

Fixed point results for r - (η, ξ, ψ) -contractive mappings of type (I), (II) and (III)

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Abstract. In this paper, we introduce some classes of r - (η, ξ, ψ) -contractive mappings and prove results of fixed point in the setting of complete metric spaces. Some examples and an application to integral equations are given to illustrate the usability of the obtained results.

1. Introduction

In the fixed point theory of continuous mappings, a well-known theorem of Banach [1] states that if (X, d) is a complete metric space and if f is a self-mapping on X which satisfies the inequality

$$d(fx, fy) \leq kd(x, y) \tag{1}$$

for some $k \in [0, 1)$ and all $x, y \in X$, then f has a unique fixed point z and the sequence of successive approximations $\{f^n x\}$ converges to z for all $x \in X$. On the other hand, the condition $d(fx, fy) < d(x, y)$ does not ensure that f has a fixed point. In the last decades, the Banach's theorem [1] has been extensively studied and generalized on many settings, see for example [2]–[18]. In [3] Boyd and Wong investigated mappings which satisfy the following condition:

$$d(fx, fy) \leq \psi(d(x, y)), \tag{2}$$

where ψ is an upper semicontinuous from the right function with $\psi(t) < t$ for all $t > 0$. Precisely, they proved the following theorem:

Theorem 1.1. *Let (X, d) be a complete metric space and let $f : X \rightarrow X$ satisfy (2), where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is upper semicontinuous from the right and $\psi(t) < t$ for all $t > 0$. Then, f has a unique fixed point z and the sequence of successive approximations $\{f^n x\}$ converges to z for all $x \in X$.*

Recently, Samet et al. [15] introduced a new concept of α -contractive type mappings and established various fixed point theorems for such mappings in complete metric spaces. The presented theorems extend, generalize and improve many existing results in the literature.

Motivated by [15], we introduce some classes of contractive mappings, called r - (η, ξ, ψ) -contractive mappings, and prove results of fixed point in the setting of complete metric spaces. Some examples and an application to integral equations are given to illustrate the usability of the obtained results.

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2. Preliminaries

In this section we give the background on which our study is based.

Definition 2.1. Let $f : X \rightarrow X$, $r > 0$ and $\eta, \xi : X \rightarrow [0, +\infty)$ be two functions. We say that f is r - (η, ξ) -admissible if

- (i) $\eta(x) \geq r$ for some $x \in X$ implies $\eta(fx) \geq r$,
- (ii) $\xi(x) \leq r$ for some $x \in X$ implies $\xi(fx) \leq r$.

We denote by Ψ the set of functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:

- (ψ 1) ψ is upper semicontinuous from the right,
- (ψ 2) $\psi(t) < t$ for all $t > 0$.

Definition 2.2. Let (X, d) be a metric space and $f : X \rightarrow X$ be a r - (η, ξ) -admissible mapping. Then f is a

- r - (η, ξ, ψ) -contractive mapping of type (I), if

$$\eta(x)\eta(y)d(fx, fy) \leq \xi(x)\xi(y)\psi(d(x, y))$$

holds for all $x, y \in X$ where $\psi \in \Psi$.

- r - (η, ξ, ψ) -contractive mapping of type (II), if

$$[\eta(x)\eta(y) + r]d(fx, fy) \leq [\xi(x)\xi(y) + r]\psi(d(x, y)) \quad \text{such that } r^2 + r > 1$$

holds for all $x, y \in X$ where $\psi \in \Psi$.

- r - (η, ξ, ψ) -contractive mapping of type (III), if

$$[d(fx, fy) + r]\eta(x)\eta(y) \leq [\psi(d(x, y)) + r]\xi(x)\xi(y) \quad \text{such that } r \geq 1$$

holds for all $x, y \in X$ where $\psi \in \Psi$.

Proposition 2.3. Let f be a r - (η, ξ, ψ) -contractive mapping of type (I) or (II) or (III). Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ where $x_0 \in X$. If $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$, then

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \tag{3}$$

holds for all $n \in \mathbb{N}$.

Proof. Since f is a r - (η, ξ) -admissible mapping and $\eta(x_0) \geq r$ then $\eta(x_1) = \eta(fx_0) \geq r$. By continuing this process, we get $\eta(x_n) \geq r$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, we can obtain that $\xi(x_n) \leq r$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we distinguish the following cases:

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (I). Then

$$\begin{aligned} r^2 d(x_n, x_{n+1}) &\leq \eta(x_{n-1})\eta(x_n)d(x_n, x_{n+1}) \\ &\leq \xi(x_{n-1})\xi(x_n)\psi(d(x_{n-1}, x_n)) \\ &\leq r^2 \psi(d(x_{n-1}, x_n)), \end{aligned}$$

that is, (3) holds for all $n \in \mathbb{N}$.

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (II). Then

$$\begin{aligned} (r^2 + r)d(x_n, x_{n+1}) &\leq [\eta(x_{n-1})\eta(x_{n-1}) + r]d(x_n, x_{n+1}) \\ &\leq [\xi(x_{n-1})\xi(x_n) + r]\psi(d(x_{n-1}, x_n)) \\ &\leq (r^2 + r)\psi(d(x_{n-1}, x_n)), \end{aligned}$$

that is, (3) holds for all $n \in \mathbb{N}$.

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (III). Then

$$\begin{aligned} [d(x_n, x_{n+1}) + r]^2 &\leq [d(x_n, x_{n+1}) + r]\eta(x_{n-1})\eta(x_n) \\ &\leq [\psi(d(x_{n-1}, x_n)) + r]\xi(x_{n-1})\xi(x_n) \\ &\leq [\psi(d(x_{n-1}, x_n)) + r]^2, \end{aligned}$$

that is, (3) holds for all $n \in \mathbb{N}$.

□

Proposition 2.4. *If in Proposition 2.3, $x_n \rightarrow x$ as $n \rightarrow +\infty$ and $\eta(x_n) \geq r$ and $\xi(x_n) \leq r$ for all $n \in \mathbb{N} \cup \{0\}$ implies $\eta(x) \geq r$ and $\xi(x) \leq r$, then*

$$d(x_{n+1}, fx) \leq \psi(d(x_n, x)) \tag{4}$$

holds for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_n \rightarrow x$ as $n \rightarrow +\infty$ and $\eta(x_n) \geq r$ and $\xi(x_n) \leq r$ for all $n \in \mathbb{N} \cup \{0\}$. By the assumption this implies $\eta(x) \geq r$ and $\xi(x) \leq r$. Now, we distinguish the following cases:

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (I). Then

$$\begin{aligned} r^2d(x_{n+1}, fx) &\leq \eta(x_n)\eta(x)d(x_{n+1}, fx) \\ &\leq \xi(x_n)\xi(x)\psi(d(x_n, x)) \\ &\leq r^2\psi(d(x_n, x)), \end{aligned}$$

that is, (4) holds for all $n \in \mathbb{N} \cup \{0\}$.

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (II). Then

$$\begin{aligned} (r^2 + r)d(x_{n+1}, fx) &\leq [\eta(x_n)\eta(x) + r]d(x_{n+1}, fx) \\ &\leq [\xi(x_n)\xi(x) + r]\psi(d(x_n, x)) \\ &\leq (r^2 + r)\psi(d(x_n, x)), \end{aligned}$$

that is, (4) holds for all $n \in \mathbb{N} \cup \{0\}$.

- Let f be a r - (η, ξ, ψ) -contractive mapping of type (III). Then

$$\begin{aligned} [d(x_{n+1}, fx) + r]^2 &\leq [d(x_{n+1}, fx) + r]\eta(x_n)\eta(x) \\ &\leq [\psi(d(x_n, x)) + r]\xi(x_n)\xi(x) \\ &\leq [\psi(d(x_n, x)) + r]^2, \end{aligned}$$

that is, (4) holds for all $n \in \mathbb{N} \cup \{0\}$.

□

3. Main Results

In this section, using the results of the preceding section, we give two theorems.

Theorem 3.1. *Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a continuous r - (η, ξ, ψ) -contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in X$ such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$, then f has a fixed point in X .*

Proof. Let $x_0 \in X$ be such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$. Now, by Proposition 2.3, we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

If $x_n = x_{n+1} = f x_n$ for some $n \in \mathbb{N}$, then the result is proved as x_n is a fixed point of f . In what follows we will suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Now by (5) we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n), \tag{6}$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing and so by (6) there is $s \geq 0$ such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow +\infty} \psi(d(x_{n-1}, x_n)) = s.$$

Now, we show that s must be equal to 0. In fact, if $s > 0$, then we get

$$s = \limsup_{n \rightarrow +\infty} \psi(d(x_{n-1}, x_n)) \leq \psi(s) < s,$$

which is a contradiction. Hence

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0. \tag{7}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , we have

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Now, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon + d(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Taking the limit as $k \rightarrow +\infty$ in the above inequality and using (7), we get

$$\lim_{k \rightarrow +\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{8}$$

Again, from

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)})$$

taking the limit as $k \rightarrow +\infty$, by (7) and (8) we obtain

$$\lim_{k \rightarrow +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon.$$

Again by Proposition 2.3, we have

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq \psi(d(x_{m(k)}, x_{n(k)})).$$

Taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we deduce

$$\varepsilon = \lim_{k \rightarrow +\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k \rightarrow +\infty} \psi(d(x_{m(k)}, x_{n(k)})) \leq \psi(\varepsilon) < \varepsilon,$$

which is a contradiction and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \rightarrow z$. Now, the continuity of the mapping f implies

$$fz = \lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} x_{n+1} = z$$

and so z is a fixed point of f . \square

In the following theorem we omit the continuity hypothesis of f .

Theorem 3.2. Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a r - (η, ξ, ψ) -contractive mapping of type (I) or (II) or (III) such that $\psi(0) = 0$. If the following conditions hold:

- (i) there exists $x_0 \in X$ such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$,
- (ii) if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$ and $\eta(x_n) \geq r$ and $\xi(x_n) \leq r$ for all $n \in \mathbb{N}$, then $\eta(x) \geq r$ and $\xi(x) \leq r$,

then f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\eta(x_0) \geq r$ and $\xi(x_0) \leq r$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$. Following the proof of Theorem 3.1, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow +\infty$. Now, by Proposition 2.4, we have

$$d(x_{n+1}, fz) \leq \psi(d(x_n, z)).$$

Taking the upper limit as $n \rightarrow +\infty$ in the above inequality, we get

$$d(z, fz) = \lim_{n \rightarrow +\infty} d(x_{n+1}, fz) \leq \limsup_{n \rightarrow +\infty} \psi(d(x_n, z)) \leq \psi(\lim_{n \rightarrow +\infty} d(x_n, z)) = \psi(0) = 0,$$

that is, $z = fz$. \square

Next, we give two illustrative examples.

Example 3.3. Let $X = [0, +\infty)$ endowed with the Euclidean metric d . Let $f : X \rightarrow X$ be defined by

$$f(x) = \begin{cases} \frac{1}{6}(x + x^2) & \text{if } x \in [0, 1] \\ 3 \ln x & \text{if } x \in (1, +\infty) \end{cases}$$

and $\eta, \xi : X \rightarrow [0, +\infty)$ be given by

$$\eta(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \xi(x) = \frac{1}{2} \text{ for all } x \in X.$$

Also define $\psi : [0, +\infty) \rightarrow [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$.

Now, we prove that all the hypotheses of Theorem 3.2 are satisfied and hence f has a fixed point. Moreover, the result of Boyd and Wong [3] cannot be applied to f .

Proof. Let $x \in X$, if $\eta(x) \geq \frac{1}{2}$ then $x \in [0, 1]$. On the other hand, for all $x \in [0, 1]$, we have $fx \leq 1$ and hence $\eta(fx) \geq \frac{1}{2}$. Similarly, if $\xi(x) \leq 1/2$ then $\xi(fx) \leq 1/2$. This implies that f is a r - (η, ξ) -admissible mapping. Clearly, $\eta(0) \geq 1/2$ and $\xi(0) \leq 1/2$.

Now, if $\{x_n\}$ is a sequence in X such that $\eta(x_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\{x_n\} \subseteq [0, 1]$ and hence $x \in [0, 1]$. This implies that $\eta(x) \geq \frac{1}{2}$.

For all $x, y \in [0, 1]$, we have

$$\begin{aligned} \eta(x)\eta(y)d(fx, fy) &= \frac{1}{4}|fx - fy| = \frac{1}{24}|x - y|(1 + x + y) \\ &\leq \frac{1}{8}|x - y| \\ &= \xi(x)\xi(y)\psi(d(x, y)). \end{aligned}$$

Otherwise, $\eta(x)\eta(y) = 0$ and so the inequality

$$\eta(x)\eta(y)d(fx, fy) \leq \xi(x)\xi(y)\psi(d(x, y))$$

holds trivially. This ensures that f is a r - (η, ξ, ψ) -contractive mapping of type (I) and therefore, by Theorem 3.2, f has a fixed point.

Now, let $x = e^2$ and $y = e$. Then, we have

$$d(fe^2, fe) = |3 \ln e^2 - 3 \ln e| = 3 > \frac{1}{2}|e^2 - e| = \frac{1}{2}d(e^2, e) = \psi(d(e^2, e))$$

and so the result of Boyd and Wong [3] cannot be applied to f . \square

Example 3.4. Let $X = [0, +\infty)$ endowed with the Euclidean metric d . Let $f : X \rightarrow X$ be defined by

$$f(x) = \begin{cases} \frac{1-x}{5} & \text{if } x \in [0, 1] \\ 2x & \text{if } x \in (1, +\infty) \end{cases}$$

and $\eta, \xi : X \rightarrow [0, +\infty)$ be given by

$$\eta(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \xi(x) = 1 \text{ for all } x \in X.$$

Also define $\psi : [0, +\infty) \rightarrow [0, +\infty)$ by $\psi(t) = \frac{1}{4}t$ for all $t \geq 0$.

Now, we prove that all the hypotheses of Theorem 3.2 are satisfied and hence f has a fixed point. Moreover, the result of Boyd and Wong [3] cannot be applied to f .

Proof. Proceeding as in Example 3.3, we deduce that f is a r - (η, ξ) -admissible mapping which satisfies conditions (i) and (ii) of Theorem 3.2.

Now, for all $x, y \in [0, 1]$, we have

$$\begin{aligned} [d(fx, fy) + 1]\eta(x)\eta(y) &= \left[\frac{1}{5}|x - y| + 1\right]^1 \leq \left[\frac{1}{4}|x - y| + 1\right]^1 \\ &= [\psi(d(x, y)) + 1]^{\xi(x)\xi(y)}. \end{aligned}$$

Otherwise, $\eta(x)\eta(y) = 0$ and hence we have

$$[d(fx, fy) + 1]^0 = 1 \leq \left[\frac{1}{4}|x - y| + 1\right]^1 = [\psi(d(x, y)) + 1]^{\xi(x)\xi(y)}.$$

This ensures that f is a r - (η, ξ, ψ) -contractive mapping of type (III) and so by Theorem 3.2 f has a fixed point.

On the other hand, for $x = 2$ and $y = 3$, we get

$$d(x, y) = 2|x - y| = 2 > \frac{1}{4}|3 - 2| = \psi(d(x, y))$$

and so the result of Boyd and Wong [3] cannot be applied to f . \square

4. Application to the existence of solutions of integral equations

Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ and $d : X \times X \rightarrow [0, +\infty)$ be defined by $d(x, y) = \|x - y\|_\infty$ for all $x, y \in X$. Then (X, d) is a complete metric space.

Consider the integral equation

$$x(t) = p(t) + \int_0^T S(t, s)f(s, x(s))ds \tag{9}$$

and the mapping $F : X \rightarrow X$ defined by

$$Fx(t) = p(t) + \int_0^T S(t, s)f(s, x(s))ds, \tag{10}$$

where

- (A) $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous,
- (B) $p : [0, T] \rightarrow \mathbb{R}$ is continuous,
- (C) $S : [0, T] \times [0, T] \rightarrow [0, +\infty)$ is continuous,
- (D) there exists $\psi \in \Psi$ such that ψ is nondecreasing and there exist $\theta, \pi : X \rightarrow \mathbb{R}$ such that if $\theta(x) \geq 0$ for $x \in X$, then for every $s \in [0, T]$ we have

$$0 \leq |f(s, x(s)) - f(s, y(s))| \leq |\pi(y)|\psi(|x(s) - y(s)|),$$

- (F) there exists $x_0 \in X$ such that $\theta(x_0) \geq 0$,
- (G) for all $x \in X$, where $\theta(x) \geq 0$, we have

$$\left\| \int_0^T S(t, s)|\pi(x)|ds \right\|_\infty \leq 1,$$

- (H) if $\{x_n\}$ is a sequence in X such that $\theta(x_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then $\theta(x) \geq 0$.

Theorem 4.1. Under the assumptions (A)-(H), the integral equation (9) has a solution in $X = C([0, T], \mathbb{R})$.

Proof. Let $F : X \rightarrow X$ be defined by (10) and let $x \in X$ be such that $\theta(x) \geq 0$. By the condition (D), we deduce that

$$\begin{aligned} |Fx(t) - Fy(t)| &= \left| \int_0^T S(t, s)[f(s, x(s)) - f(s, y(s))]ds \right| \\ &\leq \int_0^T S(t, s)|f(s, x(s)) - f(s, y(s))|ds \\ &\leq \int_0^T S(t, s)|\pi(y)|\psi(|x(s) - y(s)|)ds \\ &\leq \int_0^T S(t, s)|\pi(y)|\psi(\|x - y\|_\infty)ds \\ &= \psi(\|x - y\|_\infty) \left(\int_0^T S(t, s)|\pi(y)|ds \right). \end{aligned}$$

Then

$$\|Fx - Fy\|_{\infty} \leq \psi(\|x - y\|_{\infty}) \left\| \int_0^T S(t, s) |\pi(y)| ds \right\|_{\infty} \leq \psi(\|x - y\|_{\infty}).$$

Now, define $\eta, \xi : X \rightarrow [0, +\infty)$ by

$$\eta(x) = \begin{cases} 1 & \text{if } \theta(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \xi(y) = 1.$$

Consequently, for all $x, y \in X$ we have

$$\eta(x)\eta(y)d(F(x), F(y)) \leq \xi(x)\xi(y)\psi(d(x, y)).$$

It easily shows that all the hypotheses of Theorem 3.2 are satisfied and hence the mapping F has a fixed point which is a solution of the integral equation (9) in $X = C([0, T], \mathbb{R})$. \square

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