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On Weighted Slant Hankel operators

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Abstract. In this paper, we introduce and study the notion of weighted slant Hankel operator K_ϕ^β, $\phi \in L^\infty(\beta)$ on the space $L^2(\beta)$, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being a sequence of positive numbers with $\beta_0 = 1$. In addition to some algebraic properties, the commutant and the compactness of these operators are discussed.

1. Preliminaries and Introduction

Laurent operators [8] or multiplication operators $M_\phi(f \mapsto \phi f)$ on $L^2(\mathbb{T})$ induced by $\phi \in L^\infty(\mathbb{T})$, $\mathbb{T}$ being the unit circle, play a vital role in the theory of operators with their tendency of inducing various classes of operators. In the year 1911, O. Toeplitz [15] introduced the Toeplitz operators given as $T_\phi = PM_\phi$, where P is an orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ and later in 1964, Brown and Halmos [4] studied algebraic properties of these operators. We refer [1, 2, 4, 7, 9 and 12] for the applications and extensions of study to Hankel operators, slant Toeplitz operators, slant Hankel operators and k^{th}-order slant Hankel operators. In the mean time, the notions of weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ also gained momentum. Shield [14] made a systematic study of the Laurent operators on $L^2(\beta)$. We prefer to call the Laurent operator on $L^2(\beta)$ as weighted Laurent operator. Weighted Toeplitz operators, Slant weighted Toeplitz operators and weighted Hankel operaots on $L^2(\beta)$ are discussed in [10], [3] and [5, 6] respectively. In this paper, we extend the study to a new class of operators namely, weighted slant Hankel operators and describe its algebraic properties. We now begin with the notations and preliminaries that are needed in the paper.

We consider the space $L^2(\beta)$ of all formal Laurent series $f(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$, $a_n \in \mathbb{C}$, (whether or not the series converges for any value of z) for which

$$\|f\|_\beta^2 = \sum_{n\in\mathbb{Z}} |a_n|^2 {\beta_n}^2 < \infty,$$

where $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of positive numbers with $\beta_0 = 1$, $r \le \frac{\beta_n}{\beta_{n+1}} \le 1$ for $n \ge 0$ and $r \le \frac{\beta_n}{\beta_{n-1}} \le 1$ for $n \le 0$, for some $r > 0$.

$L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\left\langle f, g \right\rangle = \sum_{n\in\mathbb{Z}} a_n\, \bar{b}_n {\beta_n}^2,$$

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for $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n{}^2 < \infty$, denoted by $H^2(\beta)$ and is a subspace of $L^2(\beta)$.

The symbol $L^\infty(\beta)$ denotes the set of formal Laurent series $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \le c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \le c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. Also, $L^\infty(\beta) \subseteq L^2(\beta)$. By $H^\infty(\beta)$ we mean the set of all formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. We refer [14] as well as the references therein, for the details of the spaces $L^2(\beta)$, $H^2(\beta)$, $L^\infty(\beta)$ and $H^\infty(\beta)$. If $\beta_n = 1$ for each $n \in \mathbb{Z}$, and the functions under consideration are complex-valued measurable functions defined over the unit cicle $\mathbb{T}$ then these spaces coincide with the classical spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$, $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$. We reserve the symbols $\overline{\phi}, \phi^*$ and $\widetilde{\phi}$ corresponding to $\phi \in L^\infty(\beta)$ given by $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, $a_n \in \mathbb{C}$, to represent the expressions $\overline{\phi}(z) = \sum_{n\in\mathbb{Z}} \overline{a}_n z^{-n}$, $\phi^*(z) = \sum_{n\in\mathbb{Z}} \overline{a}_n z^n$ and $\widetilde{\phi}(z) = \sum_{n\in\mathbb{Z}} a_{-n} z^n$.

Let W be an operator on $L^2(\beta)$ given by

$$We_n(z) = \begin{cases} \frac{\beta_m}{\beta_{2m}} e_m(z) & \text{if } n = 2m \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and J^β denote the reflection operator on $L^2(\beta)$ defined as $J^\beta f = \sum_{n\in\mathbb{Z}} a_n \beta_n e_{-n}$ for each $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ in $L^2(\beta)$. It is easy to see the following.

1. $J^\beta e_n = e_{-n}$ for each $n \in \mathbb{Z}$.
2. $\|W\| = 1 = \|J^\beta\|$.
3. If the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual i.e. $\beta_n = \beta_{-n}$ for each n, then $WJ^\beta = J^\beta W$.
4. $J^{\beta*} = J^\beta$ and $J^{\beta^2} = I$, the identity operator on $L^2(\beta)$.

We recall the definitions of weighted Hankel operators on $H^2(\beta)$ and $L^2(\beta)$.

Definition 1.1 ([5]). *The weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is defined as $H_\phi^\beta = P^\beta J^\beta M_\phi^\beta$, where P^β denotes the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ and M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$ induced by $\phi \in L^\infty(\beta)$.*

Definition 1.2 ([6]). *A weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is given by $S_\phi^\beta = J^\beta M_\phi^\beta$.*

It is easily seen that $\|S_\phi^\beta\| = \|M_\phi^\beta\|$ and $H_\phi^\beta = P^\beta S_\phi^\beta|_{H^2(\beta)}$. Throughout the paper, we use the notions M_ϕ^β and S_ϕ^β to represent the weighted Laurent and weighted Hankel operators on $L^2(\beta)$ induced by the symbol ϕ respectively.

2. Weighted Slant Hankel Operator

For $\phi \in L^\infty(\beta)$ with formal Laurent series expression $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, the weighted Hankel operator H_ϕ^β on $H^2(\beta)$ is also given by

$$H_\phi^\beta e_j = \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{-n-j} \beta_{-n} e_n, \quad j \ge 0.$$

and similarly the weighted Hankel operator S_ϕ^β on $L^2(\beta)$ is given by

$$S_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n\in\mathbb{Z}} a_{-n-j}\beta_{-n}e_n$$

for $j \in \mathbb{Z}$.

Now we introduce the following notion.

Definition 2.1. *The weighted slant Hankel operator K_ϕ^β induced by $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ in $L^\infty(\beta)$, is an operator on $L^2(\beta)$ defined as, for each $j \in \mathbb{Z}$,*

$$K_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n\in\mathbb{Z}} a_{-2n-j}\beta_{-n}e_n.$$

The matrix representation of K_ϕ^β with respect to the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ is

$$\left[\begin{array}{cccc|cccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & a_6\frac{\beta_2}{\beta_{-2}} & a_5\frac{\beta_2}{\beta_{-1}} & a_4\frac{\beta_2}{\beta_0} & a_3\frac{\beta_2}{\beta_1} & a_2\frac{\beta_2}{\beta_2} & a_1\frac{\beta_2}{\beta_3} & \cdots \\
\cdots & a_4\frac{\beta_1}{\beta_{-2}} & a_3\frac{\beta_1}{\beta_{-1}} & a_2\frac{\beta_1}{\beta_0} & a_1\frac{\beta_1}{\beta_1} & a_0\frac{\beta_1}{\beta_2} & a_{-1}\frac{\beta_1}{\beta_3} & \cdots \\
\cdots & a_2\frac{\beta_0}{\beta_{-2}} & a_1\frac{\beta_0}{\beta_{-1}} & a_0\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & a_{-3}\frac{\beta_0}{\beta_3} & \cdots \\
\hline
\cdots & a_0\frac{\beta_{-1}}{\beta_{-2}} & a_{-1}\frac{\beta_{-1}}{\beta_{-1}} & a_{-2}\frac{\beta_{-1}}{\beta_0} & a_{-3}\frac{\beta_{-1}}{\beta_1} & a_{-4}\frac{\beta_{-1}}{\beta_2} & a_{-5}\frac{\beta_{-1}}{\beta_3} & \cdots \\
\cdots & a_{-2}\frac{\beta_{-2}}{\beta_{-2}} & a_{-3}\frac{\beta_{-2}}{\beta_{-1}} & a_{-4}\frac{\beta_{-2}}{\beta_0} & a_{-5}\frac{\beta_{-2}}{\beta_1} & a_{-6}\frac{\beta_{-2}}{\beta_2} & a_{-7}\frac{\beta_{-2}}{\beta_3} & \cdots \\
\cdots & a_{-4}\frac{\beta_{-3}}{\beta_{-2}} & a_{-5}\frac{\beta_{-3}}{\beta_{-1}} & a_{-6}\frac{\beta_{-3}}{\beta_0} & a_{-7}\frac{\beta_{-3}}{\beta_1} & a_{-8}\frac{\beta_{-3}}{\beta_2} & a_{-9}\frac{\beta_{-3}}{\beta_3} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]$$

If we take $\phi(z) = 1$ then the corresponding weighted slant Hankel mapping is denoted by K_1^β. It is interesting to see that $J^\beta W = K_1^\beta$, as for each integer k, we have

$$\begin{aligned}
K_1^\beta e_{2k}(z) &= \frac{1}{\beta_{2k}}\sum_{n\in\mathbb{Z}} a_{-2n-2k}\beta_{-n}e_n(z) \\
&= \frac{\beta_k}{\beta_{2k}}e_{-k}(z) = J^\beta W e_{2k}(z)
\end{aligned}$$

and

$$K_1^\beta e_{2k-1}(z) = 0 = J^\beta W e_{2k-1}(z).$$

In order to justify the boundedness of the weighted slant Hankel operator, we need the following lemma.

Lemma 2.2. *For any $\phi \in L^\infty(\beta)$, $K_\phi^\beta = J^\beta W M_\phi^\beta = K_1^\beta M_\phi^\beta$.*

Proof. It again follows with a straightforward computation that

$$\begin{aligned}
J^\beta W M_\phi^\beta e_k(z) &= J^\beta W\Big(\frac{1}{\beta_k}\sum_{n\in\mathbb{Z}} a_{n-k}\beta_n e_n(z)\Big) \\
&= \frac{1}{\beta_k}J^\beta\Big(\sum_{n\in\mathbb{Z}} a_{2n-k}\beta_{2n}\frac{\beta_n}{\beta_{2n}}e_n(z)\Big) \\
&= \frac{1}{\beta_k}\sum_{n\in\mathbb{Z}} a_{-2n-k}\beta_{-n}e_n(z) = K_\phi^\beta e_k(z)
\end{aligned}$$

for each integer k. □

It is clear from the Lemma 2.2 that K^β_ϕ is a bounded operator on $L^2(\beta)$ and $\|K^\beta_\phi\| \leq \|\phi\|_\infty$.

Lemma 2.3. *For any $\phi \in L^\infty(\beta)$, $K^\beta_1 K^\beta_\phi = 0$ if and only if $\phi = 0$.*

Proof. If $K^\beta_1 K^\beta_\phi = 0$ then it gives

$$\sum_{n\in\mathbb{Z}} a_{-4n-m}\beta_{-2n}\frac{\beta_n}{\beta_{2n}}e_{-n} = 0$$

for each $m \in \mathbb{Z}$. It yields that $a_{-4n-m} = 0$ for each $m, n \in \mathbb{Z}$, which provides $a_n = 0$ for each n so that $\phi = 0$. Converse is obvious. □

We can also conclude from Lemma 2.2 that the class of all weighted slant Hankel operators on $L^2(\beta)$ forms a subspace of $\mathfrak{B}(L^2(\beta))$. One can also prove the following.

Theorem 2.4. *$\phi \mapsto K^\beta_\phi$ is an injective linear mapping from $L^\infty(\beta)$ into $\mathfrak{B}(L^2(\beta))$.*

Proof. Linearity follows using the Lemma 2.2, by which $K^\beta_\phi = J^\beta W M^\beta_\phi$. Injectiveness can be viewed as follows. $K^\beta_\phi = 0$ for $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ in $L^\infty(\beta)$ implies that $K^\beta_\phi e_n = 0$ for each $n \in \mathbb{Z}$, which provides $a_n = 0$ for each n. Hence the proof is completed. □

It is known [1, Theorem 3] that an operator K on $L^2(\mathbb{T})$ is a slant Hankel operator if and only if $M_{z^{-1}}K = KM_{z^2}$, where M_z stands for the Laurent operator on $L^2(\mathbb{T})$. We first observe that this result does not hold, in general, in case of weighted slant Hankel operator on $L^2(\beta)$. For, consider the weighted slant Hankel operator K^β_ϕ on the space $L^2(\beta)$, where the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is given by

$$\beta_n = \begin{cases} 1 & \text{if } n < 2 \\ 2 & \text{if } n \geq 2 \end{cases}$$

and $\phi(z) = z$. Then one can see that

$$M^\beta_{z^{-1}} K^\beta_z e_1(z) = z^{-2},$$

whereas

$$K^\beta_z M^\beta_{z^2} = J^\beta W M^\beta_z M^\beta_{z^2} = J^\beta M^\beta_{z^3} = K^\beta_{z^3}$$

and

$$K^\beta_{z^3} e_1(z) = 2z^{-2}.$$

However, we find that the weighted slant Hankel operator satisfies the above relation under certain conditions.

Theorem 2.5. *Let K^β_ϕ be a non-zero weighted slant Hankel operator on $L^2(\beta)$. Then $M^\beta_{z^{-1}} K^\beta_\phi = K^\beta_\phi M^\beta_{z^2}$ if and only if the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual.*

Proof. Suppose $\beta_{-n} = \beta_n$ for each $n \in \mathbb{Z}$ and $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is in $L^\infty(\beta)$. Then we have

$$\begin{aligned} K^\beta_\phi M^\beta_{z^2} e_k(z) &= \frac{\beta_{k+2}}{\beta_k} K^\beta_\phi e_{k+2}(z) \\ &= \frac{1}{\beta_k} \sum_{n\in\mathbb{Z}} a_{-2n-2-k} \beta_n e_n(z) \end{aligned}$$

and also,

$$\begin{aligned} M^\beta_{z^{-1}} K^\beta_\phi e_k(z) &= M^\beta_{z^{-1}} \Big(\frac{1}{\beta_k} \sum_{n\in\mathbb{Z}} a_{-2n-k} \beta_{-n} e_n(z) \Big) \\ &= \frac{1}{\beta_k} \sum_{n\in\mathbb{Z}} a_{-2n-k} \beta_{n-1} e_{n-1}(z) \\ &= \frac{1}{\beta_k} \sum_{n\in\mathbb{Z}} a_{-2n-2-k} \beta_n e_n(z). \end{aligned}$$

Conversely, suppose that K^β_ϕ is a non-zero weighted slant Hankel operator on $L^2(\beta)$ satisfying $M^\beta_{z^{-1}} K^\beta_\phi = K^\beta_\phi M^\beta_{z^2}$, where ϕ is given by $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then for each $k \in \mathbb{Z}$, $M^\beta_{z^{-1}} K^\beta_\phi e_k = K^\beta_\phi M^\beta_{z^2} e_k$ and this gives

$$a_{-2n-k-2} \tfrac{\beta_{-(n+1)}}{\beta_{n+1}} = a_{-2n-k-2} \tfrac{\beta_{-n}}{\beta_n} \tag{2.5.1}$$

for each $n, k \in \mathbb{Z}$. Now we split the proof into two parts.

case (i): Let $a_{2n_0} \neq 0$ for some integer n_0. Then on substituting $k = -2$ and $n = -n_0$ in equation (2.5.1), we find that

$$\frac{\beta_{n_0-1}}{\beta_{-(n_0-1)}} = \frac{\beta_{n_0}}{\beta_{-n_0}}.$$

From equation (2.5.1), on putting $k = 0$ and $n = -(n_0 + 1)$, we have

$$\frac{\beta_{n_0+1}}{\beta_{-(n_0+1)}} = \frac{\beta_{n_0}}{\beta_{-n_0}}.$$

Similarly on substituting $k = 2, \pm 4, \pm 6, \cdots$ in equation (2.5.1) and giving appropriate value to n each time so that $-2n - k - 2 = 2n_0$, we conclude that

$$\frac{\beta_n}{\beta_{-n}} = \frac{\beta_{n+1}}{\beta_{-(n-1)}}$$

for each $n \in \mathbb{Z}$. This gives $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$ being $\beta_0 = 1$.

case (ii): Suppose $a_{2n} = 0$ for each integer n then, K^β_ϕ being a non-zero weighted slant Hankel operator, we can find an integer m_0 such that $a_{2m_0-1} \neq 0$. Now on substituting $k = -1$ and $n = -m_0$ in equation (2.5.1), we get

$$\frac{\beta_{m_0-1}}{\beta_{-(m_0-1)}} = \frac{\beta_{m_0}}{\beta_{-m_0}}.$$

Now on substituting $k = 1, \pm 3, \pm 5, \cdots$ in equation (2.5.1) and giving values to n so that $-2n-k-2 = 2m_0-1$, we can again conclude that

$$\frac{\beta_n}{\beta_{-n}} = \frac{\beta_{n+1}}{\beta_{-(n-1)}}$$

for each $n \in \mathbb{Z}$, which yields that $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$.

This completes the proof. □

It is interesting to know the following application of the condition of semi-duality.

Theorem 2.6. *Let the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ semi-dual. Then WK^{β}_{ϕ} is a weighted slant Hankel operator on $L^2(\beta)$ if and only if $\phi = 0$. Further, $J^{\beta}WK^{\beta}_{\phi}$ is a weighted slant Hankel operator on $L^2(\beta)$ if and only if $\phi = 0$.*

Proof. Suppose $\beta_{-n} = \beta_n$ for each $n \in \mathbb{Z}$ and $\phi(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$ is in $L^{\infty}(\beta)$. Let $WK^{\beta}_{\phi} = K^{\beta}_{\psi}$ for some $\psi(z) = \sum\limits_{n\in\mathbb{Z}} b_n z^n$ in $L^{\infty}(\beta)$. Then for each $i \in \mathbb{Z}$, $WK^{\beta}_{\phi}e_i = K^{\beta}_{\psi}e_i$, which provides

$$a_{-4n-i}\beta_{-2n}\frac{\beta_n}{\beta_{2n}} = b_{-2n-i}\beta_{-n}$$

for each $i, n \in \mathbb{Z}$.

As the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual, the above equation on $n = 0$ gives $a_k = b_k$ for each $k \in \mathbb{Z}$. Now when substituting $i = 0$, the equation gives $a_{2k} = a_{4k}$ and on $i = 1$, it provides $a_{2k+1} = a_{4k+1}$ for each $k \in \mathbb{Z}$. Since $a_n \to 0$ as $n \to \infty$, we get for each $k \in \mathbb{Z}$, $a_{2k} = 0 = a_{2k+1}$. Thus $\phi = 0$.
The converse is obvious.
Further, on applying the similar techniques and arguments, we can prove that $J^{\beta}WK^{\beta}_{\phi}$ is a weighted slant Hankel operator on $L^2(\beta)$ if and only if $\phi = 0$. □

The spaces $L^2(\mathbb{T})$ and $L^{\infty}(\mathbb{T})$ have the property that whenever it contains any function f then it contains $f(z^2)$ as well and this fact is very helpful in the study made over these spaces. However, this fact does not hold in case of the spaces $L^2(\beta)$ and $L^{\infty}(\beta)$. Before we proceed to provide an example, we characterize a condition at which the norm of the weighted Laurent operator induced by z coincides with the inverse of the norm of its own inverse.

Theorem 2.7. $\|M^{\beta}_z\| = \|M^{\beta^{-1}}_z\|^{-1}$ *if and only if $\beta_n = 1$ for each $n \in \mathbb{Z}$.*

Proof. Sufficient part of the theorem is obvious. However, the necessary part follows using the observations made in [14] so that

$$\|M^{\beta}_z\| = \sup\frac{\beta_{n+1}}{\beta_n} \geq 1 \text{ and } \|M^{\beta^{-1}}_z\|^{-1} = \inf\frac{\beta_{n+1}}{\beta_n} \leq 1,$$

which provide

$$\sup_{n\geq 0}\frac{\beta_{n+1}}{\beta_n} = 1 \text{ and } \inf_{n<0}\frac{\beta_{n+1}}{\beta_n} = 1.$$

Hence,

$$\beta_{n+1} = \beta_n \text{ for } n \geq 0 \text{ and } \beta_{n+1} = \beta_n \text{ for } n < 0.$$

This fulfills our need. □

Example 2.8. *Consider the spaces $L^{\infty}(\beta)$ and $L^2(\beta)$, where the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is given by $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$.*
Define a formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} 0 & \text{if } n \leq 0 \\ \frac{1}{n^2 2^n} & \text{if } n \geq 1. \end{cases}$$

Then ϕ is analytic in the domain $|z| < 2$ being the radius of convergence $\frac{1}{\lim\left(\frac{1}{2^n n^2}\right)^{\frac{1}{n}}} = 2$ and is bounded as well. Also, we find that

$$\|M^{\beta}_z\| = \sup\frac{\beta_{n+1}}{\beta_n} = 2 \text{ and } \|M^{\beta^{-1}}_z\|^{-1} = \inf\frac{\beta_{n+1}}{\beta_n} = \frac{1}{2}.$$

Thus, ϕ is bounded and analytic in the domain $\|M_z^{\beta^{-1}}\|^{-1} < |z| < \|M_z^{\beta}\|$ and hence on applying [14, Theorem 10′(b)], we have $\phi \in L^\infty(\beta)$ and so $\phi \in L^2(\beta)$ as well.

Now consider $\psi(z) = \phi(z^2) = \sum_{n=1}^{\infty} a_n z^{2n}$. Then for $e_0 \in L^2(\beta)$,

$$(\psi e_0)(z) = \psi(z) = \sum_{n=1}^{\infty} \frac{2^n}{n^2} e_{2n}(z),$$

which is not in $L^2(\beta)$ as the series $\sum_{n=1}^{\infty} (\frac{1}{2^n n^2})^2 \beta_{2n}^2 = \sum_{n=1}^{\infty} \frac{2^{2n}}{n^4}$ being divergent. Hence $\psi = \phi(z^2)$ does not belong to $L^\infty(\beta)$. In fact, ψ does not belong to $L^2(\beta)$.

If we consider the space $S(\beta) = \{f \in L^2(\beta) : \phi(z^2) \in L^2(\beta)\}$. Then we see that it contains all the formal Laurent polynomials and hence is dense in $L^2(\beta)$. In general, $S(\beta)$ is not a closed subspace of $L^2(\beta)$. However, we prove the following.

Theorem 2.9. *If $S(\beta)$ is a closed subspace of $L^2(\beta)$ then $\phi(z^2) \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$.*

Proof. Let $\phi \in L^\infty(\beta)$. Let f be an arbitrary element of $L^2(\beta)$ given by $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then f can be decomposed as $f(z) = f_0(z^2) + z f_1(z^2)$, where

$$f_0(z) = \sum_{n\in\mathbb{Z}} a_{2n} z^n \text{ and } f_1(z) = \sum_{n\in\mathbb{Z}} a_{2n+1} z^n .$$

Also, $f_0, f_1 \in L^2(\beta)$ being

$$\|f_0\|_\beta^2 = \sum_{n\in\mathbb{Z}} |a_{2n}|^2 \beta_n^2 \le \sum_{n\in\mathbb{Z}} |a_{2n}|^2 \beta_{2n}^2 < \infty$$

and

$$\begin{aligned} \|f_1\|_\beta^2 &= \sum_{n\in\mathbb{Z}} |a_{2n+1}|^2 \beta_n^2 \\ &= \sum_{n<0} |a_{2n+1}|^2 \beta_n^2 + \sum_{n\ge 0} |a_{2n+1}|^2 \beta_n^2 \\ &\le \sum_{n<0} |a_{2n+1}|^2 \beta_{2n}^2 + \sum_{n\ge 0} |a_{2n+1}|^2 \beta_{2n+1}^2 \\ &\le \frac{1}{r} \sum_{n<0} |a_{2n+1}|^2 \beta_{2n+1}^2 + \sum_{n\ge 0} |a_{2n+1}|^2 \beta_{2n+1}^2 \\ &< \infty. \end{aligned}$$

Now, by using the definition of $L^\infty(\beta)$ and the fact that $S(\beta)$ is closed subspace of $L^2(\beta)$ so that $S(\beta) = L^2(\beta)$, we find that $L^2(\beta)$ contains each of the functions $f_0(z^2), f_1(z^2), \phi(z^2)f_0(z^2), \phi(z^2)f_1(z^2)$ and $\phi(z^2)zf_1(z^2)$. Therefore, $\phi(z^2)f(z) = \phi(z^2)(f_0(z^2) + zf_1(z^2)) \in L^2(\beta)$ and hence $\phi(z^2) \in L^\infty(\beta)$. □

This theorem helps us to conclude the following.

Corollary 2.10. *If the sequence $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded then for each $\phi \in L^\infty(\beta)$, $\phi(z^2) \in L^\infty(\beta)$.*

Proof. Proof follows on using result [3, Lemma 2.4]. □

3. Algebraic Properties of Weighted Slant Hankel Operator

This section is devoted to study various algebraic properties for the class of weighted slant Hankel operators on $L^2(\beta)$. We recall that a sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is called a semi-dual sequence if $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$. Throughout this section, we consider the spaces, unless otherwise stated, under the assumption that the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. The sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$, where $\beta_n = \sqrt{|n|+1}$, is one of such sequences. With these assumptions on the sequence $\beta = \{\beta_n\}$, we have the accessibility of the following facts:

1. If $\phi(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$ is in $L^\infty(\beta)$ then $\phi(z^2) = \sum\limits_{n\in\mathbb{Z}} a_n z^{2n}$ and $\widetilde{\phi}(z) = \sum\limits_{n\in\mathbb{Z}} a_{-n} z^n$ are also in $L^\infty(\beta)$.
2. $\phi(z^2) \in L^2(\beta)$ for each $\phi \in L^2(\beta)$.

We have seen, in general, that a weighted slant Hankel operator on $L^2(\beta)$ may not satisfy $M^\beta_{z^{-1}}K = KM^\beta_{z^2}$. But, with the additional assumptions made on the sequence β in this section, we can prove the following.

Theorem 3.1. *An operator K on $L^2(\beta)$ is a weighted slant Hankel operator if and only if $M^\beta_{z^{-1}}K = KM^\beta_{z^2}$.*

Proof. Necessary part follows using Theorem 2.5. We prove the converse only. For, let K be an operator on $L^2(\beta)$ satisfying $M^\beta_{z^{-1}}K = KM^\beta_{z^2}$. Let $f(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$ be an arbitrary element of $L^2(\beta)$. Put $g(z) = \widetilde{f}(z) = \sum\limits_{n\in\mathbb{Z}} b_n z^{-n} = f(z^{-1})$ and $h(z) = \widetilde{f}(z^2) = f(z^{-2})$. Then $h, z^{-1}h \in L^2(\beta)$ and hence $K(h), K(z^{-1}h) \in L^2(\beta)$. But,

$$\begin{aligned} K(h(z)) &= K(f(z^{-2})) = K(\sum_{i\in\mathbb{Z}} b_{-i} z^{2i}) \\ &= \sum_{i\in\mathbb{Z}} b_{-i} K M^\beta_{z^{2i}}(1) \\ &= \sum_{i\in\mathbb{Z}} b_{-i} M^\beta_{z^{-i}} K(1) \\ &= (\sum_{i\in\mathbb{Z}} b_{-i} z^{-i}) K(1) = f(z)K(1) \end{aligned}$$

and

$$\begin{aligned} K(z^{-1}\widetilde{f}(z^2)) &= K(z^{-1}f(z^{-2})) = \sum_{i\in\mathbb{Z}} b_i K M^\beta_{z^{-2i}} M^\beta_{z^{-1}}(1) \\ &= \sum_{i\in\mathbb{Z}} b_i M^\beta_{z^i} K M^\beta_{z^{-1}}(1) \\ &= \sum_{i\in\mathbb{Z}} b_i z^i K(z^{-1}) = f(z)K(z^{-1}). \end{aligned}$$

Thus, if we put $\phi_0 = K(1)$ and $\phi_1 = K(z^{-1})$ then we have $\phi_0 f, \phi_1 f \in L^2(\beta)$ so that $\phi_0, \phi_1 \in L^\infty(\beta)$. Therefore $\phi_0(z^{-2}) = \widetilde{\phi_0}(z^2)$ and $\phi_1(z^{-2})$ are in $L^\infty(\beta)$. Now consider the function ϕ defined as

$$\phi(z) = \phi_0(z^{-2}) + z\phi_1(z^{-2}).$$

We claim that $K = K^\beta_\phi$. For, let $f \in L^2(\beta)$ be given by $f(z) = \sum\limits_{i\in\mathbb{Z}} b_i z^i$, then we can write

$$f(z) = f_0(z^{-2}) + z^{-1} f_1(z^{-2}),$$

where $f_0(z) = \sum_{i\in\mathbb{Z}} b_{2i}z^{-i}$ and $f_1(z) = \sum_{i\in\mathbb{Z}} b_{2i-1}z^{-i}$ are in $L^2(\beta)$. Now consider

$$\begin{aligned} K^{\beta}_{\phi}f &= J^{\beta}WM^{\beta}_{\phi}f = J^{\beta}W(\phi f) \\ &= J^{\beta}W[(\phi_0(z^{-2}) + z\phi_1(z^{-2}))(f_0(z^{-2}) + z^{-1}f_1(z^{-2}))] \\ &= J^{\beta}W[\phi_0(z^{-2})f_0(z^{-2}) + \phi_1(z^{-2})f_1(z^{-2})]. \end{aligned}$$

As $f_0(z) = \sum_{i\in\mathbb{Z}} d_i z^i$, where $d_i = b_{-2i}$, a straightforward computation shows that, if $\phi_0(z) = \sum_{i\in\mathbb{Z}} c_i z^i$ then $J^{\beta}W(\phi_0(z^{-2})f_0(z^{-2})) = \phi_0(z)f_0(z)$. Similarly, $J^{\beta}W(\phi_1(z^{-2})f_1(z^{-2})) = \phi_1(z)f_1(z)$. Consequently,

$$\begin{aligned} K^{\beta}_{\phi}f &= \phi_0(z)f_0(z) + \phi_1(z)f_1(z) \\ &= K(1)f_0(z) + K(z^{-1})f_1(z) \\ &= K(f_0(z^{-2})) + K(z^{-1}f_1(z^{-2})) \\ &= Kf. \end{aligned}$$

This completes the proof. □

An immediate result that follows from Theorem 3.1 is the following.

Corollary 3.2. *The class of all weighted slant Hankel operators on $L^2(\beta)$ is strongly closed subspace of $\mathfrak{B}(L^2(\beta))$.*

Proof. Suppose $K \in \mathfrak{B}(L^2(\beta))$ and $\{\phi_n\}$ a sequence in $L^\infty(\beta)$, are such that $K^{\beta}_{\phi_n} \to A$ as $n \to \infty$. Evidently,

$$\begin{aligned} KM^{\beta}_{z^2} &= \lim_{n\to\infty} K^{\beta}_{\phi_n}M^{\beta}_{z^2} \\ &= \lim_{n\to\infty} M^{\beta}_{z^{-1}}K^{\beta}_{\phi_n} \\ &= M^{\beta}_{z^{-1}}K. \end{aligned}$$

This gives that the bounded operator K on $L^2(\beta)$ is a weighted slant Hankel operator. □

It is important to note that Theorem 3.1 fails on dropping the assumption of boundedness on $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$. This we can justify through the following example.

Example 3.3. *Consider the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$, where $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$. Clearly, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is not bounded. Let $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, where*

$$a_n = \begin{cases} \frac{1}{2^n n^2} & \text{if } n > 0 \\ 0 & \text{otherwise}. \end{cases}$$

Then, on proceeding along the lines of computations made in Example 2.8, we find that $\phi \in L^\infty(\beta)$, whereas $L^\infty(\beta)$ does not contain $\phi(z^2)$. Define K on $L^2(\beta)$ as $K = S^{\beta}_{\phi}W$, where S^{β}_{ϕ} is the weighted Hankel operator on $L^2(\beta)$. Then K is a bounded operator on $L^2(\beta)$ and one can see that for $f(z) = \sum_{n\in\mathbb{Z}} c_n z^n \in L^2(\beta)$,

$$Kf == \sum_{n\in\mathbb{Z}} c_{2n}\Big(\sum_{m\in\mathbb{Z}} a_{-m-n}\beta_{-m}e_m\Big).$$

Then for each $n \in \mathbb{Z}$,

$$Ke_n = \begin{cases} \frac{1}{\beta_n}\sum_{m\in\mathbb{Z}} a_{-m-\frac{n}{2}}\beta_{-m}e_m & \text{if } n \text{ is even} \\ 0 & \text{otherwise}. \end{cases}$$

If possible, $K = K^{\beta}_{\psi}$ *for some* $\psi \in L^{\infty}(\beta)$*. Let* $\psi(z) = \sum_{n\in\mathbb{Z}} d_n z^n$*. Then,* $Ke_{2k} = K^{\beta}_{\psi} e_{2k}$ *and this gives* $a_{-m-k} = d_{-2m-2k}$*, for each integer k and m. Thus we conclude that* $a_{-k} = d_{-2k}$ *or*

$$d_{2k} = \begin{cases} a_k & \text{if } k > 0 \\ 0 & \text{otherwise} . \end{cases}$$

This provides that $\psi(z) = \sum_{n\in\mathbb{Z}} d_n z^n = \sum_{n>0} d_{2n} z^{2n} = \sum_{n>0} a_n z^{2n} = \phi(z^2)$*, which is not in* $L^{\infty}(\beta)$*. This contradicts our assumption that* $K = K^{\beta}_{\psi}$ *for some* $\psi \in L^{\infty}(\beta)$*. Hence K can not be a weighted slant Hankel operator on* $L^2(\beta)$*. However it is easy to verify that*

$$M^{\beta}_{z^{-1}} K e_j = K M^{\beta}_{z^2} e_j$$

for each $j \in \mathbb{Z}$*.*

The notion of Laurent matrix [8], slant Hankel matrix [1], Toeplitz matrix [8] and weighted Hankel matrix [6] are discussed and used to characterize their respective named operators. In the same direction, we begin with the following.

Definition 3.4. *A doubly infinite matrix* $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$ *is said to be a weighted slant Hankel matrix with respect to a sequence* $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ *of positive real numbers if*

$$\frac{\beta_j}{\beta_{-i}} \alpha_{ij} = \frac{\beta_{j+2k}}{\beta_{-(i-k)}} \alpha_{i-k\ j+2k} \qquad ,$$

for each $i, j, k \in \mathbb{Z}$*.*

It is clear that the matrix representation of a weighted slant Hankel operator on $L^2(\beta)$ is always (without $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being semi-dual or $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ bounded) a weighted slant Hankel matrix. However, the assumptions of $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded that are made in this section, help us to prove the other way too, as seen in the following.

Theorem 3.5. *An operator K on* $L^2(\beta)$ *is a weighted slant Hankel operator on* $L^2(\beta)$ *if and only if its matrix with respect to the orthonormal basis* $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ *of* $L^2(\beta)$ *is a weighted slant Hankel matrix with respect to the sequence* $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$*.*

Proof. We need to prove the sufficient part only. For, let K be an operator on $L^2(\beta)$ with its matrix $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$ with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ satisfying

$$\frac{\beta_j}{\beta_{-i}} \alpha_{ij} = \frac{\beta_{j+2k}}{\beta_{-(i-k)}} \alpha_{i-k\ j+2k} \qquad ,$$

for each $i, j, k \in \mathbb{Z}$. Now a straightforward computation, using $\beta_n = \beta_{-n}$, shows that

$$\begin{aligned} \left\langle M^{\beta}_{z^{-1}} K e_j, e_i \right\rangle &= \left\langle K e_j, M^{\beta*}_{z^{-1}} e_i \right\rangle \\ &= \frac{\beta_i}{\beta_{i+1}} \alpha_{i+1\ j} \end{aligned}$$

and

$$\begin{aligned}\left\langle KM^{\beta}_{z^2}e_j, e_i\right\rangle &= \frac{\beta_{j+2}}{\beta_j}\left\langle Ke_{j+2}, e_i\right\rangle\\ &= \frac{\beta_{j+2}}{\beta_j}\alpha_{i\ j+2}\\ &= \frac{\beta_{j+2}}{\beta_j}\frac{\beta_j}{\beta_{-(i+1)}}\frac{\beta_{-i}}{\beta_{j+2}}\alpha_{i+1\ j}\\ &= \frac{\beta_{-i}}{\beta_{-(i+1)}}\alpha_{i+1\ j} = \frac{\beta_i}{\beta_{i+1}}\alpha_{i+1\ j}.\end{aligned}$$

As a consequence of the above observations, $KM^{\beta}_{z^2} = M^{\beta}_{z^{-1}}K$ and which on applying Theorem 3.1 yields that K is a weighted slant Hankel operator on $L^2(\beta)$. □

It can be seen through Example 3.3 that the condition of boundedness on $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is essential for the Theorem 3.5 to hold. As, the operator K discussed in Example 3.3 is not a weighted slant Hankel operator on $L^2(\beta)$, although, its matrix with respect to the orthonormal basis $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ is $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$, where

$$\alpha_{ij} = \begin{cases}\frac{1}{\beta_j}a_{-i-\frac{j}{2}}\beta_{-i} & \text{if j is even}\\ 0 & \text{if j is odd}\end{cases}$$

satisfies the condition

$$\frac{\beta_j}{\beta_{-i}}\alpha_{ij} = \frac{\beta_{j+2k}}{\beta_{-(i-k)}}\alpha_{i-k\ j+2k},$$

for each $i, j, k \in \mathbb{Z}$. Thus, the matrix $[\alpha_{ij}]_{i,j\in\mathbb{Z}}$ of K is a weighted slant Hankel matrix with respect to the sequence $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ without K being a weighted slant Hankel operator on $L^2(\beta)$.

If $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is in $L^\infty(\beta)$ then for any integer m, by the symbol $\phi(z^m)$, we mean the formal Laurent series given by $\phi(z^m) = \sum_{n\in\mathbb{Z}} a_n z^{mn}$. Although $\phi(z^m)$ may or may not be in $L^\infty(\beta)$ but we have the following.

Lemma 3.6. *If $\phi(z^m) = \phi(z)$ for any integer $m \neq 1$ then ϕ is constant.*

Proof. If $\phi(z^m) = \phi(z)$ then $\phi(z^m) \in L^\infty(\beta)$ and we have

$$\sum_{n\in\mathbb{Z}} a_n\beta_{nm}e_{nm}(z) = \sum_{n\in\mathbb{Z}} a_n\beta_n e_n(z).$$

This yields that $a_k = 0$ when k is not a multiple of m and $a_n = a_{nm} = a_{nm^2} = \cdots = a_{nm^p}$ for each non-zero integer n and $p > 0$. Therefore, for any integer l(however large) and for any non-zero integer n,

$$(l+1)|a_n|^2 = \sum_{k=0}^{l} |a_{nm^k}|^2 \leq \sum_{i\in\mathbb{Z}} |a_i|^2 < \infty.$$

This gives $a_n = 0$ for each non-zero integer n and hence $\phi(z) = a_0$. □

We use the observations made in [3] and [6] that provide $M^{\beta}_{\phi(z)}W = WM^{\beta}_{\phi(z^2)}$ and $J^{\beta}M^{\beta}_{\phi} = M^{\beta}_{\tilde{\phi}}J^{\beta}$ for each $\phi \in L^\infty(\beta)$ respectively to obtain the following for the weighted slant Hankel operators.

Proposition 3.7. *For $\phi, \psi \in L^\infty(\beta)$,*

1. $M^{\beta}_{\phi}K^{\beta}_{\psi} = K^{\beta}_{\psi}M^{\beta}_{\tilde{\phi}(z^2)}$. *In fact, $M^{\beta}_{\phi}K^{\beta}_{\psi}$ is a weighted slant Hankel operator.*

2. $M^\beta_\phi K^\beta_\psi = K^\beta_\psi M^\beta_\phi$ *if and only if* $\psi(z)\phi(z) = \psi(z)\phi(z^{-2})$. *In particular, if* ψ *is invertible then* $M^\beta_\phi K^\beta_\psi = K^\beta_\psi M^\beta_\phi$ *if and only if* ϕ *is constant.*

Proof. It is easy to see that

$$\begin{aligned} M^\beta_\phi K^\beta_\psi &= M^\beta_\phi (J^\beta W M^\beta_\psi) \\ &= J^\beta M^\beta_{\widetilde{\phi}(z)} W M^\beta_\psi \\ &= J^\beta W M^\beta_{\widetilde{\phi}(z^2)} M^\beta_\psi \\ &= (J^\beta W M^\beta_\psi) M^\beta_{\widetilde{\phi}(z^2)} = K^\beta_\psi M^\beta_{\widetilde{\phi}(z^2)}. \end{aligned}$$

In fact, we find that $M^\beta_\phi K^\beta_\psi = K^\beta_\psi M^\beta_{\widetilde{\phi}(z^2)} = K^\beta_{\widetilde{\phi}(z^2)\psi}$.

The proof of (2) follows as $M^\beta_\phi K^\beta_\psi = K^\beta_\psi M^\beta_\phi$, which is equivalent to $K^\beta_{\phi\psi} = K^\beta_{\widetilde{\phi}(z^2)\psi}$ or $\phi\psi = \widetilde{\phi}(z^2)\psi$. In particular, if ψ is invertible then the latter means $\phi(z) = \widetilde{\phi}(z^2) = \phi(z^{-2})$, which on applying Lemma 3.6 gives ϕ a constant. Converse is obvious. □

Proposition 3.8. *For* $\phi, \psi \in L^\infty(\beta)$,

1. $K^\beta_\phi K^\beta_\psi = 0$ *if and only if* $\widetilde{\phi}(z^2)\psi = 0$.
2. $K^\beta_\phi K^\beta_\psi = K^\beta_\psi K^\beta_\phi$ *if and only if* $\widetilde{\phi}(z^2)\psi - \widetilde{\psi}(z^2)\phi = 0$.
3. $S^\beta_\phi K^\beta_\psi = K^\beta_\psi S^\beta_\phi$ *if and only if* $\widetilde{\phi}(z^2)\psi - \phi\widetilde{\psi} = 0$.

Proof. On applying Lemma 2.2, we can verify that

$$\begin{aligned} K^\beta_\phi K^\beta_\psi &= (J^\beta W M^\beta_\phi)(J^\beta W M^\beta_\psi) \\ &= J^\beta W J^\beta W M^\beta_{\widetilde{\phi}(z^2)\psi} \\ &= J^\beta W K^\beta_{\widetilde{\phi}(z^2)\psi} \\ &= K^\beta_1 K^\beta_{\widetilde{\phi}(z^2)\psi} \end{aligned}$$

and

$$\begin{aligned} K^\beta_\phi K^\beta_\psi - K^\beta_\psi K^\beta_\phi &= K^\beta_1 K 1^\beta M^\beta_{\widetilde{\phi}(z^2)\psi} - K^\beta_1 K^\beta_1 M^\beta_{\widetilde{\psi}(z^2)\phi} \\ &= K^\beta_1 K^\beta_{(\widetilde{\phi}(z^2)\psi - \widetilde{\psi}(z^2)\phi)}, \end{aligned}$$

which on using Lemma 2.3 complete the proof of (1) and (2). For the proof of (3), we use the facts

$$\begin{aligned} S^\beta_\phi K^\beta_\psi &= (J^\beta M^\beta_\phi)(J^\beta W M^\beta_\psi) \\ &= M_{\widetilde{\phi}} W M^\beta_\psi = W M^\beta_{\widetilde{\phi}(z^2)\psi} \end{aligned}$$

and

$$K^\beta_\psi S^\beta_\phi = W M^\beta_{\widetilde{\psi}\phi},$$

which on applying Theorem 2.4 yield that $S^\beta_\phi K^\beta_\psi = K^\beta_\psi S^\beta_\phi$ if and only if $J^\beta W M^\beta_{\widetilde{\phi}(z^2)\psi - \widetilde{\psi}\phi} = 0$ if and only if $\widetilde{\phi}(z^2)\psi - \widetilde{\psi}\phi = 0$. □

Now we discuss when the product of two weighted slant Hankel operators is a weighted slant Hankel operator.

Theorem 3.9. *A necessary and sufficient condition for the product of two weighted slant Hankel operators K^{β}_{ϕ} and K^{β}_{ψ} on $L^2(\beta)$ to be a weighted slant Hankel operator on $L^2(\beta)$ is that $\widetilde{\phi}(z^2)\psi = 0$.*

Proof. Let K^{β}_{ϕ} and K^{β}_{ψ} be two weighted slant Hankel operators on $L^2(\beta)$. Then result follows on applying Theorem 2.6, as $K^{\beta}_{\phi}K^{\beta}_{\psi} = J^{\beta}WK^{\beta}_{\widetilde{\phi}(z^2)\psi}$. □

4. Compact Operators

This section is devoted to study the compactness of the weighted slant Hankel operators on $L^2(\beta)$, whereas we continue with the assumptions that the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. We need some constructions to proceed. We recall, because of the above assumptions on the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ that $f(z^2) \in L^2(\beta)$ whenever $f \in L^2(\beta)$. Consider a linear transformation $V : L^2(\beta) \to L^2(\beta)$, defined as

$$(Vf)(z) = f(z^2) = \sum_{n\in\mathbb{Z}} a_n\beta_{2n}e_{2n}$$

for each $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ in $L^2(\beta)$. Then $Ve_n = \frac{\beta_{2n}}{\beta_n}e_{2n}$ for each integer n. We highlight some properties of this mapping and find that this mapping plays an important role to discuss the compactness of weighted slant Hankel operators.

Property 1. *V is a bounded opertor on $L^2(\beta)$ with $\|V\| = \sup_{n\in\mathbb{Z}} \frac{\beta_{2n}}{\beta_n}$.*

Proof. For $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in L^2(\beta)$,

$$\begin{aligned} \|Vf\|^2_{\beta} &= \sum_{n\in\mathbb{Z}} |a_n|^2\beta^2_{2n} \\ &= \sum_{n\in\mathbb{Z}} |a_n|^2\beta^2_n\frac{|\beta_{2n}|^2}{|\beta_n|^2} \\ &\le G^2\sum_{n\in\mathbb{Z}} |a_n|^2\beta^2_n = G^2\|f\|^2_{\beta}, \end{aligned}$$

where $G = \sup_{n\in\mathbb{Z}} \frac{\beta_{2n}}{\beta_n}$. Hence, V is a bounded operator on $L^2(\beta)$ with $\|V\| \le G$. Also,

$$\begin{aligned} \|V\| &\ge \|Ve_n\|_{\beta} \\ &= \frac{\beta_{2n}}{\beta_n} \end{aligned}$$

for each $n \in \mathbb{Z}$. This yields that $\|V\| = G$. □

Property 2. *WV=I, the identity operator on $L^2(\beta)$ and VW is given by*

$$VWe_n = \begin{cases} e_n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Proof is apparent by the definition of V and W. □

Property 3. *Let $\phi \in L^\infty(\beta)$. Then $VM^\beta_\phi = M^\beta_{\phi(z^2)}V$.*

Proof. For each $m \in \mathbb{Z}$,

$$\begin{aligned} VM^\beta_\phi e_m &= \frac{1}{\beta_m}\sum_{n\in\mathbb{Z}} a_n\beta_{2n+2m}e_{2n+2m} \\ &= (\sum_{n\in\mathbb{Z}} a_n z^{2n})\frac{\beta_{2m}}{\beta_m}e_{2m} = M^\beta_{\phi(z^2)}Ve_m. \end{aligned}$$

Hence the result follows. □

Property 4. *$K^\beta_{\phi(z^2)}V = S^\beta_\phi$, a weighted Hankel operator on $L^2(\beta)$.*

Proof. It follows immediately as

$$\begin{aligned} K^\beta_{\phi(z^2)}V &= J^\beta W M^\beta_{\phi(z^2)}V \\ &= J^\beta WVM^\beta_\phi \\ &= J^\beta M^\beta_\phi = S^\beta_\phi. \end{aligned}$$

□

If $\beta = \{\beta_n\}_{n\in\mathcal{Z}}$ is a semi-dual sequence then a necessary and sufficient condition for an operator A on $L^2(\beta)$ to be a weighted Hankel operator is that $M^\beta_z A = AM^\beta_{z^{-1}}$ [6], we use this to prove the following.

Property 5. *$K^\beta_\phi V$ is a weighted Hankel operator on $L^2(\beta)$.*

Proof. For each $m \in \mathbb{Z}$,

$$\begin{aligned} M^\beta_{z^{-1}}(K^\beta_\phi V)e_j &= \frac{1}{\beta_j}\sum_{n\in\mathbb{Z}} a_{-2n-2-2j}\beta_n e_n \\ &= (K^\beta_\phi V)M^\beta_z e_j. \end{aligned}$$

Hence, $M^\beta_{z^{-1}}(K^\beta_\phi V) = (K^\beta_\phi V)M^\beta_z$ so that $K^\beta_\phi V = S^\beta_\psi$ for some $\psi \in L^\infty(\beta)$. □

Property 6. *Let $\phi \in L^\infty(\beta)$. Then*

1. *$WM^\beta_\phi V$ is a weighted Laurent operator on $L^2(\beta)$. In fact, if $\phi = \sum_{n\in\mathbb{Z}} a_n z^n$ then $WM^\beta_\phi V = M^\beta_\psi$, where $\psi = \sum_{n\in\mathbb{Z}} a_{2n}z^n$.*
2. *$WM^\beta_\phi M^\beta_z V$ is a weighted Laurent operator on $L^2(\beta)$. Further, if $\phi = \sum_{n\in\mathbb{Z}} a_n z^n$ then $WM^\beta_\phi M^\beta_z V = M^\beta_\xi$, where $\xi = \sum_{n\in\mathbb{Z}} a_{2n-1}z^n$.*

Proof. We prove(1) only. We find that

$$\begin{aligned} M^\beta_z(WM^\beta_\phi V) &= WM^\beta_{z^2}M^\beta_\phi V \\ &= (WM^\beta_\phi V)M^\beta_z. \end{aligned}$$

Hence, by [14, Theorem 3(a)], $WM^{\beta}_{\phi}V$ is a weighted Laurent operator on $L^2(\beta)$. In fact, if $\phi \in L^{\infty}(\beta)$ is given by $\phi = \sum_{n\in\mathbb{Z}} a_n z^n$, then for $n \in \mathbb{Z}$,

$$\begin{aligned} WM^{\beta}_{\phi}Ve_n &= WM^{\beta}_{\phi}\Big(\frac{\beta_{2n}}{\beta_n}e_{2n}\Big) \\ &= W\Big(\frac{1}{\beta_n}\sum_{m\in\mathbb{Z}} a_m\beta_{m+2n}e_{m+2n}\Big) \\ &= \frac{1}{\beta_n}\sum_{m\in\mathbb{Z}} a_{2m}\beta_{2m+2n}\frac{\beta_{m+n}}{\beta_{2m+2n}}e_{m+n} \\ &= \frac{1}{\beta_n}\sum_{m\in\mathbb{Z}} a_{2m}\beta_{m+n}e_{m+n} \\ &= \frac{1}{\beta_n}\sum_{m\in\mathbb{Z}} a_{2m}z^{m+n} = M^{\beta}_{\psi}e_n, \end{aligned}$$

where $\psi = \sum_{m\in\mathbb{Z}} a_{2m}z^m$ is in $L^{\infty}(\beta)$.
Similarly we can prove (2). □

Property 7. *Let $\phi, \psi \in L^{\infty}(\beta)$. Then*

1. *$K^{\beta}_{\phi}K^{\beta}_{\psi}V^2$ is a weighted Laurent operator on $L^2(\beta)$.*
2. *$W^2M^{\beta}_{\eta}M^{\beta}_{z^i}V^2$ is a weighted Laurent operator on $L^2(\beta)$ for each $i \in \mathbb{Z}$.*

Proof. Let $\phi, \psi \in L^{\infty}(\beta)$. Then, as $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, we have $\eta \in L^{\infty}(\beta)$, where $\eta(z) = \widetilde{\phi}(z^2)\psi(z)$. Also, $K^{\beta}_{\phi}K^{\beta}_{\psi} = (J^{\beta}WM^{\beta}_{\phi})(J^{\beta}WM^{\beta}_{\phi}) = (J^{\beta})^2W^2M^{\beta}_{\widetilde{\phi}(z^2)\psi} = W^2M^{\beta}_{\eta}$. Hence, $K^{\beta}_{\phi}K^{\beta}_{\psi}V^2 = W^2M^{\beta}_{\eta}V^2$. Moreover, if we take $\eta(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, then along the lines of the computations made in Property 6, we get

$$\begin{aligned} W^2M^{\beta}_{\eta}V^2e_n &= W(WM^{\beta}_{\eta}V)(Ve_n) \\ &= W\Big(\sum_{m\in\mathbb{Z}} a_{2m}z^m\Big)(Ve_n) \\ &= \Big(\sum_{m\in\mathbb{Z}} a_{4m}z^m\Big)e_n. \end{aligned}$$

This completes the proof of (1).

To prove (2), let i be any integer. Now, if $\eta(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ then $z^i\eta(z) = \sum_{n\in\mathbb{Z}} a_{n-i}z^n$. Hence, on replacing η by $z^i\eta$, we find that

$$W^2M^{\beta}_{z^i\eta}V^2e_n = \Big(\sum_{m\in\mathbb{Z}} a_{4m-i}z^m\Big)e_n.$$

for each $n \in \mathbb{Z}$. Hence the result. □

In [6], it is proved that the weighted Laurent operator M^{β}_{ϕ} on $L^2(\beta)$ is compact if and only if $\phi = 0$. Using this result and the properties of V, we prove the following.

Theorem 4.1. *The weighted slant Hankel operator K^{β}_{ϕ} on $L^2(\beta)$ is compact if and only if $\phi = 0$.*

Proof. Suppose that the weighted slant Hankel operator K^{β}_{ϕ} on $L^2(\beta)$ is compact. Using Property 6(1), we get that $J^{\beta}K^{\beta}_{\phi}V = J^{\beta}(J^{\beta}WM^{\beta}_{\phi})V = WM^{\beta}_{\phi}V = M^{\beta}_{\psi}$ is a compact weighted Laurent operator, where $\psi = \sum_{m\in\mathbb{Z}} a_{2m}z^m$. Hence $\psi = 0$ or $a_{2n} = 0$ for each $n \in \mathbb{Z}$.

Again, $J^{\beta}K^{\beta}_{\phi}M^{\beta}_{z}V$ is compact and this time applying the Property 6(2), we conclude that $a_{2n-1} = 0$ for each $n \in \mathbb{Z}$. Hence, $\phi = 0$.
The converse is obvious. □

This theorem immediately provides the following for $\phi, \psi \in L^{\infty}(\beta)$.

Corollary 4.2. *$S^{\beta}_{\phi}K^{\beta}_{\psi}$ is compact if and only if $\widetilde{\phi}(z^2)\psi = 0$.*

Proof. As J^{β} is an invertible operator, $S^{\beta}_{\phi}K^{\beta}_{\psi}$ is compact if and only if $J^{\beta}S^{\beta}_{\phi}K^{\beta}_{\psi} = K^{\beta}_{\widetilde{\phi}(z^2)\psi}$ is compact. The latter holds if and only if $\widetilde{\phi}(z^2)\psi = 0$. □

Corollary 4.3. *$K^{\beta}_{\phi}S^{\beta}_{\psi}$ is compact if and only if $\widetilde{\phi}\psi = 0$.*

Proof. This follows, as the compactness of $K^{\beta}_{\phi}S^{\beta}_{\psi}$ is the same as that of $J^{\beta}K^{\beta}_{\phi}S^{\beta}_{\psi} = K^{\beta}_{\widetilde{\phi}\psi}$. □

In the next result we discuss the compactness of the product of the weighted slant Hankel operators on $L^2(\beta)$.

Theorem 4.4. *Let $\phi, \psi \in L^{\infty}(\beta)$. Then the following are equivalent.*

1. *$K^{\beta}_{\phi}K^{\beta}_{\psi}$ is a compact operator on $L^2(\beta)$.*
2. *$\widetilde{\phi}(z^2)\psi = 0$.*
3. *$S^{\beta}_{\phi}K^{\beta}_{\psi}$ is compact.*
4. *$K^{\beta}_{\phi}K^{\beta}_{\psi} = 0$.*

Proof. As $K^{\beta}_{\phi}K^{\beta}_{\psi} = W^2M^{\beta}_{\eta}$, where $\eta = \widetilde{\phi}(z^2)\psi$, we get (4) from (2). Further (4) always implies (1) and also (2) is equivalent to (3) by Corollary 4.2.

Hence we only need to prove that (1) implies (2). For this suppose $K^{\beta}_{\phi}K^{\beta}_{\psi}$ is a compact operator on $L^2(\beta)$. Then, $W^2M^{\beta}_{\eta}V^2$ and $W^2M^{\beta}_{\eta}M^{\beta}_{z^i}V^2$, $i = 1, 2, 3$ are compact operators.

Now, if $\eta(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, then on applying Property 7 (1), $W^2M^{\beta}_{\eta}V^2$ is a compact weighted Laurent operator given by

$$W^2M^{\beta}_{\eta}V^2e_n = \Big(\sum_{m\in\mathbb{Z}} a_{4m}z^m\Big)e_n$$

for each $n \in \mathbb{Z}$ and hence $a_{4m} = 0$ for each $n \in \mathbb{Z}$.

Again, using Property 7(2) and applying the same arguments as earlier for each $W^2M^{\beta}_{\eta}M^{\beta}_{z^i}V^2$, $i = 1, 2, 3$, we get $a_{4m-1} = a_{4m-2} = a_{4m-3} = 0$ for each $n \in \mathbb{Z}$. Thus $\eta = 0$ equivalently $\widetilde{\phi}(z^2)\psi = 0$. □

Some immediate results that follow from this theorem are the following.

Corollary 4.5. *For any $\psi \in L^{\infty}(\beta)$, $K^{\beta}_{1}K^{\beta}_{\psi}$ is compact if and only if $\psi = 0$.*

Proof. It is a particular case of the theorem when $\phi = 1$. □

Corollary 4.6. *For any* $\phi, \psi \in L^\infty(\beta)$, $K^\beta_\phi K^\beta_\psi - K^\beta_\psi K^\beta_\phi$ *is compact if and only if* $\widetilde{\phi}(z^2)\psi - \widetilde{\psi}(z^2)\phi = 0$.

Proof. Result follows immediately, as we have seen in Proposition 3.8(2) that $K^\beta_\phi K^\beta_\psi - K^\beta_\psi K^\beta_\phi = K^\beta_1 K^\beta_{\widetilde{\phi}(z^2)\psi - \widetilde{\psi}(z^2)\phi}$. □

We have proved that the only compact weighted slant Hankel operator on $L^2(\beta)$, when the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is semi-dual and $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded, is the zero operator. However, there may exist a plenty of compact weighted slant Hankel operators on $L^2(\beta)$ once we drop the restriction of boundedness on $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$. For, consider the space $L^2(\beta)$ with the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ taken as in Example 3.3, i.e. $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$, which is semi-dual but $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is not bounded. Let $\phi = c$ (constant). Then,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \|K^\beta_c e_j\|^2_\beta &= c^2 \sum_{m\in\mathbb{Z}} \frac{\beta^2_m}{\beta^2_{2m}} \\ &= c^2 \sum_{m\in\mathbb{Z}} \frac{1}{2^{2|m|}} \\ &= c^2\Big(\sum_{m=-\infty}^{-1} \frac{1}{2^{-2m}} + \sum_{m=0}^{\infty} \frac{1}{2^{2m}}\Big) < \infty\end{aligned}$$

so that K^β_c is a non-zero Hilbert Schmidt and hence a non-zero compact operator on $L^2(\beta)$. One can also check by similar calculations that $K^\beta_{z^i}$ is also a non-zero compact operator on $L^2(\beta)$ for each integer i.

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