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Quasi Grüss' type inequalities for continuous functions of selfadjoint operators in Hilbert spaces

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Abstract. Some inequalities of Grüss' type for vectors and continuous functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given. Applications for power and logarithmic functions are provided as well.

1. Introduction

In 1935, G. Grüss [21] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\leq\frac{1}{4}\left(\Phi-\phi\right)(\Gamma-\gamma), \tag{1}$$

where $f, g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $1/4$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki et al. [25, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,\ldots,a_n)$, $b=(b_1,\ldots,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,\ldots,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s),$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

For a simple proof of (1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [25].

For other related results see the papers [1]-[3], [4]-[6], [7]-[9], [10]-[17], [20], [28], [30] and the references therein.

In [11], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

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Theorem 1.1. ([11]) *Let $(H,\langle .,.\rangle)$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R},\mathbb{C}$) and $e\in H,\|e\|=1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e-x,x-\varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e-y,y-\gamma e\rangle \geq 0$$

hold, then we have the inequality

$$\left|\langle x,y\rangle -\langle x,e\rangle \langle e,y\rangle \right| \leq \frac{1}{4}\left|\Phi -\varphi \right|\cdot \left|\Gamma -\gamma \right|.$$

The constant $1/4$ is best possible in the sense that it can not be replaced by a smaller constant.

For other results of this type, see the recent monograph [14] and the references therein.

2. Grüss' type operator inequalities

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [22, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right)=\Phi(f)^*$;

(iii) $\left\|\Phi(f)\right\|=\left\|f\right\|:=\sup_{t\in Sp(A)}\left|f(t)\right|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [22] and the references therein. For other results, see [27], [24] and [29].

The following operator version of the Grüss inequality was obtained by Mond and Pečarić in [26]:

Theorem 2.1. ([26]) *Let C_j, $j\in\{1,...,n\}$ be selfadjoint operators on the Hilbert space $(H,\langle .,.\rangle)$ and such that $m_j\cdot 1_H\leq C_j\leq M_j\cdot 1_H$ for $j\in\{1,\ldots,n\}$, where 1_H is the identity operator on H. Further, let $g_j,h_j:\left[m_j,Mj\right]\to\mathbb{R}$, $j\in\{1,\ldots,n\}$ be functions such that*

$$\varphi\cdot 1_H\leq g_j\left(C_j\right)\leq \Phi\cdot 1_H \quad \text{and} \quad \gamma\cdot 1_H\leq h_j\left(C_j\right)\leq \Gamma\cdot 1_H$$

for each $j\in\{1,\ldots,n\}$.

If $x_j\in H$, $j\in\{1,\ldots,n\}$ are such that $\sum_{j=1}^n\left\|x_j\right\|^2=1$, then

$$\left|\sum_{j=1}^n\left\langle g_j\left(C_j\right)h_j\left(C_j\right)x_j,x_j\right\rangle-\sum_{j=1}^n\left\langle g_j\left(C_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^n\left\langle h_j\left(C_j\right)x_j,x_j\right\rangle\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).$$

If C_j, $j \in \{1,...,n\}$ are selfadjoint operators such that $Sp\left(C_j\right) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$ and if $g,h : [m,M] \longrightarrow \mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\left|\sum_{j=1}^{n}\left\langle g\left(C_j\right)h\left(C_j\right)x_j,x_j\right\rangle - \sum_{j=1}^{n}\left\langle g\left(C_j\right)x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle h\left(C_j\right)x_j,x_j\right\rangle\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{2}$$

where $x_j \in H$, $j \in \{1,...,n\}$ are such that $\sum_{j=1}^{n}\left\|x_j\right\|^2 = 1$ and $\varphi = \min_{t\in[m,M]} g(t)$, $\Phi = \max_{t\in[m,M]} g(t)$, $\gamma = \min_{t\in[m,M]} h(t)$ and $\Gamma = \max_{t\in[m,M]} h(t)$.

In particular, if the selfadjoint operator C satisfy the condition $Sp(C) \subseteq [m,M]$ for some scalars $m < M$, then

$$\left|\langle g(C)h(C)x,x\rangle - \langle g(C)x,x\rangle \cdot \langle h(C)x,x\rangle\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),$$

for any $x \in H$ with $\|x\| = 1$.

We say that the functions $f,g : [a,b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s)) \geq (\leq)\, 0 \text{ for each } t,s \in [a,b].$$

It is obvious that, if f,g are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

In the recent paper [18] the following *Čebyšev type inequality* for operators has been obtained:

Theorem 2.2. ([18]) *Let A_j be selfadjoint operators with $Sp\left(A_j\right) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$. If $f,g : [m,M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m,M]$, then*

$$\sum_{j=1}^{n}\left\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\right\rangle \geq (\leq) \sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle,$$

for each $x_j \in H$, $j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2 = 1$.

In the recent paper [19] we obtained amongst other the following refinement of the Grüss inequality (2):

Theorem 2.3. ([19]) *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m < M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned}&\left|\langle f(A)g(A)x,x\rangle - \langle f(A)x,x\rangle \cdot \langle g(A)x,x\rangle\right| \\ &\leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\left\|g(A)x\right\|^2 - \langle g(A)x,x\rangle^2\right]^{1/2} \left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

This inequality has the multivariate version as follows

Theorem 2.4. ([19]) *Let A_j be selfadjoint operators with $Sp\left(A_j\right) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$. If $f,g : [m,M] \longrightarrow \mathbb{R}$ are continuous and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$\begin{aligned}&\left|\sum_{j=1}^{n}\left\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\right\rangle - \sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle \cdot \sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle\right| \\ &\leq \frac{1}{2}\cdot(\Gamma-\gamma)\left[\sum_{j=1}^{n}\left\|g\left(A_j\right)x_j\right\|^2 - \left(\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle\right)^2\right]^{1/2} \\ &\left(\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right)\end{aligned}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \left\|x_j\right\|^2 = 1$ *where* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

In order to provide some new vector Grüss' type inequalities for continuous functions of selfadjoint operators in Hilbert spaces, we need the following facts concerning the spectral representation of such functions.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda,$$

which in terms of vectors can be written as

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y\rangle, \tag{3}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

3. Quasi Grüss' type inequalities

In this section we provide various bounds for the magnitude of the difference

$$\langle f(A)x, y\rangle - \langle x, y\rangle \langle f(A)x, x\rangle$$

under different assumptions on the continuous function, the selfadjoint operator $A : H \to H$ and the vectors $x, y \in H$ with $\|x\| = 1$.

Theorem 3.1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Assume that $x, y \in H, \|x\| = 1$ are such that there exists $\gamma, \Gamma \in \mathbb{C}$ with either*

$$\operatorname{Re}\langle \Gamma x - y, y - \gamma x\rangle \geq 0$$

or, equivalently

$$\left\|y - \frac{\gamma + \Gamma}{2} x\right\| \leq \frac{1}{2}\left|\Gamma - \gamma\right|.$$

1. *If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\begin{aligned} &\left|\langle f(A)x, y\rangle - \langle x, y\rangle \langle f(A)x, x\rangle\right| \\ &\leq \max_{\lambda\in[m,M]} \left|\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right| \bigvee_{m}^{M}(f) \\ &\leq \max_{\lambda\in[m,M]} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} \left(\left\|y\right\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2} \bigvee_{m}^{M}(f) \\ &\leq \frac{1}{2}\left(\left\|y\right\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2} \bigvee_{m}^{M}(f) \leq \frac{1}{4}\left|\Gamma - \gamma\right| \bigvee_{m}^{M}(f). \end{aligned} \tag{4}$$

2. *If $f:[m,M]\to\mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on $[m,M]$, then we have the inequality*

$$\begin{aligned}&\left|\langle f(A)x,y\rangle-\langle x,y\rangle\langle f(A)x,x\rangle\right| \\ &\le L\int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|d\lambda \\ &\le L\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda \\ &\le L\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2} \\ &\le \frac{1}{2}(M-m)L\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\le\frac{1}{4}\left|\Gamma-\gamma\right|(M-m)L.\end{aligned} \tag{5}$$

3. *If $f:[m,M]\to\mathbb{R}$ is a continuous monotonic nondecreasing function on $[m,M]$, then we have the inequality*

$$\begin{aligned}\left|\langle f(A)x,y\rangle-\langle x,y\rangle\langle f(A)x,x\rangle\right| &\le \int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|df(\lambda) \\ &\le \left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}df(\lambda) \\ &\le \left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\langle(f(M)1_H-f(A))x,x\rangle^{1/2}\langle(f(A)-f(m)1_H)x,x\rangle^{1/2} \\ &\le \frac{1}{2}\left[f(M)-f(m)\right]\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\le\frac{1}{4}\left|\Gamma-\gamma\right|\left[f(M)-f(m)\right].\end{aligned}$$

Proof. First of all, we notice that by the Schwarz inequality in H we have for any $u,v,e\in H$ with $\|e\|=1$ that

$$|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle|\le\left(\|u\|^2-|\langle u,e\rangle|^2\right)^{1/2}\left(\|v\|^2-|\langle v,e\rangle|^2\right)^{1/2}. \tag{6}$$

Now on utilizing (6), we can state that

$$\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|\le\left(\|E_\lambda x\|^2-|\langle E_\lambda x,x\rangle|^2\right)^{1/2}\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2} \tag{7}$$

for any $\lambda\in[m,M]$.

Since E_λ are projections and $E_\lambda\ge 0$ then

$$\begin{aligned}\|E_\lambda x\|^2-|\langle E_\lambda x,x\rangle|^2&=\langle E_\lambda x,x\rangle-\langle E_\lambda x,x\rangle^2 \\ &=\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\le\frac{1}{4}\end{aligned}$$

for any $\lambda\in[m,M]$ and $x\in H$ with $\|x\|=1$.

Also, by making use of the Grüss' type inequality in inner product spaces obtained by the author in [11] we have

$$\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\le\frac{1}{2}\left|\Gamma-\gamma\right|. \tag{8}$$

Combining the relations (7)-(8) we deduce the following inequality that is of interest in itself

$$\begin{aligned}&\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right| \\ &\le\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2} \\ &\le\frac{1}{2}\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\le\frac{1}{4}\left|\Gamma-\gamma\right|\end{aligned} \tag{9}$$

for any $\lambda \in [m,M]$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \max_{t\in[a,b]}\left|p(t)\right| \bigvee_a^b (v),$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a,b]$.

Utilising this property of the Riemann-Stieltjes integral and the inequality (9) we have

$$\begin{aligned}
&\left|\int_{m-0}^{M} \left[\langle E_\lambda x,y\rangle - \langle E_\lambda x,x\rangle\langle x,y\rangle\right] df(\lambda)\right| \qquad (10)\\
&\leq \max_{\lambda\in[m,M]}\left|\langle E_\lambda x,y\rangle - \langle E_\lambda x,x\rangle\langle x,y\rangle\right| \bigvee_m^M (f)\\
&\leq \max_{\lambda\in[m,M]} \left(\langle E_\lambda x,x\rangle\langle (1_H-E_\lambda)x,x\rangle\right)^{1/2}\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\bigvee_m^M (f)\\
&\leq \frac{1}{2}\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\bigvee_m^M (f) \leq \frac{1}{4}\left|\Gamma-\gamma\right|\bigvee_m^M (f)
\end{aligned}$$

for x and y as in the assumptions of the theorem.

Now, integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation (3) we have

$$\begin{aligned}
&\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle - \langle E_\lambda x,x\rangle\langle x,y\rangle\right] df(\lambda) \qquad (11)\\
&= \left[\langle E_\lambda x,y\rangle - \langle E_\lambda x,x\rangle\langle x,y\rangle\right] f(\lambda)\Big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda)\,d\left[\langle E_\lambda x,y\rangle - \langle E_\lambda x,x\rangle\langle x,y\rangle\right]\\
&= \langle x,y\rangle\int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x,x\rangle - \int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x,y\rangle\\
&= \langle x,y\rangle\langle f(A)x,x\rangle - \langle f(A)x,y\rangle
\end{aligned}$$

which together with (10) produces the desired result (4).

Now, recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$\left|f(s)-f(t)\right| \leq L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b \left|p(t)\right| dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have from (9) that

$$\begin{aligned}&\left|\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right]df(\lambda)\right| \\ &\leq L\int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|d\lambda \\ &\leq L\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda.\end{aligned} \tag{12}$$

If we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation (3) we have successively

$$\begin{aligned}&\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda \\ &\leq\left[\int_{m-0}^{M}\langle E_\lambda x,x\rangle d\lambda\right]^{1/2}\left[\int_{m-0}^{M}\langle(1_H-E_\lambda)x,x\rangle d\lambda\right]^{1/2} \\ &=\left[\langle E_\lambda x,x\rangle\lambda|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle E_\lambda x,x\rangle\right]^{1/2}\left[\langle(1_H-E_\lambda)x,x\rangle\lambda|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\langle(1_H-E_\lambda)x,x\rangle\right] \\ &=\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2}.\end{aligned} \tag{13}$$

On utilizing (13), (12) and (11) we deduce the first three inequalities in (5).

The fourth inequality follows from the fact that

$$\langle(M1_H-A)x,x\rangle\langle(A-m1_H)x,x\rangle\leq\frac{1}{4}\left[\langle(M1_H-A)x,x\rangle+\langle(A-m1_H)x,x\rangle\right]^2=\frac{1}{4}(M-m)^2.$$

The last part follows from (8).

Further, from the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b\left|p(t)\right|dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\int_a^b\left|p(t)\right|dv(t).$$

Utilising this property and the inequality (9) we have successively

$$\begin{aligned}\left|\int_{m-0}^{M}\left[\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right]df(\lambda)\right|&\leq\int_{m-0}^{M}\left|\langle E_\lambda x,y\rangle-\langle E_\lambda x,x\rangle\langle x,y\rangle\right|df(\lambda) \\ &\leq\left(\|y\|^2-\left|\langle y,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}df(\lambda).\end{aligned}$$

Applying the Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with

monotonic integrators and the spectral representation (3) we have

$$\begin{aligned}
&\int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} df(\lambda) \\
&\leq \left[\int_{m-0}^{M} \langle E_\lambda x, x\rangle df(\lambda)\right]^{1/2} \left[\int_{m-0}^{M} \langle (1_H - E_\lambda) x, x\rangle df(\lambda)\right]^{1/2} \\
&= \left[\langle E_\lambda x, x\rangle f(\lambda)\Big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda) d\langle E_\lambda x, x\rangle\right]^{1/2} \left[\langle (1_H - E_\lambda) x, x\rangle f(\lambda)\Big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda) d\langle (1_H - E_\lambda) x, x\rangle\right]^{1/2} \\
&= \langle (f(M) 1_H - f(A)) x, x\rangle^{1/2} \langle (f(A) - f(m) 1_H) x, x\rangle^{1/2} \\
&\leq \frac{1}{2}\left[f(M) - f(m)\right]
\end{aligned}$$

and the proof is complete. □

Remark 3.2. If we drop the conditions on x, y, we can obtain from the inequalities (4)-(5) the following results that can be easily applied for particular functions:

1. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$\left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| \leq \frac{1}{2}\|x\|^2 \left(\|y\|^2 \|x\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2} \bigvee_{m}^{M} (f)$$

for any $x, y \in H, x \neq 0$.

2. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality

$$\begin{aligned}
\left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| &\leq L\left(\|y\|^2 \|x\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2} \left[\langle (M1_H - A) x, x\rangle \langle (A - m1_H) x, x\rangle\right]^{1/2} \\
&\leq \frac{1}{2}(M - m) L \|x\|^2 \left(\|y\|^2 \|x\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2}
\end{aligned}$$

for any $x, y \in H, x \neq 0$.

3. If $f : [m, M] \to \mathbb{R}$ is a continuous monotonic nondecreasing function on $[m, M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A) x, y\rangle \|x\|^2 - \langle x, y\rangle \langle f(A) x, x\rangle\right| \leq \left(\|y\|^2 \|x\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2} \\
&\times \left[\langle (f(M) 1_H - f(A)) x, x\rangle \langle (f(A) - f(m) 1_H) x, x\rangle\right]^{1/2} \\
&\leq \frac{1}{2}\left[f(M) - f(m)\right] \|x\|^2 \left(\|y\|^2 \|x\|^2 - \left|\langle y, x\rangle\right|^2\right)^{1/2}
\end{aligned}$$

for any $x, y \in H, x \neq 0$.

The following lemma may be stated.

Lemma 3.3. *Let $u : [a, b] \to \mathbb{R}$ and $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$. The following statements are equivalent:*

(i) The function $u - \frac{\varphi+\Phi}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$, is $\frac{1}{2}(\Phi - \varphi)$–Lipschitzian;

(ii) We have the inequality:

$$\varphi \leq \frac{u(t) - u(s)}{t - s} \leq \Phi \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t \neq s;$$

(iii) We have the inequality:

$$\varphi(t-s) \leq u(t) - u(s) \leq \Phi(t-s) \quad \text{for each} \quad t,s \in [a,b] \quad \text{with } t > s.$$

Following [23], we can introduce the concept:

Definition 3.4. The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ, Φ) –Lipschitzian on $[a,b]$.

Notice that in [23], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ, Φ) –Lipschitzian functions.

Proposition 3.5. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty$$

then u is (γ, Γ) –Lipschitzian on $[a,b]$.

We are able now to provide the following corollary:

Corollary 3.6. *With the assumptions of Theorem 3.1 and if $f : [m,M] \to \mathbb{R}$ is a (φ, Φ)-Lipschitzian function then we have*

$$\begin{aligned}
\left|\langle f(A)x, y\rangle - \langle x, y\rangle \langle f(A)x, x\rangle\right| &\leq \frac{1}{2}(\Phi - \varphi)\int_{m-0}^{M} \left|\langle E_\lambda x, y\rangle - \langle E_\lambda x, x\rangle \langle x, y\rangle\right| d\lambda \\
&\leq \frac{1}{2}(\Phi - \varphi)\left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} d\lambda \\
&\leq \frac{1}{2}(\Phi - \varphi)\left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \langle (M1_H - A)x, x\rangle^{1/2} \langle (A - m1_H)x, x\rangle^{1/2} \\
&\leq \frac{1}{4}(M-m)(\Phi - \varphi)\left(\|y\|^2 - |\langle y, x\rangle|^2\right)^{1/2} \\
&\leq \frac{1}{8}\left|\Gamma - \gamma\right|(M-m)(\Phi - \varphi).
\end{aligned}$$

The proof follows from the second part of Theorem 3.1 applied for the $\frac{1}{2}(\Phi - \varphi)$-Lipschitzian function $f - \frac{\Phi+\varphi}{2}\cdot e$ by performing the required calculations in the first term of the inequality. The details are omitted.

4. Applications for Grüss' type inequalities

The following result provides some Grüss' type inequalities for two function of two selfadjoint operators.

Proposition 4.1. *Let A, B be two selfadjoint operators in the Hilbert space H with the spectra $Sp(A), Sp(B) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Assume that $g : [m,M] \to \mathbb{R}$ is a continuous function and denote $n := \min_{t\in[m,M]} g(t)$ and $N := \max_{t\in[m,M]} g(t)$.*

1. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle g(B)x,x\rangle\right| \\
&\leq \max_{\lambda\in[m,M]}\left|\langle E_\lambda x,g(B)x\rangle-\langle E_\lambda x,x\rangle\langle x,g(B)x\rangle\right|\bigvee_m^M(f) \\
&\leq \max_{\lambda\in[m,M]}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\bigvee_m^M(f) \\
&\leq \frac{1}{2}\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\bigvee_m^M(f)\leq\frac{1}{4}(N-n)\bigvee_m^M(f)
\end{aligned}$$

for any $x\in H,\|x\|=1$.

2. If $f:[m,M]\to\mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle g(B)x,x\rangle\right| \qquad (14)\\
&\leq L\int_{m-0}^{M}\left|\langle E_\lambda x,g(B)x\rangle-\langle E_\lambda x,x\rangle\langle x,g(B)x\rangle\right|d\lambda \\
&\leq L\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}d\lambda \\
&\leq L\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\langle(M1_H-A)x,x\rangle^{1/2}\langle(A-m1_H)x,x\rangle^{1/2} \\
&\leq \frac{1}{2}(M-m)L\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2} \\
&\leq \frac{1}{4}(N-n)(M-m)L
\end{aligned}$$

for any $x\in H,\|x\|=1$.

3. If $f:[m,M]\to\mathbb{R}$ is a continuous monotonic nondecreasing function on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\langle f(A)x,g(B)x\rangle-\langle f(A)x,x\rangle\langle g(B)x,x\rangle\right| \qquad (15)\\
&\leq \int_{m-0}^{M}\left|\langle E_\lambda x,g(B)x\rangle-\langle E_\lambda x,x\rangle\langle x,g(B)x\rangle\right|df(\lambda) \\
&\leq \left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x,x\rangle\langle(1_H-E_\lambda)x,x\rangle\right)^{1/2}df(\lambda) \\
&\leq \left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2}\langle(f(M)1_H-f(A))x,x\rangle^{1/2}\langle(f(A)-f(m)1_H)x,x\rangle^{1/2} \\
&\leq \frac{1}{2}[f(M)-f(m)]\left(\|g(B)x\|^2-\left|\langle g(B)x,x\rangle\right|^2\right)^{1/2} \\
&\leq \frac{1}{4}(N-n)[f(M)-f(m)]
\end{aligned}$$

for any $x\in H,\|x\|=1$.

Proof. We notice that, since $n:=\min_{t\in[m,M]}g(t)$ and $N:=\max_{t\in[m,M]}g(t)$, then $n\leq\langle g(B)x,x\rangle\leq N$ which implies that $\langle g(B)x-nx,Mx-g(B)x\rangle\geq 0$ for any $x\in H,\|x\|=1$. On applying Theorem 3.1 for $y=Bx,\Gamma=N$ and $\gamma=n$ we deduce the desired result. □

Remark 4.2. We observe that if the function f takes real values and is a (φ, Φ)-Lipschitzian function on $[m, M]$, then the inequality (14) can be improved as follows

$$\begin{aligned}
&\left|\langle f(A)x, g(B)x\rangle - \langle f(A)x, x\rangle\langle g(B)x, x\rangle\right| \qquad (16)\\
&\leq \frac{1}{2}(\Phi-\varphi)\int_{m-0}^{M}\left|\langle E_\lambda x, g(B)x\rangle - \langle E_\lambda x, x\rangle\langle x, g(B)x\rangle\right| d\lambda\\
&\leq \frac{1}{2}(\Phi-\varphi)\left(\left\|g(B)x\right\|^2 - \left|\langle g(B)x, x\rangle\right|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x, x\rangle\langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} d\lambda\\
&\leq \frac{1}{2}(\Phi-\varphi)\left(\left\|g(B)x\right\|^2 - \left|\langle g(B)x, x\rangle\right|^2\right)^{1/2}\langle (M1_H - A)x, x\rangle^{1/2}\langle (A - m1_H)x, x\rangle^{1/2}\\
&\leq \frac{1}{4}(M-m)(\Phi-\varphi)\left(\left\|g(B)x\right\|^2 - \left|\langle g(B)x, x\rangle\right|^2\right)^{1/2}\\
&\leq \frac{1}{8}(N-n)(M-m)(\Phi-\varphi)
\end{aligned}$$

for any $x \in H, \|x\| = 1$.

5. Applications

By choosing different examples of elementary functions into the above inequalities, one can obtain various Grüss' type inequalities of interest.

For instance, if we choose $f, g : (0,\infty) \to (0,\infty)$ with $f(t) = t^p, g(t) = t^q$ and $p, q > 0$, then for any selfadjoint operators A, B with $Sp(A), Sp(B) \subseteq [m, M] \subset (0,\infty)$ we get from (15) the inequalities

$$\begin{aligned}
|\langle A^p x, B^q x\rangle - \langle A^p x, x\rangle\langle B^q x, x\rangle| &\leq p\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x, x\rangle\langle (1_H - E_\lambda)x, x\rangle\right)^{1/2}\lambda^{p-1}d\lambda\\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\langle (M^p 1_H - A^p)x, x\rangle^{1/2}\langle (A^p - m^p 1_H)x, x\rangle^{1/2}\\
&\leq \frac{1}{2}(M^p - m^p)\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \leq \frac{1}{4}(M^q - m^q)(M^p - m^p)
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $\{E_\lambda\}_\lambda$ is the spectral family *of* A.

The same choice of functions considered in the inequality (16) produce the result

$$\begin{aligned}
|\langle A^p x, B^q x\rangle - \langle A^p x, x\rangle\langle B^q x, x\rangle| &\leq \frac{1}{2}\Delta_p\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\int_{m-0}^{M}\left(\langle E_\lambda x, x\rangle\langle (1_H - E_\lambda)x, x\rangle\right)^{1/2} d\lambda\\
&\leq \frac{1}{2}\Delta_p\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\langle (M^p 1_H - A^p)x, x\rangle^{1/2}\langle (A^p - m^p 1_H)x, x\rangle^{1/2}\\
&\leq \frac{1}{4}(M-m)\Delta_p\left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\\
&\leq \frac{1}{8}(M^q - m^q)(M-m)\Delta_p
\end{aligned}$$

where

$$\Delta_p := p \times \begin{cases} M^{p-1} - m^{p-1} \text{ if } p \geq 1 \\ \frac{M^{1-p} - m^{1-p}}{M^{1-p} m^{1-p}} \text{ if } 0 < p < 1. \end{cases}$$

for any $x \in H$ with $\|x\| = 1$.

Now, if we choose $f(t) = \ln t, t > 0$ and keep the same g then we have the inequalities

$$\begin{aligned}
&|\langle \ln Ax, B^q x\rangle - \langle \ln Ax, x\rangle \langle B^q x, x\rangle| \qquad (17)\\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} \lambda^{-1} d\lambda\\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \langle (\ln M 1_H - \ln A) x, x\rangle^{1/2} \langle (\ln A - \ln m 1_H) x, x\rangle^{1/2}\\
&\leq \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \ln \sqrt{\frac{M}{m}}\\
&\leq \frac{1}{2} (M^q - m^q) \ln \sqrt{\frac{M}{m}}
\end{aligned}$$

and

$$\begin{aligned}
&|\langle \ln Ax, B^q x\rangle - \langle \ln Ax, x\rangle \langle B^q x, x\rangle| \qquad (18)\\
&\leq \frac{1}{2} \left(\frac{M-m}{mM}\right) \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \int_{m-0}^{M} \left(\langle E_\lambda x, x\rangle \langle (1_H - E_\lambda) x, x\rangle\right)^{1/2} d\lambda\\
&\leq \frac{1}{2} \left(\frac{M-m}{mM}\right) \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2} \langle (M 1_H - A) x, x\rangle^{1/2} \langle (A - m 1_H) x, x\rangle^{1/2}\\
&\leq \frac{1}{4} \frac{(M-m)^2}{mM} \left(\|B^q x\|^2 - |\langle B^q x, x\rangle|^2\right)^{1/2}\\
&\leq \frac{1}{8} (M^q - m^q) \frac{(M-m)^2}{mM}
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

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