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Harmonic Bergman spaces on the complement of a lattice

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Abstract. We investigate harmonic Bergman spaces $b^p = b^p(\Omega)$, $0 < p < \infty$, where $\Omega = \mathbb{R}^n \setminus \mathbb{Z}^n$ and prove that $b^q \subset b^p$ for $n/(k+1) \leq q < p < n/k$. In the planar case we prove that b^p is non empty for all $0 < p < \infty$. Further, for each $0 < p < \infty$ there is a non-trivial $f \in b^p$ tending to zero at infinity at any prescribed rate.

1. Introduction

We denote the space of all complex valued harmonic functions on a domain $V \subset \mathbb{R}^n$ by $h(V)$, with topology of locally uniform convergence. For $0 < p < \infty$ we set $b^p(V) = L^p(V) \cap h(V)$. With respect to L^p (quasi)norm these spaces are Frechet spaces for $0 < p < 1$ and Banach spaces for $p \geq 1$. Let $\Gamma = \mathbb{Z}^n$, $\Omega = \mathbb{R}^n \setminus \Gamma$. In the planar case the analytic Bergman spaces $B^p(\Omega)$ were studied in [1], in this paper we investigate harmonic Bergman spaces $b^p(\Omega)$.

For $x \in \mathbb{R}^n$ and $r > 0$ $B(x,r)$ denotes the open ball of radius r centered at x. We set, for $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$. Also, $Q(z,a) = \{w : \|w - z\|_\infty < a/2\}$ denotes an open cube centered at $z \in \mathbb{R}^n$ of side length $a > 0$ and $\dot{Q}(z,a) = Q(z,a) \setminus \{z\}$. In the planar case we also use notation $D(z,r) = \{w : |z - w| < r\}$, $D_r = D(0,r)$. The n dimensional Lebesgue measure is denoted by dm. Letter C denotes a constant, its value can vary from one occurrence to the next. For future reference we state some known facts.

Proposition 1.1. *If $f : \mathbb{R}^n \to \mathbb{C}$ is a harmonic function, not identically equal to zero, then $f \notin b^p(\mathbb{R}^n)$, $p > 0$. Moreover:*

$$\left(\int_{B(x,R)} |f(y)|^p dy \right)^{1/p} \geq C_{p,n} R^{n/p} |f(x)|, \qquad x \in \mathbb{R}^n. \tag{1}$$

Proof. Indeed, (1) follows from subharmonic behavior of $|f|^p$ for $0 < p < \infty$, see [3]. Therefore

$$\left(\int_{\mathbb{R}^n} |f(y)|^p dy \right)^{1/p} \geq \lim_{R \to +\infty} C_{p,n} |f(x)| R^{n/p} = +\infty$$

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whenever $f(x) \neq 0$ for some $x \in \mathbb{R}^n$. □

It is a standard fact that for $f \in b^p(V)$, $V \subset \mathbb{R}^n$, $0 < p < +\infty$ we have

$$|f(x)| \leq C_{p,n} \frac{\|f\|_p}{r^{n/p}}, \qquad \text{where} \quad r = d(x, V^c). \tag{2}$$

In fact, using (1.1), we get

$$|f(x)|^p \leq \frac{C_{n,p}}{m(B(x,r))} \int_{B(x,r)} |f|^p dm \leq C_{n,p} r^{-n} \|f\|_p^p,$$

and (2) easily follows. Note that the this allows one to conclude that convergence in $b^p(V)$ implies locally uniform convergence on V.

We need certain facts about expansions of harmonic functions near singularities, for details see [2].

Suppose $n \geq 3$, $a \in V \subset \mathbb{R}^n$, and $f \in h(V \setminus \{a\})$. Then there are homogeneous harmonic polynomials p_m and q_m of degree m such that

$$f(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}. \tag{3}$$

A classification of singularities follow from this expansion: f has a removable singularity at a if and only if $\lim_{x \to a} |x-a|^{n-2} \left|f(x)\right| = 0$, f has a pole at a of order $M + n - 2$ if and only if $0 < \limsup_{x \to a} |x-a|^{M+n-2} \left|f(x)\right| < \infty$ and finally point a is an essential singularity if and only if $\limsup_{x \to a} |x-a|^N \left|f(x)\right| = \infty$ for every positive integer N.

When $n = 2$ the situation is slightly different, in that case there are homogeneous harmonic polynomials p_m and q_m of degree m on $\mathbb{R}^2$ such that

$$f(z) = \sum_{m=0}^{\infty} p_m(z-a) + q_0 \log|z-a| + \sum_{m=1}^{\infty} \frac{q_m(z-a)}{|z-a|^{2m}} \tag{4}$$

The presence of the logarithmic factor makes a difference between analytic and harmonic case, see for example Proposition 2.3 below.

In the above situation f has a removable singularity at a iff $\lim_{z \to a} \frac{f(z)}{\log|z-a|} = 0$, it has a fundamental pole at a if and only if $0 < \lim_{z \to a} \left|\frac{f(z)}{\log|z-a|}\right| < \infty$, it has a pole at a of order M if and only if $0 < \limsup_{z \to a} |z-a|^M \left|f(z)\right| < \infty$ and finally f has an essential singularity at a if and only if $\limsup_{z \to a} |z-a|^N \left|f(z)\right| = \infty$ for every positive integer N.

There is an alternative, but equivalent way to expand $u \in h(V \setminus \{a\})$, $V \subset \mathbb{C}$, namely to use analytic and conjugate analytic functions. We assume, for simplicity, that $a = 0$. Then we have

$$u(z) = a_0 + b_0 \log|z| + \sum_{n \neq 0} (c_n z^n + d_n \overline{z}^n), \qquad 0 < |z| < r. \tag{5}$$

Note that $a_0 = a_0(u)$, $b_0 = b_0(u)$, $c_n = c_n(u)$ and $d_n = d_n(u)$.

Proposition 1.2. *The functionals a_0, b_0, c_n and d_n, $n \neq 0$, are continuous on the Frechet space $h(V')$, $V' = V \setminus \{0\}$.*

Proof. Using

$$b_0(u) = \frac{1}{2\pi} \int_{C_\rho} \frac{\partial u}{\partial n} ds, \qquad 0 < \rho < \text{dist}(0, \partial V), \tag{6}$$

where C_ρ is the circle centered at 0 of radius ρ, we conclude, using continuity of derivatives on the space $h(V')$ that b_0 is continuous on $h(V')$. Now we fix $0 < \rho_1 < \rho_2 < \mathrm{dist}(0, \partial V)$. For any $k \neq 0$ we have

$$\phi_k(u) = \frac{1}{2\pi\rho_1} \int_{C_{\rho_1}} u(z) z^{-k} ds = c_k(u) + \rho_1^{-2k} d_{-k}(u) \tag{7}$$

and

$$\psi_k(u) = \frac{1}{2\pi\rho_2} \int_{C_{\rho_2}} u(z) \overline{z^k} ds = \rho_2^{2k} c_k(u) + d_{-k}(u). \tag{8}$$

Both ϕ_k and ψ_k are continuous on $h(V')$, since (7) and (8) represent a system of linear equations with determinant $1 - (\rho_2/\rho_1)^k \neq 0$ it follows immediately that c_k and d_k are continuous. The case of a_0 is left to the reader. □

2. Inclusions between b^p spaces

We start with an auxiliary proposition.

Proposition 2.1. *Assume $f \in b^p(V')$, where $V' = V \setminus \{a\}$ for some $a \in V \subset \mathbb{R}^n$. Then*

$$|f(x)| = o(|x - a|^{-n/p}), \qquad x \to a. \tag{9}$$

In particular, a is either a removable singularity of f or a pole of order $k < n/p$. If $n \geq 3$ and $p \geq \frac{n}{n-2}$, then a is a removable singularity.

Proof. Applying (2) to $V = B(x, |x - a|)$ one gets (9) and that suffices in view of the above classification of isolated singularities. □

Combining the last proposition and Proposition 1.1 we obtain the following:

Corollary 2.2. *If $f \in b^p(\Omega)$, $p \geq \frac{n}{n-2}$ and $n \geq 3$, then f is identically zero.*

Our first result demonstrates a basic difference between harmonic and analytic Bergman spaces on Ω in the planar case, namely $B^p(\Omega) = \{0\}$ for $p \geq 2$, see [1]. However we have:

Proposition 2.3. *If $n = 2$, then $b^p(\Omega) \neq \{0\}$ for $0 < p < \infty$.*

Proof. The function $f(z) = \log|z - 1| - 2\log|z| + \log|z + 1|$ is harmonic in Ω and, by Lagrange's theorem, $|f(z)| = O(|z|^{-2})$ as $z \to \infty$. Therefore $f \in b^2(\Omega)$.

Similarly, $f(z) = \log|z + 1| - \log|z|$ is harmonic in Ω and, by Lagrange's theorem, $|f(z)| = O(|z|^{-1})$. Therefore $f \in b^p(\Omega)$ for $2 < p < \infty$.

Finally, for $0 < p < 2$ the analytic Bergman spaces $B^p(\Omega)$ are non-empty, in fact they contain nontrivial rational functions, see [1]. □

Lemma 2.4. *Let $k \in \mathbb{N}$ and $n/(k+1) \leq q < p < n/k$. Then there is a constant $C = C_{p,q,n}$ such that*

$$\|u\|_{b^p(\dot{Q}(a,1))} \leq C\|u\|_{b^q(\dot{Q}(a,3/2))} \qquad \textit{for every} \quad u \in b^q(\dot{Q}(a, 3/2)), \quad a \in \Gamma.$$

Proof. This lemma states that the restriction operator $R : b^q(\dot{Q}(a, 3/2)) \to b^p(\dot{Q}(a, 1))$ given by $Ru = u|_{\dot{Q}(a,1)}$ is continuous. Since both spaces $b^q(\dot{Q}(a, 3/2))$ and $b^p(\dot{Q}(a, 1))$ are complete it suffices, by the closed graph theorem, to prove that R maps $b^q(\dot{Q}(a, 3/2))$ into $b^p(\dot{Q}(a, 1))$. Let $u \in b^q(\dot{Q}(a, 3/2))$. Since $q \geq n/(k + 1)$ Proposition 3 implies that the order of pole of u at a is at most k. Therefore, $|u(z)|^p = O(|a - z|^{-kp})$ where $kp < n$. Hence $|u|^p$ is integrable in a neighborhood of a and that implies $u \in b^p(\dot{Q}(a, 1))$. □

The main result of this section is the following result.

Theorem 2.5. *If $n/(k+1) \leq q < p < n/k$ ($k = 1, 2, \ldots$), then $b^q(\Omega) \subset b^p(\Omega)$.*

Proof. Set $Q_\omega = Q(\omega, 1)$ for $\omega \in \Gamma$. Let $u \in b^q(\Omega)$. The poles of u have orders at most k hence $u(z) = O(|z-\omega|^{-k})$ as $z \to \omega$. Therefore $u|_{Q_\omega} \in L^p(Q_\omega)$. Using Lemma 1 we get

$$\begin{aligned}
\|u\|_p^p = \int_\Omega |u|^p dm &= \sum_{\omega\in\Gamma} \int_{\dot{Q}_\omega} |u|^p dm \leq C \sum_{\omega\in\Gamma} \left(\int_{\dot{Q}(\omega,3/2)} |u|^q dm \right)^{p/q} \\
&\leq C \left(\sum_{\omega\in\Gamma} \int_{\dot{Q}(\omega,3/2)} |u|^q dm \right)^{p/q} \\
&\leq 4^{p/q} C \left(\sum_{\omega\in\Gamma} \int_{\dot{Q}_\omega} |u|^q dm \right)^{p/q} = 4^{p/q} C \|u\|_q^p
\end{aligned}$$

because $p/q \geq 1$ and almost every point in $\mathbb{C}$ lies in precisely 4 squares $Q(\omega, 3/2)$. □

We note that the above proof can be used to prove Theorem 1 from [1], in fact it presents a simplification of the proof given in [1].

3. Asymptotics at infinity of functions in $b^p(\Omega)$

One might conjecture that on the set $\Omega_\epsilon = \{z \in \mathbb{C} : d(z, \Gamma) > \epsilon\}$ we can control the size of functions $f \in b^p(\Omega)$, for example that we can prove $f(z) = O(|z|^{-2/p})$, $|z| \to \infty$, $z \in \Omega_\epsilon$. However, this is never true in general. The following theorem was proved in the case $0 < p < 2$ for analytic Bergman spaces $B^p(\Omega)$ in [1], and the same method of proof works in the present situation. We present this proof for reader's convenience.

Theorem 3.1. *Implication $f \in b^p(\Omega) \Rightarrow f(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$, $z \in \Omega_\epsilon$ does not hold for any $0 < p < \infty$, $\alpha > 0$, $0 < \epsilon < 1/\sqrt{2}$.*

Proof. Assume this implication holds for some $0 < p < \infty$, $\alpha > 0$ and $0 < \epsilon < 1/\sqrt{2}$. One easily proves that

$$h_{\epsilon,\alpha} = \{f \in h(\Omega_\epsilon) : \|f\|_{\epsilon,\alpha} = \sup_{z\in\Omega_\epsilon} |z|^\alpha |f(z)| < +\infty\}$$

is a Banach space. The restriction operator $R : b^p(\Omega) \to h_{\epsilon,\alpha}$ has closed graph because convergence in both (quasi)-norms $\|\cdot\|_p$ and $\|\cdot\|_{\epsilon,\alpha}$ implies pointwise convergence. Hence R is bounded, that is $\|f\|_{\epsilon,\alpha} \leq C\|f\|_p$ for all $f \in b^p(\Omega)$. Let us pick a non-trivial $f \in b^p(\Omega)$. Then

$$\begin{aligned}
|f(z_0)| &= |f_n(z_0 - n)| \leq |z_0 - n|^{-\alpha} \|f_n\|_{\epsilon,\alpha} \leq C|z_0 - n|^{-\alpha} \|f_n\|_p \\
&= C|z_0 - n|^{-\alpha} \|f\|_p
\end{aligned}$$

for all $n \in \mathbb{N}$, $z_0 \in \Omega_\epsilon$ (f_n denotes a function $f_n(z) = f(z+n)$). This gives, as $n \to \infty$, $f(z_0) = 0$, hence $f(z) = 0$ on Ω_ϵ and therefore on Ω as well. Contradiction. □

Remark 3.2. *The same proof works for a function $\phi(|z|)$ instead of $|z|^{-\alpha}$, where $\phi(r)$ is strictly positive and $\lim_{r\to+\infty} \phi(r) = 0$.*

4. Some generalizations and open problems

We alert reader to possible generalizations and open problems, these are parallel to those mentioned in [1]. One can define mixed norm spaces $b^{p,q}(\Omega)$ using (quasi)-norms

$$\|f\|_{p,q} = \left\{ \sum_{\omega \in \Gamma} \left(\int \int_{Q(\omega,1)} |f(z)|^p dxdy \right)^{q/p} \right\}^{1/q}, \qquad 0 < p, q < \infty.$$

Note that $b^{p,p}(\Omega) = b^p(\Omega)$. Some of our results generalize to the $b^{p,q}$ spaces, without any substantial changes in the proofs. For example:

$$b^{q,r} \subset b^{p,r}, \qquad \frac{2}{n+1} \le q < p < \frac{2}{n} \quad 0 < r < +\infty \tag{10}$$

Finally, we mention some natural questions on $b^p(\Omega)$ spaces.

1. Is there a bounded projection from $L^p(\Omega)$ onto $b^p(\Omega)$? This problem is related to the problem of finding the dual space of $b^p(\Omega)$, see [5] for the problem in the context of analytic functions.

2. Describe the dual of $b^p(\Omega)$.

3. Is $b^p(\Omega)$ isomorphic to l^p? We note that there is a vast amount of literature related to classical Banach spaces, see [4].

4. Are there sequences z_n in Ω such that $\|f\|_p^p \sim \sum_{n=1}^{\infty} d(z_n, \Gamma)^2 |f(z_n)|^p$?

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