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The maximal rank of a kind of partial banded block matrix subject to linear equations

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Abstract. We establish the formulas of the maximal rank of a 3×3 partial banded block matrix

$$\begin{pmatrix} M_{11} & M_{12} & X \\ M_{21} & Y & M_{23} \\ Z & M_{32} & M_{33} \end{pmatrix}$$

where X, Y, and Z are three variant quaternion matrices subject to linear matrix equations$A_1X = C_1,\ XB_1 = C_2,\ A_2Y = D_1,\ YB_2 = D_2,\ A_3Z = E_1,\ ZB_3 = E_2$. In order to demonstrate the feasibility of the result obtained, we present a necessary and sufficient condition for the solvability to the cubic system $A_1X = C_1,\ XB_1 = C_2,\ A_2Y = D_1,\ YB_2 = D_2,\ A_3Z = E_1,\ ZB_3 = E_2, XYZ = J$ over the quaternion algebra.

1. Introduction

Throughout this paper, we denote the real number field by $\mathbb{R}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0,\ a_1,\ a_2,\ a_3\ \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate dimension by I, the rank of matrix A by $r(A)$, and a reflexive inverse of matrix A over $\mathbb{H}$ by A^+ which satisfies simultaneously $AA^+A = A$, $A^+AA^+ = A^+$. Moreover, R_A and L_A stand for the two projectors $R_A = I - AA^+$, $L_A = I - A^+A$ induced by A, where A^+ is any but fixed reflexive inverse of A. Clearly, R_A and L_A are idempotent and one of its reflexive inverses is itself.

In matrix theory, the solvability of matrix equations is one of the important topics. In recent years, some authors ([2]-[9]) investigate the extremal ranks of the general solutions of systems subject to consistent

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systems and provide applications over the quaternion algebra $\mathbb{H}$. In this paper, We formulate the maximal rank of a kind of 3×3 partial banded block matrix

$$M(X,Y,Z)=\begin{bmatrix} M_{11} & M_{12} & X \\ M_{21} & Y & M_{23} \\ Z & M_{32} & M_{33} \end{bmatrix}, \tag{1.1}$$

where $M_{ij}[(i,j)=(1,1);(1,2);(2,1);(2,3);(3,2);(3,3)]\in\mathbb{H}^{m_i\times n_j}$ are known, $X\in\mathbb{H}^{m_1\times n_3}$, $Y\in\mathbb{H}^{m_2\times n_2}$ and $Z\in\mathbb{H}^{m_3\times n_1}$ are three independent variant matrices subject to consistent system

$$A_1X=C_1,\ XB_1=C_2,\ A_2Y=D_1,\ YB_2=D_2,\ A_3Z=E_1,\ ZB_3=E_2 \tag{1.2}$$

over $\mathbb{H}$. As an application of the result derived, in Section 3, we present a necessary and sufficient condition for the solvability to the cubic system

$$A_1X=C_1,\ XB_1=C_2,\ A_2Y=D_1,\ YB_2=D_2,\ A_3Z=E_1,\ ZB_3=E_2,\ XYZ=J \tag{1.3}$$

over $\mathbb{H}$ by the rank equalities. It demonstrates that the simple non-linear matrix equations (1.3) are solvable.

2. Maximal rank of (1.1) subject to system (1.2)

By Lemma 2.1 in [5], we can obtain the following lemma.

Lemma 2.1. *Let $A_1\in\mathbb{H}^{p_1\times m_1}, A_2\in\mathbb{H}^{p_2\times m_2}, A_3\in\mathbb{H}^{p_3\times m_3}, B_1\in\mathbb{H}^{n_3\times q_1}, B_2\in\mathbb{H}^{n_2\times q_2}, B_3\in\mathbb{H}^{n_1\times q_3}$, $C_1\in\mathbb{H}^{p_1\times n_3}, C_2\in\mathbb{H}^{m_1\times q_1}, D_1\in\mathbb{H}^{p_2\times n_2}, D_2\in\mathbb{H}^{m_2\times q_2}, E_1\in\mathbb{H}^{p_3\times n_1}, E_2\in\mathbb{H}^{m_3\times q_3}$ be known, and $X\in\mathbb{H}^{m_1\times n_3}, Y\in\mathbb{H}^{m_2\times n_2}, Z\in\mathbb{H}^{m_3\times n_1}$ be unknown. Then the following statements are equivalent:*
(1) *The system (1.2) is consistent.*
(2)

$$R_{A_1}C_1=0,\ C_2L_{B_1}=0,\ A_1C_2=C_1B_1.$$

$$R_{A_2}D_1=0,\ D_2L_{B_2}=0,\ A_2D_2=D_1B_2.$$

$$R_{A_3}E_1=0,\ E_2L_{B_3}=0,\ A_3E_2=E_1B_3.$$

(3)

$$A_1C_2=C_1B_1,\ r[A_1,C_1]=r(A_1),\ r\begin{bmatrix} B_1 \\ C_2 \end{bmatrix}=r(B_1).$$

$$A_2D_2=D_1B_2,\ r[A_2,D_1]=r(A_2),\ r\begin{bmatrix} B_2 \\ D_2 \end{bmatrix}=r(B_2).$$

$$A_3E_2=E_1B_3,\ r[A_3,E_1]=r(A_3),\ r\begin{bmatrix} B_3 \\ E_2 \end{bmatrix}=r(B_3).$$

In that case, the general solution of (1.2) can be expressed as

$$X=A_1^+C_1+L_{A_1}C_2B_1^+ +L_{A_1}UR_{B_1}, \tag{2.4}$$

$$Y=A_2^+D_1+L_{A_2}D_2B_2^+ +L_{A_2}VR_{B_2}, \tag{2.5}$$

$$Z=A_3^+E_1+L_{A_3}E_2B_3^+ +L_{A_3}WR_{B_3}, \tag{2.6}$$

where U, V,and W are arbitrary matrices over $\mathbb{H}$ with compatible dimensions.

Lemma 2.2. *(Lemma 2.4 in [6]) Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$, $E \in \mathbb{H}^{l\times i}$, $U \in \mathbb{H}^{p\times q}$,and $V \in \mathbb{H}^{q\times t}$. Then the following rank equalities are true:*
(a)

$$r(CL_A) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(A), \quad r[R_B A] = r\begin{bmatrix} B & A \end{bmatrix} - r(B),$$

(b)

$$r[B, AL_C] = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C), \quad r\begin{bmatrix} C \\ R_B A \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B),$$

(c)

$$r[R_B AL_C] = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C),$$

$$r\begin{bmatrix} A & BL_D \\ R_E C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E).$$

Lemma 2.2 below plays an important role in simplifying ranks of various kinds of block matrices. Tian in [10] has given the following Lemma over a field. The result can be generalized to $\mathbb{H}$.

Lemma 2.3. *Let $f(X_1, X_2, X_3) = A + B_1X_1C_1 + B_2X_2C_2 + B_3X_3C_3$ be a matrix expression over $\mathbb{H}$. Then the maximal rank of $f(X_1, X_2, X_3)$ can be shown as the following:*

$$\max_{X_1,X_2,X_3} r[f(X_1, X_2, X_3)] = \min\left\{ \begin{array}{c} r\begin{bmatrix} A & B_1 & B_2 & B_3 \end{bmatrix}, r\begin{bmatrix} A \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}, r\begin{bmatrix} A & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix}, \\ r\begin{bmatrix} A & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} \end{array} \right\}.$$

Now we consider the maximal rank of $M(X, Y, Z)$ subject to the consistent systems (1.2) and (1.3). For convenience of representation, the following notations are adopted:

$$\left\{ \begin{array}{l} J_1 = \{X \in \mathbb{H}^{m_1\times n_3} \,|A_1X = C_1,\ XB_1 = C_2\} \\ J_2 = \{Y \in \mathbb{H}^{m_2\times n_2} \,|A_2Y = D_1,\ YB_2 = D_2\} \\ J_3 = \{Z \in \mathbb{H}^{m_3\times n_1} \,|A_3Z = E_1,\ ZB_3 = E_2\} \end{array} \right\}. \tag{2.7}$$

Theorem 2.4. *The maximal rank of the matrix expression (1.1) subject to the consistent system (1.2) is given in the following:*

$$\max_{(X\in J_1,\ Y\in J_2,\ Z\in J_3)} r[M(X, Y, Z)] = \min\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\},$$

where

$$s_1 = r\begin{bmatrix} A_1M_{11} & A_1M_{12} & C_1 \\ A_2M_{21} & D_1 & A_2M_{23} \\ E_1 & A_3M_{32} & A_3M_{33} \end{bmatrix} - r(A_1) - r(A_2) - r(A_3) + m_1 + m_2 + m_3,$$

$$s_2 = r\begin{bmatrix} M_{11}B_3 & M_{12}B_2 & C_2 \\ M_{21}B_3 & D_2 & M_{23}B_1 \\ E_2 & M_{32}B_2 & M_{33}B_1 \end{bmatrix} - r(B_1) - r(B_2) - r(B_3) + n_1 + n_2 + n_3,$$

$$s_3 = r\begin{bmatrix} M_{11} & M_{12} & C_2 \\ A_2M_{21} & D_1 & A_2M_{23}B_1 \\ E_1 & A_3M_{32} & A_3M_{33}B_1 \end{bmatrix} - r(B_1) - r(A_2) - r(A_3) + m_2 + m_3 + n_3,$$

$$s_4 = r\begin{bmatrix} A_1M_{11} & A_1M_{12}B_2 & C_1 \\ M_{21} & D_2 & M_{23} \\ E_1 & A_3M_{32}B_2 & A_3M_{33} \end{bmatrix} - r(A_1) - r(B_2) - r(A_3) + m_1 + n_2 + m_3,$$

$$s_5 = r\begin{bmatrix} A_1M_{11}B_3 & A_1M_{12} & C_1 \\ A_2M_{21}B_3 & D_1 & A_2M_{23} \\ E_2 & M_{32} & M_{33} \end{bmatrix} - r(A_1) - r(A_2) - r(B_3) + m_1 + m_2 + n_1,$$

$$s_6 = r\begin{bmatrix} A_1M_{11}B_3 & A_1M_{12}B_2 & C_1 \\ M_{21}B_3 & D_2 & M_{23} \\ E_2 & M_{32}B_2 & M_{33} \end{bmatrix} - r(A_1) - r(B_2) - r(B_3) + m_1 + n_1 + n_2,$$

$$s_7 = r\begin{bmatrix} M_{11}B_3 & M_{12} & C_2 \\ A_2M_{21}B_3 & D_1 & A_2M_{23}B_1 \\ E_2 & M_{32} & M_{33}B_1 \end{bmatrix} - r(B_1) - r(A_2) - r(B_3) + m_2 + n_1 + n_3,$$

$$\text{and } s_8 = r\begin{bmatrix} M_{11} & M_{12}B_2 & C_2 \\ M_{21} & D_2 & M_{23}B_1 \\ E_1 & A_3M_{32}B_2 & A_3M_{33}B_1 \end{bmatrix} - r(B_1) - r(B_2) - r(A_3) + m_3 + n_2 + n_3.$$

Proof. Substituting (2.4), (2.5) and (2.6) into (1.1)

$$M(X,Y,Z) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & 0 & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} Y \begin{bmatrix} 0 & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} Z \begin{bmatrix} I & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} & M_{12} & A_1^+C_1 + L_{A_1}C_2B_1^+ \\ M_{21} & A_2^+D_1 + L_{A_2}D_2B_2^+ & M_{23} \\ A_3^+E_1 + L_{A_3}E_2B_3^+ & M_{32} & M_{33} \end{bmatrix}$$
$$+ \begin{bmatrix} L_{A_1} \\ 0 \\ 0 \end{bmatrix} U \begin{bmatrix} 0 & 0 & R_{B_1} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{A_2} \\ 0 \end{bmatrix} V \begin{bmatrix} 0 & R_{B_2} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{A_3} \end{bmatrix} W \begin{bmatrix} R_{B_3} & 0 & 0 \end{bmatrix}.$$

Let

$$\begin{bmatrix} M_{11} & M_{12} & A_1^+C_1 + L_{A_1}C_2B_1^+ \\ M_{21} & A_2^+D_1 + L_{A_2}D_2B_2^+ & M_{23} \\ A_3^+E_1 + L_{A_3}E_2B_3^+ & M_{32} & M_{33} \end{bmatrix} = \widehat{A},$$

$$\begin{bmatrix} L_{A_1} \\ 0 \\ 0 \end{bmatrix} = \widehat{B_1}, \begin{bmatrix} 0 \\ L_{A_2} \\ 0 \end{bmatrix} = \widehat{B_2}, \begin{bmatrix} 0 \\ 0 \\ L_{A_3} \end{bmatrix} = \widehat{B_3},$$

$$\begin{bmatrix} 0 & 0 & R_{B_1} \end{bmatrix} = \widehat{C_1}, \begin{bmatrix} 0 & R_{B_2} & 0 \end{bmatrix} = \widehat{C_2}, \text{ and } \begin{bmatrix} R_{B_3} & 0 & 0 \end{bmatrix} = \widehat{C_3}.$$

Then

$$\max_{(X\in J_1,\, Y\in J_2,\, Z\in J_3)} r[M(X,Y,Z)] = \max_{U,V,W} r\left[\widehat{A} + \widehat{B_1}U\widehat{C_1} + \widehat{B_2}V\widehat{C_2} + \widehat{B_3}W\widehat{C_3}\right]$$

$$= \min\left\{ \begin{array}{c} r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_2} & \widehat{B_3} \end{bmatrix}, r\begin{bmatrix} \widehat{A} \\ \widehat{C_1} \\ \widehat{C_2} \\ \widehat{C_3} \end{bmatrix}, r\begin{bmatrix} \widehat{A} & \widehat{B_2} & \widehat{B_3} \\ \widehat{C_1} & 0 & 0 \end{bmatrix}, r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_3} \\ \widehat{C_2} & 0 & 0 \end{bmatrix}, \\ r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_2} \\ \widehat{C_3} & 0 & 0 \end{bmatrix}, r\begin{bmatrix} \widehat{A} & \widehat{B_1} \\ \widehat{C_2} & 0 \\ \widehat{C_3} & 0 \end{bmatrix}, r\begin{bmatrix} \widehat{A} & \widehat{B_2} \\ \widehat{C_1} & 0 \\ \widehat{C_3} & 0 \end{bmatrix}, r\begin{bmatrix} \widehat{A} & \widehat{B_3} \\ \widehat{C_1} & 0 \\ \widehat{C_2} & 0 \end{bmatrix} \end{array} \right\},$$

where

$$r\begin{bmatrix} \widehat{A} & \widehat{B_3} \\ \widehat{C_1} & 0 \\ \widehat{C_2} & 0 \end{bmatrix} = r\begin{bmatrix} M_{11} & M_{12} & A_1^+C_1 + L_{A_1}C_2B_1^+ & 0 \\ M_{21} & A_2^+D_1 + L_{A_2}D_2B_2^+ & M_{23} & 0 \\ A_3^+E_1 + L_{A_3}E_2B_3^+ & M_{32} & M_{33} & L_{A_3} \\ 0 & 0 & R_{B_1} & 0 \\ 0 & R_{B_2} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} M_{11} & M_{12} & A_1^+C_1 + L_{A_1}C_2B_1^+ & 0 & 0 & 0 \\ M_{21} & A_2^+D_1 + L_{A_2}D_2B_2^+ & M_{23} & 0 & 0 & 0 \\ A_3^+E_1 + L_{A_3}E_2B_3^+ & M_{32} & M_{33} & I_{m_3} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 & B_1 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & A_3 & 0 & 0 \end{bmatrix} - r(B_1) - r(B_2) - r(A_3)$$

$$= r\begin{bmatrix} M_{11} & 0 & 0 & 0 & -\left(A_1^+C_1 + L_{A_1}C_2B_1^+\right)B_1 & -M_{12}B_2 \\ M_{21} & 0 & 0 & 0 & -M_{23}B_1 & -\left(A_2^+D_1 + L_{A_2}D_2B_2^+\right)B_2 \\ 0 & 0 & 0 & I_{m_3} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 & B_1 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 & B_2 \\ -A_3\left(A_3^+E_1 + L_{A_3}E_2B_3^+\right) & 0 & 0 & A_3 & A_3M_{33}B_1 & A_3M_{32}B_2 \end{bmatrix}$$
$$- r(B_1) - r(B_2) - r(A_3).$$

Note that

$$\left(A_1^+C_1 + L_{A_1}C_2B_1^+\right)B_1 = C_2,$$

$$\left(A_2^+D_1 + L_{A_2}D_2B_2^+\right)B_2 = D_2,$$

$$A_3\left(A_3^+E_1 + L_{A_3}E_2B_3^+\right) = E_1,$$

so

$$r\begin{bmatrix} \widehat{A} & \widehat{B_3} \\ \widehat{C_1} & 0 \\ \widehat{C_2} & 0 \end{bmatrix} = s_8.$$

Similarly,

$$r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_2} & \widehat{B_3} \end{bmatrix} = s_1,\ r\begin{bmatrix} \widehat{A} \\ \widehat{C_1} \\ \widehat{C_2} \\ \widehat{C_3} \end{bmatrix} = s_2,\ r\begin{bmatrix} \widehat{A} & \widehat{B_2} & \widehat{B_3} \\ \widehat{C_1} & 0 & 0 \end{bmatrix} = s_3,\ r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_3} \\ \widehat{C_2} & 0 & 0 \end{bmatrix} = s_4,$$

$$r\begin{bmatrix} \widehat{A} & \widehat{B_1} & \widehat{B_2} \\ \widehat{C_3} & 0 & 0 \end{bmatrix} = s_5,\ r\begin{bmatrix} \widehat{A} & \widehat{B_1} \\ \widehat{C_2} & 0 \\ \widehat{C_3} & 0 \end{bmatrix} = s_6,\ r\begin{bmatrix} \widehat{A} & \widehat{B_2} \\ \widehat{C_1} & 0 \\ \widehat{C_3} & 0 \end{bmatrix} = s_7.$$

This completes the proof of Theorem 2.4. □

3. Some solvability conditions of system (1.3)

In order to demonstrate the power of Theorem 2.4, in this section, we apply it to the solvability conditions to cubic matrix equations (1.3).

Theorem 3.1. *Let the linear matrix equations (1.2) be consistent and the solution set of (1.2) be (2.7). Then only one of following conclusions*

$$\left\{ r\begin{bmatrix} A_1J & 0 & C_1 \\ 0 & D_1 & -A_2 \\ E_1 & A_3 & 0 \end{bmatrix} = r[A_2] + r[A_3],\ r[A_1] = m_1 \right\}, \tag{3.8}$$

$$\left\{ r\begin{bmatrix} JB_3 & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_2 & B_2 & 0 \end{bmatrix} = r[B_1] + r[B_2],\ r[B_3] = n_1 \right\}, \tag{3.9}$$

$$\left\{ r\begin{bmatrix} J & 0 & C_2 \\ 0 & D_1 & -A_2B_1 \\ E_1 & A_3 & 0 \end{bmatrix} = r[A_2] + r[A_3],\ r[B_1] = n_3 \right\}, \tag{3.10}$$

$$\left\{ r\begin{bmatrix} A_1J & C_1D_2 \\ E_1 & A_3B_2 \end{bmatrix} = r[B_2],\ r[A_1] = m_1,\ r[A_3] = m_3 \right\}, \tag{3.11}$$

$$\left\{ r\begin{bmatrix} A_1JB_2 & C_1 \\ D_1E_2 & A_2 \end{bmatrix} = r[A_2],\ r[A_1] = m_1,\ r[B_3] = n_1 \right\}, \tag{3.12}$$

$$\left\{ r\begin{bmatrix} A_1JB_3 & C_1D_2 \\ E_2 & B_2 \end{bmatrix} = r[B_2],\ r[A_1] = m_1,\ r[B_3] = n_1 \right\}, \tag{3.13}$$

$$\left\{ r\begin{bmatrix} JB_3 & C_2 \\ D_1E_2 & A_2B_1 \end{bmatrix} = r[B_1],\ r[A_2] = m_2,\ r[B_3] = n_1 \right\}, \tag{3.14}$$

$$or\ \left\{ r\begin{bmatrix} J & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_1 & A_3B_2 & 0 \end{bmatrix} = r[B_1] + r[B_2],\ r[A_3] = m_3 \right\} \tag{3.15}$$

is true, the cubic matrix equations (1.3) are consistent too, and the solution set of (1.3) is equal to that of (1.2).

Proof. Use the following rank formula

$$r(J - XYZ) = r\begin{bmatrix} J & 0 & X \\ 0 & Y & -I_{n_3} \\ Z & I_{m_3} & 0 \end{bmatrix} - m_3 - n_3.$$

Let M_{11} in Theorem 2.4 be replaced by J, M_{23} by $-I_{n_3}$, M_{32} by I_{m_3}, M_{12}, M_{21}, M_{33} by zero matrices with dimensions, $m_2 = n_3, n_2 = m_3$ respectively. According to Theorem 2.4, we can obtain the following equality:

$$\max_{(X\in J_1,\, Y\in J_2,\, Z\in J_3)} r(J - XYZ) = \min\left\{ \begin{array}{l} r\begin{bmatrix} A_1J & 0 & C_1 \\ 0 & D_1 & -A_2 \\ E_1 & A_3 & 0 \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1, \\ r\begin{bmatrix} JB_3 & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_2 & B_2 & 0 \end{bmatrix} - r[B_1] - r[B_2] - r[B_3] + n_1, \\ r\begin{bmatrix} J & 0 & C_2 \\ 0 & D_1 & -A_2B_1 \\ E_1 & A_3 & 0 \end{bmatrix} - r[B_1] - r[A_2] - r[A_3] + m_2, \\ r\begin{bmatrix} A_1J & C_1D_2 \\ E_1 & A_3B_2 \end{bmatrix} - r[A_1] - r[B_2] - r[A_3] + m_1 + m_3, \\ r\begin{bmatrix} A_1JB_3 & C_1 \\ D_1E_2 & A_2 \end{bmatrix} - r[A_1] - r[A_2] - r[B_3] + m_1 + n_1, \\ r\begin{bmatrix} A_1JB_3 & C_1D_2 \\ E_2 & B_2 \end{bmatrix} - r[A_1] - r[B_2] - r[B_3] + m_1 + n_1, \\ r\begin{bmatrix} JB_3 & C_2 \\ D_1E_2 & A_2B_1 \end{bmatrix} - r[B_1] - r[A_2] - r[B_3] + m_2 + n_1, \\ r\begin{bmatrix} J & 0 & C_2 \\ 0 & D_2 & -B_1 \\ E_1 & A_3B_2 & 0 \end{bmatrix} - r[B_1] - r[B_2] - r[A_3] + m_3 \end{array} \right\}.$$

It is obvious that

$$\max_{(X\in J_1,\, Y\in J_2,\, Z\in J_3)} r(J - XYZ) \geq 0.$$

If one of the conditions from (3.8) to (3.15) is satisfied, then

$$\max_{(X\in J_1,\, Y\in J_2,\, Z\in J_3)} r(J - XYZ) = 0.$$

Conversely, assume that

$$r\begin{bmatrix} A_1J & 0 & C_1 \\ 0 & D_1 & -A_2 \\ E_1 & A_3 & 0 \end{bmatrix} - r[A_1] - r[A_2] - r[A_3] + m_1 = 0.$$

Note that

$$r\begin{bmatrix} A_1J & 0 & C_1 \\ 0 & D_1 & -A_2 \\ E_1 & A_3 & 0 \end{bmatrix} \geq r[A_2] + r[A_3]$$

and

$$r[A_1] \leq m_1.$$

Therefore, the condition (3.8) is satisfied.
Similarly, assume that one of conditions from (3.9) to (3.15) is satisfied, then

$$\max_{(X\in J_1,\, Y\in J_2,\, Z\in J_3)} r(J - XYZ) = 0.$$

This corresponds to

$$\underset{(X\in J_1,\, Y\in J_2,\, Z\in J_3)}{XYZ} \equiv J.$$

It demonstrates that the cubic matrix equations (1.3) are consistent, and the solution set of (1.3) covers that of (2.7). On the other hand, the cubic matrix equations (1.3) contain the linear matrix equations (1.2), and the solution set of (1.3) covers that of (2.7), Therefore, the solution set of (1.3) is equal to that of (2.7). □

4. Conclusion

In this paper, we derive the maximal rank of a kind of 3×3 partial banded block variant matrix (1.1) subject to the consistent system (1.2). Moreover, in order to demonstrate the feasibility of the result obtained, Theorem 2.4 is applied to derive the necessary and sufficient conditions for the solvability to the cubic system (1.3) by the rank equalities. It means the non-linear matrix equations (1.3) are solvable.

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References

[1] T.W. Hungerford, Algebra, Spring-Verlag, New York Inc., 1980.
[2] S. K. Mitra, The matrix equations $AX = C$, $XB = D$, Linear Algebra Appl. 59 (1984) 171-181.
[3] N. Cohen, C.R. Johnson, L.Rodman and H.J.Woerdeman, Ranks of completions of partial matrices, Oper.Theory: Adv.Appl.40(1989),165-185.
[4] H.J. Woerdeman, Minimal rank completions of partial banded matrices, *Linear Multilinear Algebra.* 36 (1993): 59-69.
[5] Q.W. Wang, S.W. Yu, C.Y. Lin, Extreme ranks of a linear quaternion matrix expression subject to triple quaternion matrix equations with applications, *Appl. Math. Comput*, 195 (2008): 733-744.
[6] Q.W. Wang, G.J. Song, C.Y. Lin, Extreme ranks of the solution to a consistent system of linear quaternion matrix equations with an application, *Appl. Math. Comput.* 189 (2007): 1517-1532.
[7] Q.W. Wang, The general solution to a system of real quaternion matrix equations, *Computers and Mathematics with Applications* 49 (2005): 665–675.
[8] Q.W. Wang, A System of Four Matrix Equations over von Neumann Regular Rings and Its Applications, *Acta Math. Sinica, English Series* 21(2) (2005): 323-334.
[9] C.Y. Lin, Q.W. Wang, The minimal and maximal ranks of the general solution to a system of matrix equations over an arbitrary division ring, *Math. Sci. Res. J. 10 (3)* (2006): 57-65.
[10] Y. Tian, Completing block matrices with maximal and minimal ranks, *Linear Algebra Appl.* 321 (2000): 327-325.
[11] Q.W. Wang, H.S. Zhang, S.W. Yu, On the real and pure imaginary solutions to the quaternion matrix equations $AXB + CYD = E$, Electronic Journal of Linear Algebra. **17** (2008), 343–358.
[12] Q.W. Wang, G.J. Song, Maximal and minimal ranks of the common solution of some linear matrix equations over an arbitrary division ring with applications, Algebra Colloquium, **16(2)** (2009), 293–308.
[13] Q.W. Wang, C.K. Li, Ranks and the least-norm of the general solution to a system of quaternion matrix equations, Linear Algebra and Its Applications, **430** (2009), 1626–1640.
[14] Q.W. Wang, S.W. Yu, The real solution to a system of quaternion matrix equations with applications, Communications in Algebra, 37(6) (2009), 2060–2079.
[15] Q.W. Wang, H.S. Zhang, G.J. Song, A new solvable condition for a pair of the generalized Sylvester equations, Electronic Journal of Linear Algebra. **18** (2009), 289–301.
[16] Q.W. Wang, S.W. Yu, Extreme ranks of real matrices in solution of the quaternion matrix equations $AXB = C$ with applications, Algebra Colloquium, **17(2)** (2010), 345–360.
[17] I. A. Khan, Q.W. Wang, G.J. Song, Minimal ranks of some quaternion matrix expressions with applications, Applied Mathematies and Computation, **217** (2010), 2031–2040.
[18] Q.W. Wang, Jing. Jiang, Extreme ranks of (skew-)Hermitian solutions to a quaternion matrix equation, Electronic Journal of Linear Algebra, **20** (2010), 552–573.

[19] Q. Zhang, Q.W. Wang, The {P,Q}-(skew)Symmetric extremal rank solutions to a system of quaternion matrix equations, Applied Mathematies and Computation, **217** (2011), 9286–9296.
[20] Q.W. Wang, Y. Zhou, Q. Zhang, Ranks of the common solution to six quaternion matrix equation, Acta Math. Applic. Sinica (English Series). **27**(3) (2011), 443–462.
[21] Q.W. Wang, X. Liu, S.W. Yu, The common bisymmetric nonnegative definite solutions with extreme ranks and inertias to a pair of matrix equations, Appl. Math. Comput. **218**(6) (2011), 2761–2771.
[22] X. Zhang, Q.W. Wang, X. Liu, Inertias and ranks of some Hermitian matrix functions with applications, Cent. Eur. J. Math. **10**(1) (2012), 329–351.
[23] Q.W. Wang, X. Zhang, J.W.V.D. Woude, A new simultaneous decomposition of a matrix quaternity over an arbitrary ring with applications, Communications in Algebra, **40** (2012), 2309–2342.
[24] Q.W. Wang, Z.H. He, Some matrix equations with applications, Linear and Multilinear Algebra, DOI:10.1018/03081087.2011.648635.
[25] Q.W. Wang, X. Zhang, Z.H. He, On the Hermitian structures of the solutions to a pair of matrix equations, Linear and Multilinear Algebra, DOI:10.1018/03081087.2012.663370.
[26] Z.H. He, Q.W. Wang, A real quaternion matrix equation with applications, Linear and Multilinear Algebra, DOI:10.1018/03081087.2012.703129.