_{n,k} - a_{n,k+1}| = \sum_{k=0}^{n}(a_{n,k} - a_{n,k+1}) = a_{n,0} - a_{n,n+1} = a_{n,0}, \text{ as } a_{n+1,0} = 0,$$

is also true. Consequently Theorem 2.1 partially generalized Theorem 1.6.

3. Lemmas

We shall use the following lemmas in the proof of the theorem:

Lemma 3.1. ([10]) *If* $f \in Lip(1, p)$, $p > 1$ *then*

$$\|\sigma_n(f;x) - s_n(f;x)\|_p = O(n^{-1}). \tag{18}$$

Lemma 3.2. ([10]) *Let, for* $0 < \alpha \leq 1$ *and* $p > 1$, $f \in Lip(\alpha, p)$. *Then*

$$\|f(x) - s_n(f;x)\|_p = O(n^{-\alpha})\ \ \forall n > 0. \tag{19}$$

Lemma 3.3. *Let T have monotone rows and satisfy (9). Then, for* $0 < \alpha < 1$,

$$\sum_{k=1}^{n} a_{n,k}k^{-\alpha} = O(n^{-\alpha}) \tag{20}$$

Proof. Let $r := [n/2]$. Then, we have

$$\sum_{k=0}^{n} a_{n,k}k^{-\alpha} = \sum_{k=1}^{r} a_{n,k}k^{-\alpha} + \sum_{k=r+1}^{n} a_{n,k}k^{-\alpha}.$$

Case I. If $a_{n,k}$ is non-decreasing in k. Then, using (13), we get

$$\begin{aligned}\sum_{k=1}^{n} a_{n,k}k^{-\alpha} &\leq a_{n,r}\sum_{k=1}^{r} k^{-\alpha} + (r+1)^{-\alpha}\sum_{k=r+1}^{n} a_{n,k} \\ &\leq a_{n,r}\sum_{k=0}^{r} k^{-\alpha} + (r+1)^{-\alpha} t_n \\ &= O((n+1)^{-1})O(n^{1-\alpha}) + O(n^{-\alpha}) = O(n^{-\alpha}).\end{aligned}$$

Case II. If $a_{n,k}$ is non-increasing in k. Then, using (13), we get

$$\sum_{k=1}^{n} a_{n,k}k^{-\alpha} \leq a_{n,0}\sum_{k=1}^{r} k^{-\alpha} + O(n^{-\alpha}) = O(n^{-\alpha}).$$

□

4. Proof of Theorem 2.1

Proof. Case I. If $p > 1$, $0 < \alpha < 1$. The proof runs similar to the case (i) of Theorem 1.6 [6]. Let $a_{n,k}$ be AMS in k. Then

$$\tau_n(f;x) - f(x) = \sum_{k=0}^{n} a_{n,k}s_k(f;x) - t_n f(x) + (t_n - 1)f(x)$$

$$= \sum_{k=0}^{n} a_{n,k}(s_k(f;x) - f(x)) + (t_n - 1)f(x). \tag{21}$$

Using (12) and Lemmas (3.2) and (3.3),

$$\begin{aligned}\|\tau_n(f;x) - f(x)\|_p &\leq \sum_{k=0}^{n} a_{n,k}\|s_k(f;x) - f(x)\|_p + |(t_n - 1)|_p\|f(x)\|_p \\ &= \sum_{k=1}^{n} a_{n,k}O(k^{-\alpha}) + O(n^{-\alpha}) = O(n^{-\alpha}).\end{aligned}$$

Case III. If $p > 1$, $\alpha = 1$, we have

$$\tau_n(f;x) - f(x) = \tau_n(f;x) - s_n(f;x) + s_n(f;x) - f(x).$$

Now, using Lemma 3.2, we get

$$\|\tau_n(f;x) - f(x)\|_p \leq \|\tau_n(f;x) - s_n(f;x)\|_p + \|s_n(f;x) - f(x)\|_p$$

$$= \|\tau_n(f;x) - s_n(f;x)\|_p + O(n^{-1}). \tag{22}$$

Now to prove our theorem, it remains to show that

$$\|\tau_n(f;x) - s_n(f;x)\|_p = O(n^{-1}). \tag{23}$$

Now, we write

$$\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k}s_k(f;x) = \sum_{k=0}^{n} a_{n,k}\left(\sum_{i=0}^{k} u_i(f;x)\right) = \sum_{k=0}^{n} A_{n,k}u_k(f;x),$$

and thus, as $A_{n,o} = t_n$, we have

$$\tau_n(f;x) - s_n(f;x) = \sum_{k=1}^{n}(A_{n,k} - A_{n,0})u_k(f;x) + (t_n - 1)f(x).$$

Hence by Abeł's transformation, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{A_{n,k} - A_{n,0}}{k}\right)ku_k(f;x) &= \sum_{k=1}^{n} b_{n,k}ku_k(f;x)\\ &= \sum_{k=1}^{n-1}(\Delta_k b_{n,k})\left(\sum_{j=1}^{k} ju_j(f;x)\right) + b_{n,n}\sum_{j=1}^{n} ju_j(f;x).\end{aligned}$$

Thus by triangle inequality, we find

$$\begin{aligned}\|\tau_n(f;x) - f(x)\|_p &\le \sum_{k=1}^{n-1}|\Delta_k b_{n,k}|\left\|\sum_{j=1}^{k} ju_j(f;x)\right\|_p + |b_{n,n}|\left\|\sum_{j=1}^{n} ju_j(f;x)\right\|_p + |t_n - 1|\ \|f(x)\|_p\\ &= \sum_{k=1}^{n-1}|\Delta_k b_{n,k}|\left\|\sum_{j=1}^{k} ju_j(f;x)\right\|_p + |b_{n,n}|\left\|\sum_{j=1}^{n} ju_j(f;x)\right\|_p + O(n^{-1}). \qquad (24)\end{aligned}$$

Now

$$\sigma_n(f;x) - s_n(f;x) = \frac{1}{n+1}\sum_{m=0}^{n} s_m(f;x) - s_n(f;x) = \frac{1}{n+1}\sum_{m=0}^{n} s_m(f;x) - \sum_{k=0}^{k} u_k(f;x) = -\frac{1}{n+1}\sum_{j=1}^{n} ju_j(f;x).$$

Therefore by Lemma 3.1, we have

$$\left\|\sum_{j=1}^{n} ju_j(f;x)\right\|_p = (n+1)\|\sigma_n(f;x) - s_n(f;x)\|_p = (n+1)O(n^{-1}) = O(1). \qquad (25)$$

We note that

$$|b_{n,n}| = \left|\frac{A_{n,n} - A_{n,0}}{n}\right| = \frac{|A_{n,0} - A_{n,n}|}{n} = \frac{|t_n - a_{n,n}|}{n} = O(n^{-1}).$$

Thus

$$\left\|b_{n,n}\sum_{j=1}^{n} ju_j(f;x)\right\|_p = |b_{n,n}|\left\|\sum_{j=1}^{n} ju_j(f;x)\right\|_p = O(n^{-1}). \qquad (26)$$

As in case (ii) of proof of Theorem 1.6 ([6], p. 672), we write

$$\Delta_k b_{n,k} = \frac{1}{k(k+1)}\left[(k+1)a_{n,k} - \sum_{r=0}^{k} a_{n,r}\right]. \qquad (27)$$

Next we shall verify by mathematical induction that

$$\left|\sum_{r=0}^{k} a_{n,r} - (k+1)a_{n,k}\right| \le \sum_{r=0}^{k-1}(r+1)|a_{n,r} - a_{n,r+1}|. \qquad (28)$$

If $k = 1$, then

$$\left|\sum_{r=0}^{1} a_{n,r} - 2a_{n,1}\right| = |a_{n,0} - a_{n,1}|.$$

Thus (28) holds.
Now let us suppose that (28) holds for $k = m$ i.e.

$$\left|\sum_{r=0}^{m} a_{n,r} - (m+1)a_{n,k}\right| \leq \sum_{r=0}^{m-1}(r+1)|a_{n,r} - a_{n,r+1}|, \tag{29}$$

and we have to show that (28) is true for $k = m + 1$.
For $k = m + 1$ and using (29), we get

$$\begin{aligned}
\left|\sum_{r=0}^{m+1} a_{n,r} - (m+2)a_{n,m+1}\right| &= \left|\sum_{r=0}^{m} a_{n,r} - (m+1)a_{n,m+1}\right| \\
&= \left|\sum_{r=0}^{m} a_{n,r} - (m+1)a_{n,m} + (m+1)a_{n,m} - (m+1)a_{n,m+1}\right| \\
&\leq \sum_{r=0}^{m-1}(r+1)|a_{n,r} - a_{n,r+1}| + (m+1)|a_{n,m} - a_{n,m+1}| \\
&= \sum_{r=0}^{(m+1)-1} (r+1)|a_{n,r} - a_{n,r+1}|,
\end{aligned}$$

which shows that (28) is true for $k = m + 1$. Thus (28) holds good $\forall\ k \in N$.
Using (13), (17),(27) and (28), we find

$$\begin{aligned}
\sum_{k=1}^{n} |\Delta_k b_{n,k}| &= \sum_{k=1}^{n} \left|\Delta_k[k^{-1}(A_{n,k} - A_{n,0})]\right| = \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\left|(k+1)a_{n,k} - \sum_{r=0}^{k} a_{n,r}\right| \\
&= \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\left|\sum_{r=0}^{k} a_{n,r} - (k+1)a_{n,k}\right| \\
&\leq \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\sum_{r=0}^{k-1}(r+1)|a_{n,r} - a_{n,r+1}| = \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|a_{n,m-1} - a_{n,m}| \\
&\leq \sum_{m=1}^{n} m|\Delta_m a_{n,m-1}|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} = \sum_{k=0}^{n-1}|\Delta_k a_{n,k}| = O(a_{n,0}) = O(n^{-1}).
\end{aligned} \tag{30}$$

Combining (24), (25, (26) and (30) yields (23). From (23) and (22), we get

$$\|\tau_n(f;x) - f(x)\|_p = O(n^{-1}).$$

Case II. If $p > 1$, $\alpha = 1$. For this, we first prove that the condition

$$\sum_{k=0}^{n-1}(n-k)|\Delta_k a_{n,k}| = O(1) \Longrightarrow$$

$$B_n \equiv \sum_{k=1}^{n} |\Delta_k b_{n,k}| = \sum_{k=1}^{n} \left|\Delta_k\{k^{-1}(A_{n,k} - A_{n,0})\}\right| = O(n^{-1}). \tag{31}$$

For this, using (28) as in case (iii), we have

$$B_n = \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\left|(k+1)a_{n,k} - \sum_{r=0}^{k} a_{n,r}\right| \leq \sum_{k=1}^{n} k^{-1}(k+1)^{-1}\sum_{r=0}^{k-1}(r+1)|a_{n,r} - a_{n,r+1}|$$

$$\leq \left(\sum_{k=1}^{r} + \sum_{k=r}^{n}\right) k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|a_{n,m-1}| = B_1 + B_2,\ say. \tag{32}$$

Now, using (14) and interchanging the order of summation, we get

$$\begin{aligned}
B_1 &\equiv \sum_{k=1}^{r} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|\Delta_m a_{n,m-1}| \leq \sum_{m=1}^{r} m|\Delta_m a_{n,m-1}|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} \\
&= \sum_{m=1}^{r}|\Delta_m a_{n,m-1}| = \sum_{m=n-r+1}^{n}|\Delta_{n-m}a_{n,n-m}| \leq \sum_{m=r-1}^{n}|\Delta_{n-m}a_{n,n-m}|\left(\frac{m}{r-1}\right) \\
&\leq \frac{1}{r-1}\sum_{m=1}^{n} m|\Delta_{n-m}a_{n,n-m}| = \frac{1}{r-1}\sum_{k=0}^{n-1}(n-k)|\Delta_k a_{n,k}|
\end{aligned}$$

$$= \frac{1}{r-1}O(1) = O(n^{-1}). \tag{33}$$

On the other hand

$$\begin{aligned}
B_2 &\equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=1}^{k} m|\Delta_m a_{n,m-1}| \\
&\leq \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\left[\left(\sum_{m=1}^{r} + \sum_{m=r}^{k}\right) m|\Delta_m a_{n,m-1}|\right]
\end{aligned}$$

$$= B_{n,1} + B_{n,2}, say. \tag{34}$$

Using arguments as B_1 and (14), we obtain

$$\begin{aligned}
B_{n,1} &\equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=1}^{r} m|\Delta_m a_{n,m-1}| \\
&\leq \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=1}^{r}|\Delta_m a_{n,m-1}| = \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=n-r+1}^{n}|\Delta_{n-m}a_{n,n-m}| \\
&\leq \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=r-2}^{n}|\Delta_{n-m}a_{n,n-m}|\frac{m}{r-2} \\
&\leq \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}\sum_{m=1}^{n} m|\Delta_{n-m}a_{n,n-m}| \\
&= \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}\sum_{k=0}^{n-1}(n-k)|\Delta_k a_{n,k}|
\end{aligned}$$

$$= \frac{1}{r-2}\sum_{k=r}^{n}(k+1)^{-1}O(1) = O(1/n), \tag{35}$$

again using (14) and interchanging the order of summation, we have

$$
\begin{aligned}
B_{n,2} &\equiv \sum_{k=r}^{n} k^{-1}(k+1)^{-1}\sum_{m=r}^{k} m|\Delta_m a_{n,m-1}| \le \sum_{k=r}^{n}(k+1)^{-1}\sum_{m=r}^{k}|\Delta_m a_{n,m-1}| \\
&\le \frac{1}{r+1}\sum_{m=r}^{n}|\Delta_m a_{n,m-1}|\sum_{k=m}^{n}1 = \frac{1}{r+1}\sum_{m=r}^{n}(n-m+1)|\Delta_m a_{n,m-1}| \\
&= \frac{1}{r+1}\sum_{k=r-1}^{n-1}(n-k)|\Delta_k a_{n,k}| = \frac{1}{r+1}O(1) = O(n^{-1}). \qquad (36)
\end{aligned}
$$

From (32), (33), (34), (35) and (36), we get (31).
Thus (22), (24), (25), the estimate of $b_{n,n}$ and Lemma 3.2 again yield (10).
Case IV. If $p = 1$, $0 < \alpha < 1$, using (21), $a_{n,n+1} = 0$ and the Abel's transformation, we obtain

$$
\begin{aligned}
\tau_n(f;x) - f(x) &= \sum_{k=0}^{n} a_{n,k}(s_k(f;x) - f(x)) + (1-t_n)f(x) \\
&= \sum_{k=0}^{n-1}(\Delta_k a_{n,k})\left\{\sum_{r=0}^{k}(s_r(f;x) - f(x))\right\} + (a_{n,n} - a_{n,n+1})\sum_{r=0}^{n}(s_r(f;x) - f(x)) + (1-t_n)f(x) \\
&= \sum_{k=0}^{n}(\Delta_k a_{n,k})\left\{\sum_{r=0}^{k}(s_r(f;x) - f(x))\right\} + (1-t_n)f(x) \\
&= \sum_{k=0}^{n}(\Delta_k a_{n,k})(k+1)(\sigma_k(f;x) - f(x)) + (1-t_n)f(x).
\end{aligned}
$$

Hence, by condition (13), (16) and Lemma3.1, we find

$$
\begin{aligned}
\|\tau_n(f;x) - f(x)\|_1 &\le \sum_{k=0}^{n} k\,|\Delta_k a_{n,k}|\,\|(\sigma_k(f;x) - f(x))\|_1 + |(1-t_n)|\,\|f(x)\|_1 \\
&= O\left\{\sum_{k=0}^{n} k^{1-\alpha}\,|\Delta_k a_{n,k}|\right\} + O(n^{-\alpha}) = O(n^{1-\alpha})\sum_{k=0}^{n}|\Delta_k a_{n,k}| + O(n^{-\alpha}) \\
&= O(n^{1-\alpha})O(a_{n,0}) = O(n^{1-\alpha})O(n^{-1}) + O(n^{-\alpha}) = O(n^{-\alpha}).
\end{aligned}
$$

This completes the proof of case (iv) and consequently the proof of Theorem 2.1 is complete. □

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