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Index of Graded Filiform and Quasi Filiform Lie Algebras

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Abstract. The filiform and the quasi-filiform Lie algebras form a special class of nilpotent Lie algebras. The aim of this paper is to compute the index and provide regular vectors of this two classes of nilpotent Lie algebras. We consider graded filiform Lie algebras L_n, Q_n, then n-dimensional filiform Lie algebras for $n < 8$, also graded quasi-filiform Lie algebras and finally Lie algebras whose nilradical is Q_{2n}.

1. Introduction

The index of a Lie algebra has applications to invariant theory and is of interest in deformation and quantum group theory. A Lie algebra is said to be Frobenius if the index is 0 which is equivalent to say that there is a functional in the dual such that the bilinear form B_F, defined by $B_F(x, y) = F([x, y])$, is nondegenerate. Frobenius algebras were first studied by Ooms in [20]. He proved that the universal enveloping algebra of the Lie algebra is primitive, that is it admits a faithful simple module, provided that the Lie algebra is Frobenius and that the converse holds when the Lie algebra is algebraic. Most of the studies of index concerned simple Lie algebras or their subalgebras. They have been considered by many authors [5, 7–10, 21, 24, 25]. Notice that simple Lie algebra can never be Frobenius but many subalgebras are. In this paper we focus on the computation of the index for nilpotent Lie algebras, mainly the class of filiform and quasi-filiform Lie algebras.

The filiform Lie algebras were introduced by M. Vergne (see [26]), she classified them up to dimension 6 and also characterized the graded filiform Lie algebras. In the last decades several papers were dedicated to classification of filiform Lie algebras of larger dimensions. In particular, L_n plays an important role in the study of filiform and nilpotent Lie algebras. It is known that any n-dimensional filiform Lie algebra may be obtained by deformation of the one of the filiform Lie algebras L_n. The classification of naturally graded quasi-filiform Lie algebras is known. They have the characteristic sequence $(n - 2, 1, 1)$ where n is the dimension of the Lie algebra. The aim of this paper is to give an extended version of our paper [1] and to focus on filiform Lie algebras and quasi-filiform Lie algebras. We compute the index and provide the regular vectors of n-dimensional filiform Lie algebras for $n < 8$ and quasi-filiform Lie algebras. In the first Section, we summarize the index theory of Lie algebras. Then, in Section 2, we review the nilpotent and filiform Lie algebras theories. Section 3, is dedicated to the two graded filiform Lie algebras L_n and Q_n. In Section 4, we consider the classification up to dimension 8 and compute for each filiform Lie algebra

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its index and the set of all regular vectors. In Section 5 we compute the index of graded quasi-filiform Lie algebras, and provide corresponding regular vectors. In the last section we compute the index of Lie algebras whose nilradical is Q_{2n}.

2. Lie algebras Index

Throughout this paper $\mathbb{K}$ is an algebraically closed field of characteristic 0. In this Section, we summarize the index theory of Lie algebras. Let $\mathcal{G}$ be an n-dimensional Lie algebra. Let $x \in \mathcal{G}$, we denote by adx the endomorphism of $\mathcal{G}$ defined by $adx(y) = [x, y]$ for all $y \in \mathcal{G}$.

Definition 2.1. *A Lie algebras $\mathcal{G}$ over $\mathbb{K}$ is a pair consisting of a vector space $\mathbb{V} = \mathcal{G}$ and a skew-symmetric bilinear map $[\ ,\ ] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ $(x, y) \to [x, y]$ satisfying the Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \forall x, y, z \in \mathcal{G}.$$

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{K}$ provided with the Zariski topology, $\mathcal{G}$ be a Lie algebra and $\mathcal{G}^*$ its dual. Then $\mathcal{G}$ actes on $\mathcal{G}^*$ as follows:

$$\begin{aligned} &\mathcal{G} \times \mathcal{G}^* \to \mathcal{G}^* \\ &\ (x, f) \mapsto x \cdot f \end{aligned}$$

where $\forall y \in \mathcal{G} : (x \cdot f)(y) = f([x, y])$.

Let $f \in \mathcal{G}^*$ and Φ_f be a skew-symmetric bilinear form defined by

$$\begin{aligned} &\Phi_f : \mathcal{G} \times \mathcal{G} \to \mathbb{K} \\ &\qquad (x, y) \mapsto \Phi_f(x, y) = f([x, y]). \end{aligned}$$

We denote the kernel of the map Φ_f by $\mathcal{G}^f$,

$$\mathcal{G}^f = \{x \in \mathcal{G} : f([x, y]) = 0 \ \ \forall y \in \mathcal{G}\}. \tag{1}$$

Definition 2.2. *The index of a Lie algebra $\mathcal{G}$ is the integer*

$$\chi_{\mathcal{G}} = \inf\left\{\dim\ \mathcal{G}^f ; f \in \mathcal{G}^*\right\}.$$

A linear functional $f \in \mathcal{G}^*$ is called regular if $\dim \mathcal{G}^f = \chi_{\mathcal{G}}$. The set of all regular linear functionals is denoted by $\mathcal{G}^*_r$.

Remark 2.3. *The set $\mathcal{G}^*_r$ of all regular linear functionals is a nonempty Zariski open set.*

Let $\{x_1, \cdots, x_n\}$ be a basis of $\mathcal{G}$. We can express the index using the matrix $([x_i, x_j])_{1\le i<j\le n}$ as a matrix over the ring $S(\mathcal{G})$, (see [6]). We have the following proposition.

Proposition 2.4. *The index of an n-dimensional Lie algebra $\mathcal{G}$ is the integer*

$$\chi(\mathcal{G}) = n - Rank_{R(\mathcal{G})}\left(\left[x_i, x_j\right]\right)_{1\le i<j\le n}$$

where $R(\mathcal{G})$ is the quotient field of the symmetric algebra $S(\mathcal{G})$.

Remark 2.5. *The index of an n-dimensional abelian Lie algebra is n.*

Proposition 2.6. *Let $\mathcal{G}_0$ be a Lie algebra, and $\mathcal{G}$ be a central extension of $\mathcal{G}_0$ by a $1-$dimensional Lie algebra $\mathcal{L} = c\mathbb{C}$, then $\chi(\mathcal{G}) = \chi(\mathcal{G}_0) + 1$. Moreover if f is a regular vector of $\mathcal{G}$, then $f = g + \rho c^*$, where g is a regular vector of $\mathcal{G}_0$ and $\rho \in \mathbb{C}$.*

Proof. Indeed, we have

$$\begin{cases} [x,c] = 0 \quad \forall x \in \mathcal{G}_0, \\ [c,c] = 0. \end{cases}$$

Then the matrix associated to $\mathcal{G}$ is of the form $M = \begin{pmatrix} M_{\mathcal{G}_0} & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $Rank(\mathcal{G}) = Rank(\mathcal{G}_0)$. Therefore

$$\chi(\mathcal{G}) = \chi(\mathcal{G}_0) + 1.$$

Let g be a regular vector of $\mathcal{G}_0$. Then $\dim \mathcal{G}_0^g = \chi(\mathcal{G}_0)$ and $f = g + \rho c^*$ is a regular vector of $\mathcal{G}$.
We know that $\mathcal{G}^f = \{x \in \mathcal{G}, \ f([x,y]) = 0, \forall y \in \mathcal{G}\}$.
We set

$$x = x_0 + \lambda c \text{ and } y = y_{0+}\mu c.$$

Then

$$f([x,y]) = g([x,y]) + \rho c^*([x,y]) = g([x_0,y_0]).$$

We have

$$g([x_0,y_0]) = 0 \text{ if } x_0 \in \mathcal{G}_0, \forall y \in \mathcal{G}_0.$$

Therefore, $\mathcal{G}^f = \mathcal{G}_0^g + c\mathbb{C}$. □

Remark 2.7. *In the sequel, we use the following procedure to compute regular vectors. We recall that if* $\dim \mathcal{G}^f = \chi(\mathcal{G})$ *then* f *is a regular vector of* $\mathcal{G}$*, where* $\chi(\mathcal{G}) = \min\left\{\dim \mathcal{G}^f, f \in \mathcal{G}^*\right\}$ *and* $\mathcal{G}^f = \{x \in \mathcal{G} : f([x,y]) = 0 \ \forall y \in \mathcal{G}\}$.

The equation $f([x,y]) = 0$ *implies* $\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s x_s^* \left([x_i, x_j]\right) = 0$.
It is equivalent to
$\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s C_{ij}^s = 0$, *where* C_{ij}^s *are the structure constants with respect to the basis* $\{x_i\}_i$. *Then for all* j, *we have* $\sum_{s=1}^n \sum_{i=1}^n a_i p_s C_{ij}^s = 0$. *It leads to*

$$\left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij} \begin{pmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

We denote the element $\left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij}$ *by* M *and assume* $C_{ij}^s = -C_{ji}^s$.
We search the minors of order $n - \chi(\mathcal{G})$ *of non-zero determinant of the matrix* M.
The matrix $M = \left(\sum_{s=1}^n p_s C_{ij}^s\right)_{ij}$ *is the same matrix as the multiplication table in which we replace* x_s *by* p_s.

Definition 2.8. *A Lie algebra* $\mathcal{G}$ *over an algebraically closed field of characteristic* 0 *is said to be Frobenius if there exists a linear form* $f \in \mathcal{G}^*$ *such that the bilinear form* Φ_f *on* $\mathcal{G}$ *is nondegenerate.*

In [9], the author described all the Frobenius algebraic Lie algebras $\mathcal{G} = R + N$ whose nilpotent radical N is abelian in the following two cases: the reductive Levi subalgebra R acts on N irreducibly and R is simple. He classified all the algebraic Frobenius algebras up to dimension 6. See also [20] and [21] for further computations.

We discuss now the evolution by deformation of the index of a Lie algebra. About deformation theory we refer to [12, 18, 19]. Let $\mathbb{V}$ be a $\mathbb{K}$-vector space and $\mathcal{G}_0 = (\mathbb{V}, [\ ,\ ]_0)$ be a Lie algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in $\mathbb{K}$ and $\mathbb{V}[[t]]$ be the set of formal power series whose coefficients are elements of $\mathbb{V}$. A *formal Lie deformation* of $\mathcal{G}_0$ is given by the $\mathbb{K}[[t]]$-bilinear map $[\ ,\ ]_t : \mathbb{V}[[t]] \times \mathbb{V}[[t]] \to \mathbb{V}[[t]]$ of the form $[\ ,\ ]_t = \sum_{i\geq 0} [\ ,\ ]_i t^i$, where each $[\ ,\ ]_i$ is a $\mathbb{K}$-bilinear map $[\ ,\ ]_i : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, satisfying the skew-symmetry and the Jacobi identity.

Proposition 2.9. *The index of a Lie algebra decreases by one parameter formal deformation.*

Proof. The rank of the matrix $\left(\left[x_i, x_j\right]\right)_{ij}$ increases by deformation, consequently the index decreases. $\square$

3. Nilpotent and Filiform Lie algebras

In this Section, we review the theory of nilpotent and filiform Lie algebras. Let $\mathcal{G}$ be a Lie algebra. We set $C^0\mathcal{G} = \mathcal{G}$ and $C^k\mathcal{G} = [C^{k-1}\mathcal{G}, \mathcal{G}]$, for $k > 0$. A Lie algebra $\mathcal{G}$ is said to be nilpotent if there exists an integer p such that $C^p\mathcal{G} = 0$. The smallest p such that $C^p\mathcal{G} = 0$ is called the nilindex of $\mathcal{G}$. Then, a nilpotent Lie algebra has a natural filtration given by the central descending sequence:

$$\mathcal{G} = C^0\mathcal{G} \supseteq C^1\mathcal{G} \supseteq \cdots C^{p-1}\mathcal{G} \supseteq C^p\mathcal{G} = 0.$$

We have the following characterization of nilpotent Lie algebras (Engel's Theorem).

Theorem 3.1. *A Lie algebra $\mathcal{G}$ is nilpotent if and only if the operator adx is nilpotent for all x in $\mathcal{G}$.*

In the study of nilpotent Lie algebras, filiform Lie algebras, which were introduced by M. Vergne, play an important role. An n-dimensional nilpotent Lie algebra is called *filiform* if its nilindex p equals $n-1$. The filiform Lie algebras are the nilpotent algebras with the largest nilindex. If $\mathcal{G}$ is an n-dimensional filiform Lie algebra, then we have $dimC^i\mathcal{G} = n - i \quad \text{for} \quad 2 \leq i \leq n$.

Another characterization of filiform Lie algebras uses characteristic sequences $c(\mathcal{G}) = sup\{c(x) : x \in \mathcal{G} \setminus [\mathcal{G}, \mathcal{G}]\}$, where $c(x)$ is the sequence, in decreasing order, of dimensions of characteristic subspaces of the nilpotent operator adx.

Definition 3.2. *An n-dimensional nilpotent Lie algebra is filiform if its characteristic sequence is of the form $c(\mathcal{G}) = (n-1, 1)$.*

4. Index of Graded filiform Lie algebras

The classification of filiform Lie algebras was given by Vergne ([26]) up to dimension 6 and then extended to dimension 11 by several authors (see [2, 3, 14, 17, 23]).

In the general case there is two classes L_n and Q_n of filiform Lie algebras which plays an important role in the study of the algebraic varieties of filiform and more generally nilpotent Lie algebras.

Let $\{x_1, \cdots, x_n\}$ be a basis of the $\mathbb{K}$-vector space L_n, the Lie algebra structure of L_n is defined by the following non-trivial brackets : $[x_1, x_i] = x_{i+1} \quad i = 2, ..., n-1$.

Let $\{x_1, \cdots, x_{n=2k}\}$ be a basis of the $\mathbb{K}$-vector space Q_n, the Lie algebra structure of Q_n is defined by the following non-trivial brackets.

$$\begin{aligned} Q_n : [x_1, x_i] &= x_{i+1} \quad i = 2, ..., n-1, \\ [x_i, x_{n-i+1}] &= (-1)^{i+1} x_n \quad i = 2, ..., k \quad \text{where } n = 2k. \end{aligned}$$

The classification of n-dimensional graded filiform Lie algebras yields two isomorphic classes L_n and Q_n when n is odd, and only the Lie algebra L_n when n is even.

It turns out that any filiform Lie algebra is isomorphic to a Lie algebra obtained as a deformation of a Lie algebra L_n.

4.1. Index of Filiform Lie algebras

We aim to compute the index of L_n and Q_n and regular vectors.

Index of L_n :.

Let $\{x_1, x_2, ..., x_n\}$ be a fixed basis of the vector space $\mathbb{V} = L_n$ and $\left\{x_1^*, ..., x_n^*\right\}$ be a basis of the dual space. Set $f = \sum_{i\geq 1} p_i x_i^* \in \mathbb{V}^*$.

Proposition 4.1. *For $n \geq 3$, the index of the n-dimensional filiform Lie algebra L_n is $\chi(L_n) = n - 2$. The regular vectors of L_n are of the form $f = p_1\, x_1^* + p_2\, x_2^* + p_s\, x_s^*$ where $s \in \{3, ..., n\}$ and $p_s \neq 0$.*

Proof. Since the corresponding matrix to the Lie algebra L_n is of the form

$$\begin{pmatrix} 0 & x_3 & \cdots & x_n & 0 \\ -x_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and its rank is 2, then $\chi(L_n) = n - 2$. The second assertion is obtained by a direct calculation:
We set $x = \sum_{i=1}^n a_i x_i$, $y = \sum_{j=1}^n b_j x_j$, $f = \sum_{s=1}^n p_s x_s^*$ and $\mathcal{G}^f = \{x \in \mathcal{G} : f([x, y]) = 0 \ \ \forall y \in \mathcal{G}\}$.
Then $f([x, y]) = 0$ implies $\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^n a_i b_j p_s x_s^* \left([x_i, x_j]\right) = 0$. It is equivalent to

$$\sum_{s=1}^{n} \sum_{j=2}^{n-1} a_1 b_j p_s x_s^* \left([x_1, x_j]\right) - \sum_{i=2}^{n-1} a_i b_1 p_s x_s^* \left([x_1, x_j]\right) = 0.$$

Then we obtain $\sum_{s=1}^n \sum_{i=2}^{n-1} (a_1 b_i - a_i b_1)\, p_s x_s^*(x_{i+1}) = 0$. The equation $\sum_{i=2}^{n-1} (a_1 b_i - a_i b_1)\, p_{i+1} = 0$ should hold for all b_i. It leads to the following system

$$\begin{cases} a_1 p_{i+1} = 0, \ \ 2 \leq i \leq n-1, \\ \sum_{i=2}^{n-1} a_i p_{i+1} = 0. \end{cases}$$

Therefore, one of the p_i satisfies $p_i \neq 0$ where $i \in \{3, ..., n\}$. □

Index of Q_n :.

Proposition 4.2. *For $n = 2k$ and $k \geq 2$, the index of the n-dimensional filiform Lie algebra Q_n is $\chi(Q_n) = 2$. The regular vectors of Q_n are of the form $f = \sum_{i=1}^n p_i x_i^*$ with $p_n \neq 0$.*

Proof. Since the corresponding matrix to the Lie algebra Q_n is of the form

$$\begin{pmatrix} 0 & x_3 & x_4 & \cdots & x_{n-1} & x_n & 0 \\ -x_3 & 0 & 0 & \cdots & 0 & -x_n & 0 \\ -x_4 & 0 & 0 & \cdots & x_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-1} & 0 & -x_n & \cdots & 0 & 0 & 0 \\ -x_n & x_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and its rank is $n - 2$, then $\chi(Q_n) = 2$. The second assertion is obtained by the following calculation.
Let $\{x_1, x_2, ..., x_n\}$ be a fixed basis of Q_n, $x = \sum_{i=1}^n a_i x_i$, $y = \sum_{j=1}^n b_j x_j$, $f = \sum_{s=1}^n p_s x_s^*$ and $\mathcal{G}^f = \{x \in \mathcal{G} : f([x, y]) = 0 \ \ \forall y \in \mathcal{G}\}$.

The equation $f([x,y]) = 0$ may be written as $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} a_i b_j p_s x_s^*\left([x_i,x_j]\right) = 0$. It is equivalent to $\sum_{i=2}^{n-1}(a_1b_i - a_ib_1)\,p_{i+1} + \sum_{i=2}^{n-1}(-1)^{i+1}(a_1b_{n-i+1} - a_{n-i+1}b_i)\,p_n = 0$. Then

$$\begin{cases} \sum_{i=2}^{n-1}(a_1p_{i+1})\,b_i = 0, \\ -b_1\sum_{i=2}^{n-1} a_ip_{i+1} = 0, \\ \sum_{i=2}^{n-1}(-1)^{i+1}(a_1p_n)\,b_{n-i+1} = 0, \\ -\sum_{i=2}^{n-1}(-1)^{i+1}(a_{n-i+1}p_n)\,b_i = 0. \end{cases}$$

Canceling the first and the last columns and the corresponding lines, leads to the following minor

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -x_n \\ 0 & 0 & \cdots & x_n & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -x_n & \cdots & 0 & 0 \\ x_n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Hence, we obtain $f = \sum_{i=1}^{n} p_ix_i^*$, with $p_n \neq 0$. □

Using Proposition 2.9, we obtain the following result.

Corollary 4.3. *The index of a filiform Lie algebra is less or equal to $n-2$.*

Proof. Any filiform Lie algebra $\mathcal{N}$ is obtained as a deformation of the Lie algebra L_n, since $\chi(L_n) = n-2$ and using Proposition 2.9, one has $\chi(\mathcal{N}) \leq n-2$. □

5. Index of Filiform Lie algebras of dimension ≤ 8

In this section, we compute the indexes of n-dimensional Filiform Lie algebras with $n < 8$. Let $\mathcal{G}$ be an n-dimensional Lie algebra. We set $\{x_1, x_2, \ldots\ldots, x_n\}$ be a fixed basis of $\mathbb{V} = \mathcal{G}$, $\left\{x_1^*, x_2^*, \ldots, x_n^*\right\}$ is the dual basis and $f = \sum_{i\geq 1} p_ix_i^*$.

5.1. *Filiform Lie algebras of dimension less than 6*

Any n-dimensional Lie algebras with $n \leq 5$ is isomorphic to one of the following Lie algebras.

Dimension 1 and 2 We have only the abelian Lie algebras.

Dimension 3

$\mathcal{F}_3^1 : [x_1,x_2] = x_3$.

Dimension 4

$\mathcal{F}_4^1 : [x_1,x_2] = x_3$, $[x_1,x_3] = x_4$.

Dimension 5

$\mathcal{F}_5^1 : [x_1,x_i] = x_{i+1}$, for $i = 2,3,4$.

$\mathcal{F}_5^2 : [x_1,x_i] = x_{i+1}$, for $i = 2,3,4$ and $[x_2,x_3] = x_5$.

The computations of the index using Proposition 2.4 lead to the following result.

Proposition 5.1. *The indexes of n-dimensional filiform Lie algebras with $n \leq 5$ are*

$$\chi\left(\mathcal{F}_3^1\right) = 1, \quad \chi\left(\mathcal{F}_4^1\right) = 2, \quad \chi\left(\mathcal{F}_5^1\right) = 3, \quad \chi\left(\mathcal{F}_5^2\right) = 1.$$

The regular vectors of $\mathcal{F}_n^1$ for $n = 3,4,5$ are of the form $f = \sum_{i=1}^{5} p_ix_i^*$ with one of $p_i \neq 0$, $i \in \{3,4,5\}$.
The regular vectors of $\mathcal{F}_5^2$ are of the form $f = \sum_{i=1}^{5} p_ix_i^*$ with $p_i \neq 0$, $i \in \{3,4,5\}$.

Proof. The filiform Lie algebras $\mathcal{F}_3^1, \mathcal{F}_4^1$ and $\mathcal{F}_5^1$ are of type L_n. For $\mathcal{F}_5^2$, the corresponding matrix is of rank 4, then the index is one. The regular vector are obtained by direct calculation. □

5.2. Filiform Lie algebras of dimension 6

Any n-dimensional Lie algebras with $n = 6$ is isomorphic to one of the following Lie algebras.
$\mathcal{F}_6^1 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$.
$\mathcal{F}_6^2 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_6$.
$\mathcal{F}_6^3 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_5] = x_6$, and $[x_3, x_4] = -x_6$.
$\mathcal{F}_6^4 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_5$, and $[x_2, x_4] = x_6$.
$\mathcal{F}_6^5 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_2, x_3] = x_5 - x_6, [x_2, x_4] = x_6, [x_2, x_5] = x_6, [x_3, x_4] = -x_6$.

Proposition 5.2. *The indexes of 6-dimensional filiform Lie algebras are*

$$\chi\left(\mathcal{F}_6^i\right) = 2 \textit{ for } i = 2, 4, 3, 5.$$
$$\chi\left(\mathcal{F}_6^1\right) = 4$$

The regular vectors of $\mathcal{F}_6^1$ are of the form $f = \sum_{i=1}^6 p_i x_i^*$with one of $p_i \neq 0,\ i = \{3, ..., 6\}$ (class of L_n).
The regular vectors of $\mathcal{F}_6^2$ are of the form $f = p_1 x_1^* + p_2 x_2^* + p(x_3^* + x_4^* + x_5^*) + p_5 x_5^*$.
The regular vectors of $\mathcal{F}_6^4$ are of the form $f = \sum_{i=1}^5 p_i x_i^*$ with $p_6 = 0$.
The regular vectors of $\mathcal{F}_6^i$ for $i = 3, 5$ are of the form $f = \sum_{i=1}^6 p_i x_i^*$ with one of $p_i \neq 0,\ i \in \{3, ...6\}$.

5.3. Filiform Lie algebras of dimension 7

Any n-dimensional Lie algebras with $n = 7$ is isomorphic to one of the following Lie algebras.
$\mathcal{F}_7^1 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5$, $[x_1, x_6] = \alpha x_7$, $[x_2, x_3] = (1+\alpha) x_5$, $[x_2, x_4] = (1+\alpha) x_6$, $[x_3, x_4] = x_7$.
$\mathcal{F}_7^2 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_5$, $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$.
$\mathcal{F}_7^3 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_5 + x_6$, $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$.
$\mathcal{F}_7^4 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_7$, $[x_2, x_5] = x_7$, $[x_3, x_4] = -x_7$.
$\mathcal{F}_7^5 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6 + x_7$, $[x_2, x_4] = x_7$.
$\mathcal{F}_7^6 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_7$.
$\mathcal{F}_7^7 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$, $[x_2, x_3] = x_7$.
$\mathcal{F}_7^8 : [x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4, 5, 6$ (class of L_n).

Proposition 5.3. *The indexes of 7-dimensional filiform Lie algebras are*

$$\chi\left(\mathcal{F}_7^i\right) = 3 \textit{ for } i = 2, 3, 5, 6, 7, \quad \chi\left(\mathcal{F}_7^4\right) = 1,$$
$$\chi\left(\mathcal{F}_7^1\right) = \begin{cases} 1 \textit{ if } \alpha \neq \{0, -1\} \\ 3 \textit{ if } \alpha = 0, \end{cases}$$
$$\chi\left(\mathcal{F}_7^8\right) = 5.$$

The regular vectors of $\mathcal{F}_7^i$ are given by the following table

item	*regular vectors*
$i = 1$	$f = \sum_{i=1}^5 p_i x_i^* + p(x_6^* + x_7^*),\ \textit{if } \alpha = 0$ $f = \sum_{i=1}^6 p_i x_i^* \textit{ with } p_i \neq 0,\ \textit{if } \alpha \neq 0$
$i = 2$	$f = p_1 x_1^* + p_2 x_2^* + p(x_4^* + x_5^* + x_6^*) \textit{ with } p \neq 0$
$i = 3$	$f = \sum_{i=1}^4 p_i x_i^*$
$i = 4$	$f = \sum_{i=1}^7 p_i x_i^* \textit{ with } p_4 = 0, p_3 = 0$
$i = 5$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_5^* + x_6^*)$
$i = 6$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p(x_4^* + x_5^*)$
$i = 7$	$f = p_1 x_1^* + p_2 x_2^* + p_3 x_3^* + p_4 x_4^* + p(x_6^* + x_7^*)$
$i = 8$	$f = \sum_{i=1}^7 p_i x_i^* \textit{with one of } p_i \neq 0\ \ i \in \{3, ...7\}$

6. Index of Graded quasi-filiform Lie algebras

The classification of naturally graded quasi-filiform Lie algebras is known and given in [15]. They have the characteristic sequence $(n-2,1,1)$ where n is the dimension of the Lie algebra.

Definition 6.1. *[15] An n-dimensional nilpotent Lie algebra $\mathcal{G}$ is said to be quasi-filiform if $C^{n-3}\mathcal{G} \neq 0$ and $C^{n-2}\mathcal{G} = 0$, where $C^0\mathcal{G} = \mathcal{G}$, $C^i\mathcal{G} = \left[\mathcal{G}, C^{i-1}\mathcal{G}\right]$.*

In the following, we describe the classification of naturally quasi-graded filiform Lie algebras.
Let $\mathcal{B} = \{x_0 x_2, ..., x_{n-1}\}$ be a basis of $\mathcal{G}$:

6.1. Naturally graded Quasi-filiform Lie algebras

We consider the following classes of n-dimensional Lie algebras which are naturally graded quasi-filiform Lie algebras.
We set
Split : $L_{n-1} \oplus \mathbb{C}$ $(n \geq 4)$:
$[x_0, x_i] = x_{i+1}. \quad 1 \leq i \leq n-3.$
$Q_{n-1} \oplus \mathbb{C}$ $(n \geq 7,\ n$ odd$)$,
$[x_0, x_i] = x_{i+1}. \quad 1 \leq i \leq n-3,$
$[x_i, x_{n-2-i}] = (-1)^{i-1} x_{n-2}. \quad 1 \leq i \leq \frac{n-3}{2}.$
Principal : $\mathcal{L}_{(n,r)}$ $\left(n \geq 5,\ r \text{ odd},\ 3 \leq r \leq 2\left[\frac{n-1}{2}\right] - 1\right)$:
$[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq n-3,$
$[x_i, x_{r-i}] = (-1)^{i-1} x_{n-1}, \quad 1 \leq i \leq \frac{r-1}{2},$
$Q_{(n,r)}$ $(n \geq 7,\ n$ odd, r odd, $3 \leq r \leq n-4)$:
$[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq n-3,$
$[x_i, x_{r-i}] = (-1)^{i-1} x_{n-1}, \quad 1 \leq i \leq \frac{r-1}{2},$
$[x_i, x_{n-2-i}] = (-1)^{i-1} x_{n-2}, \quad 1 \leq i \leq \frac{n-3}{2}.$
Terminal : $\mathcal{T}_{(n,n-3)}$ $(n$ even, $n \geq 6)$:
$[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq n-3,$
$[x_{n-1}, x_1] = \frac{n-4}{2} x_{n-2},$
$[x_i, x_{n-3-i}] = (-1)^{i-1}(x_{n-3} + x_{n-1}), \quad 1 \leq i \leq \frac{n-4}{2},$
$[x_i, x_{n-2-i}] = (-1)^{i-1} \frac{n-2-2i}{2} x_{n-2}, \quad 1 \leq i \leq \frac{n-4}{2},$
$\mathcal{T}_{(n,n-4)}$ $(n$ odd, $n \geq 7)$,
$[x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq n-3,$
$[x_{n-1}, x_1] = \frac{n-5}{2} x_{n-4+i}, \quad 1 \leq i \leq 2,$
$[x_i, x_{n-4-i}] = (-1)^{i-1}(x_{n-4} + x_{n-1}), \quad 1 \leq i \leq \frac{n-5}{2},$
$[x_i, x_{n-3-i}] = (-1)^{i-1} \frac{n-3-2i}{2} x_{n-2}, \quad 1 \leq i \leq \frac{n-5}{2},$
$[x_i, x_{n-2-i}] = (-1)^{i-1}(i-1) \frac{n-3-i}{2} x_{n-2}, \quad 1 \leq i \leq \frac{n-3}{2}.$
Moreover, we have the following 7-dimensional and 9-dimensional Lie algebra [11].

$$\varepsilon_{(7,3)} : \begin{cases} [x_0, x_i] = x_{i+1}, & 1 \leq i \leq 4, \\ [x_6, x_i] = x_{3+i}, & 1 \leq i \leq 2, \\ [x_1, x_2] = x_3 + x_6, \\ [x_1, x_i] = x_{i+1}, & 3 \leq i \leq 4. \end{cases}$$

$$\varepsilon^1_{(9,5)} : \begin{cases} [x_0, x_i] = x_{i+1}, \ \ 1 \leq i \leq 6, \\ [x_8, x_i] = 2x_{5+i}, \ \ 1 \leq i \leq 2, \\ [x_1, x_4] = x_5 + x_8, \ \ [x_1, x_5] = 2x_6, \\ [x_1, x_6] = 3x_7, \ \ [x_2, x_3] = -x_5 - x_8, \\ [x_2, x_4] = -x_6, \ \ [x_2, x_5] = -x_7. \end{cases}$$

$$\varepsilon^2_{(9,5)} : \begin{cases} [x_0, x_i] = x_{i+1}, \ \ 1 \le i \le 6, \\ [x_8, x_i] = 2x_{5+i}, \ \ 1 \le i \le 2, \\ [x_1, x_4] = x_5 + x_8, \ \ [x_1, x_5] = 2x_6, \\ [x_1, x_6] = x_7, \ \ [x_2, x_3] = -x_5 - x_8, \\ [x_2, x_4] = -x_6, [x_2, x_5] = x_7, [x_3, x_4] = -2x_7. \end{cases}$$

$$\varepsilon^3_{(9,5)} : \begin{cases} [x_0, x_i] = x_{i+1}, \ \ 1 \le i \le 6, \\ [x_0, x_8] = x_6, \ \ 1 \le i \le 2, \\ [x_1, x_4] = x_8, \ \ [x_3, x_4] = -3x_7, \\ [x_2, x_4] = -x_6, \ \ [x_1, x_5] = 2x_6, \\ [x_2, x_3] = -x_8, \ \ [x_2, x_5] = 2x_7. \end{cases}$$

are graded quasi-filiform Lie algebras.

Theorem 6.2. *[15] Every n-dimensional naturally graded quasi-filiform Lie algebra is isomorphic to one of the following Lie algebras :*

- *If n is even to* $L_{n-1} \oplus \mathbb{C}$, $\mathcal{T}_{(n,n-3)}$, *or* $\mathcal{L}_{(n,r)}$ *with r odd and* $3 \le r \le n-3$.
- *If n is odd to* $L_{n-1} \oplus \mathbb{C}$, $Q_{n-1} \oplus \mathbb{C}$, $\mathcal{L}_{(n,n-2)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$, *or* $Q_{(n,r)}$ *with r odd, and* $3 \le r \le n-4$. *In the case of* $n = 7$ *and* $n = 9$, *we add* $\varepsilon_{(7,3)}$, $\varepsilon^1_{(9,5)}$, $\varepsilon^2_{(9,5)}$, $\varepsilon^3_{(9,5)}$.

6.1.1. Index of graded quasi-filiform Lie algebras

In the following, we compute the indexes of graded quasi–filiform Lie algebras. Let $\mathcal{G}$ be an n-dimensional graded quasi-filiform Lie algebra.

Theorem 6.3. *Indexes of graded quasi-filiform Lie algebras are*

case where n ***is even***

1. $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.
2. $\chi(\mathcal{T}_{(n,n-3)}) = 2$.
3. $\chi(\mathcal{L}_{(n,r)}) = n - r - 1, \ \ 3 \le r \le n - 3$.

case where n ***is odd***

1. $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.
2. $\chi(Q_{n-1} \oplus \mathbb{C}) = 3$.
3. $\chi(\mathcal{L}_{(n,n-2)}) = 3$.
4. $\chi(\mathcal{T}_{(n,n-4)}) = 3$.
5. $\chi(\mathcal{L}_{(n,r)}) = n - r - 1, \ \ 3 \le r \le n - 3$.
6. $\chi(Q_{(n,r)}) = 3$.
7. $\chi\left(\varepsilon_{(7,3)}\right) = 3$.
8. $\chi\left(\varepsilon^1_{(9,5)}\right) = 3$.
9. $\chi\left(\varepsilon^i_{(9,5)}\right) = 2, \quad i = 2, 3$.

Proof. **Case where** n **is even**

The corresponding matrix to the graded quasi-filiform Lie algebra $L_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix} 0 & x_2 & \cdots & x_{n-1} & 0 & 0 \\ -x_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.
The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-3)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & x_{n-3}+x_{n-1} & \left(\frac{n-4}{2}\right)x_{n-2} & 0 & \left(\frac{n-4}{2}\right)x_{n-2} \\
-x_3 & 0 & 0 & \cdots & -\left(\frac{n-6}{2}\right)x_{n-2} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & -x_{n-3}-x_{n-1} & \left(\frac{n-6}{2}\right)x_{n-2} & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & -\left(\frac{n-4}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\left(\frac{n-4}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

Its rank is $n - 2$, then $\chi(\mathcal{T}_{(n,n-3)}) = 2$.
The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,r)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & 0 & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

For $3 \leq r \leq n-3$, its rank is $r + 1$. Then $\chi(\mathcal{L}_{(n,r)}) = n - r - 1$.
Case where n is odd :
The corresponding matrix to the graded quasi-filiform Lie algebra $L_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix}
0 & x_2 & \cdots & x_{n-1} & 0 & 0 \\
-x_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-1} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}$$

Its rank is 2, then $\chi(L_{n-1} \oplus \mathbb{C}) = n - 2$.
The corresponding matrix to the graded quasi-filiform Lie algebra $Q_{n-1} \oplus \mathbb{C}$ is of the form

$$\begin{pmatrix} 0 & x_2 & x_3 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\ -x_2 & 0 & 0 & \cdots & 0 & -x_{n-2} & 0 & 0 \\ -x_3 & 0 & 0 & \cdots & -x_{n-2} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-3} & 0 & -x_{n-2} & \cdots & 0 & 0 & 0 & 0 \\ -x_{n-2} & x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its rank is $n-3$, then $\chi(Q_{n-1} \oplus \mathbb{C}) = 3$.
The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,n-2)}$ is of the form

$$\begin{pmatrix} 0 & x_2 & x_3 & x_4 & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\ -x_2 & 0 & 0 & 0 & \cdots & 0 & -x_{n-1} & 0 & 0 \\ -x_3 & 0 & 0 & 0 & \cdots & x_{n-1} & 0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 \\ -x_{n-3} & 0 & -x_{n-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ -x_{n-2} & x_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its rank is $n-3$, then $\chi(\mathcal{L}_{(n,n-2)}) = 3$.
The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{L}_{(n,r)}$ is of the form

$$\begin{pmatrix} 0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\ -x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ -x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ -x_{n-2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

Its rank is $r+1$, then $\chi(\mathcal{L}_{(n,r)}) = n-r-1$, $3 \leq r \leq n-4$.
The corresponding matrix to the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-4)}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & x_4 & \cdots & x_{n-4} & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & 0 & \cdots & x_{n-4}+x_{n-1} & \frac{n-5}{2}x_{n-3} & 0 & 0 & -\left(\frac{n-5}{2}\right)x_{n-3} \\
-x_3 & 0 & 0 & 0 & \cdots & -\left(\frac{n-7}{2}\right)x_{n-3} & \frac{n-5}{2}x_{n-2} & 0 & 0 & -\left(\frac{n-5}{2}\right)x_{n-2} \\
-x_4 & 0 & 0 & 0 & \vdots & -2\left(\frac{n-6}{2}\right)x_{n-2} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-4} & -x_{n-4}-x_{n-1} & \left(\frac{n-7}{2}\right)x_{n-3} & 2\left(\frac{n-6}{2}\right)x_{n-2} & \cdots & 0 & 0 & 0 & 0 & 0 \\
-x_{n-3} & -\left(\frac{n-5}{2}\right)x_{n-3} & -\left(\frac{n-5}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
-x_{n-2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & \left(\frac{n-5}{2}\right)x_{n-3} & \left(\frac{n-5}{2}\right)x_{n-2} & 0 & \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Its rank is $n-3$, then $\chi\left(\mathcal{T}_{(n,n-4)}\right) = 3$.
The corresponding matrix to the graded quasi-filiform Lie algebra $Q_{\binom{}{n,r}}$ is of the form

$$\begin{pmatrix}
0 & x_2 & x_3 & \cdots & x_r & \cdots & x_{n-3} & x_{n-2} & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & -x_{n-1} & \cdots & 0 & -x_{n-2} & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_r & x_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-x_{n-3} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
-x_{n-2} & x_{n-2} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}$$

For $3 \leq r \leq n-4$, its rank is $n-3$. Then $\chi(Q_{(n,r)}) = 3$. $\square$

Remark 6.4. *There are no Frobenius quasi-filiform Lie algebra.*

6.1.2. Regular vectors

Proposition 6.5. *The regular vectors of the families $\mathcal{T}_{(n,n-3)}$, $\mathcal{T}_{(n,n-4)}$, $\mathcal{L}_{(n,r)}$ and $Q_{\binom{}{n,r}}$ are given by the following functionals f where x_i^* are the elements of the dual basis and p_i are parameters.*

1. $\mathcal{T}_{(n,n-3)}$:

$$f = \sum_{i=0}^{n-1} p_i\, x_i^* \text{ with } p_{n-2} \neq 0.$$

2. $\mathcal{T}_{(n,n-4)}$:

$$f = \sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-2} \neq 0.$$

3. $\mathcal{L}_{(n,r)}$: *n odd or even and $r < n-2$:*

$$f = \sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-1} \neq 0 \text{ and one of } p_i \neq 0 \text{ where } i \in \{r+1, ..., n-2\}.$$

4. $Q_{(n,r)}$:

$$f = \sum_{i=0}^{n-1} p_i\, x_i^* \text{ with } p_{n-2} \neq 0.$$

5. $\mathcal{L}_{(n,n-2)}$:

$$f = \sum_{i=0}^{n-1} p_i x_i^* \text{ with } p_{n-1} \neq 0.$$

Proof. $\mathcal{T}_{(n,n-3)}$:

The associate system of the graded quasi-filiform Lie algebra $\mathcal{T}_{(n,n-3)}$ is of the form:

$$\begin{cases} \sum\limits_{i=1}^{n-3} a_i p_{i+1} = 0, \\ a_0 p_2 - a_{n-4}(p_{n-3} + p_{n-1}) - \frac{n-4}{2} a_{n-3} p_{n-2} + \frac{n-4}{2} a_{n-1} p_{n-2} = 0, \\ a_0 p_{i+1} + (-1)^i a_{n-3-i}(p_{n-3} + p_{n-1}) - (-1)^i \frac{n-2-2i}{2} a_{n-2-i} p_{n-2} = 0, \quad i = 2, ...n-4, \\ a_0 p_{n-2} + \frac{n-4}{2} a_1 p_{n-2} = 0, \\ -\frac{n-4}{2} a_1 p_{n-2} = 0. \end{cases}$$

It turns out that $p_{n-2} \neq 0$ gives a solution of this system such that $\dim \mathcal{G}^f = \chi_{\mathcal{G}}$, then the regular vectors are given by : $f = \sum_{i=0}^{n-1} p_i\, x_i^*$ with $p_{n-2} \neq 0$.

$\mathcal{T}_{(n,n-4)}$:

The associate system is of the form:

$$\begin{cases} \sum\limits_{i=1}^{n-3} a_i p_{i+1} = 0, \\ a_0 p_2 - a_{n-5}(p_{n-4} + p_{n-1}) - \frac{n-5}{2} a_{n-4} p_{n-3} + \frac{n-5}{2} a_{n-1} p_{n-3} = 0, \\ a_0 p_3 + a_{n-6}(p_{n-4} + p_{n-1}) + \frac{n-7}{2} a_{n-5} p_{n-3} - \frac{n-5}{2} a_{n-4} p_{n-2} + \frac{n-5}{2} a_{n-1} p_{n-2} = 0, \\ a_0 p_{i+1} + (-1)^i a_{n-4-i}(p_{n-4} + p_{n-1}) + (-1)^i \frac{n-3-2i}{2} a_{n-3-i} p_{n-3} - (-1)^i \frac{n-3-i}{2} a_{n-2-i} p_{n-2} = 0 \\ \qquad\qquad i = 3, ...n-5, \\ a_0 p_{n-3} = 0, \\ -\frac{n-5}{2} a_1 p_{n-3} + \frac{n-5}{2} a_2 p_{n-2} = 0, \\ a_0 p_{n-3} + \frac{n-5}{2} a_1 p_{n-3} + \frac{n-5}{2} a_2 p_{n-2} = 0. \end{cases}$$

It follows that $p_{n-2} \neq 0$ gives a solution of this system such that $\dim \mathcal{G}^f = \chi_{\mathcal{G}}$, then the regular vectors are given by : $f = \sum_{i=0}^{n-1} p_i\, x_i^*$ with $p_{n-2} \neq 0$.

$\mathcal{L}_{(n,r)}$ n odd or even and $r < n-2$.

We cancel the columns $(r+1)$ until $(n-1)$ and the corresponding lines. We obtain the following minor

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -p_{n-1} \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -p_{n-1} & \cdots & 0 & 0 \\ p_{n-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is of non-zero determinant and this leads to $f = \sum_{i=0}^{n} p_i x_i^*$, with $p_{n-1} \neq 0$ and one of the p_i satisfies $p_i \neq 0$ where $i \in \{r+1, ..., n-2\}$.

The same reasoning and calculations are used for $Q_{(n,r)}$ and $\mathcal{L}_{(n,n-2)}$. $\square$

Remark 6.6. *Since $L_{n-1} \oplus \mathbb{C}$ and $Q_{n-1} \oplus \mathbb{C}$ are the central extension of L_n and Q_n, then the regular vectors could be given using Proposition 2.6.*

7. index of Lie algebras whose nilradical is L_n or Q_{2n}

Snobel and Winternitz determined the Lie algebras whose nilradical is isomorphic to the filiform Lie algebra L_n. In their work this algebra is denoted by $\mathfrak{n}_{n,1}$ and it is defined with respect to the basis $\{x_1, ..., x_n\}$ by

$$[x_i, x_n] = x_{i-1}, \quad i = 2, ..., n-1.$$

Theorem 7.1. *[11] Let τ be a solvable Lie algebra over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and having as nilradical $\mathfrak{n}_{n,1}$. Then it is isomorphic to one of the following Lie algebras.*

1. *If dim $\tau = n+1$, set $\mathcal{B} = \{x_1, ..., x_n, f\}$ be a basis of τ.*
   $\tau_{n+1,1}$ *defined as*
   $$[f, x_i] = (n-2+\beta)\, x_i, \quad i = 1, ..., n-1,$$
   $$[f, x_n] = x_n.$$
   $\tau_{n+1,2}$ *defined as*
   $$[f, x_i] = x_i, \quad i = 1, ..., n-1.$$
   $\tau_{n+1,3}$ *defined as*
   $$[f, x_i] = (n-i)\, x_i, \quad i = 1, ..., n-1,$$
   $$[f, x_n] = x_n + x_{n-1}.$$
2. *If dim $\tau = n+2$, set $\mathcal{B} = \{x_1, ..., x_n, f_1, f_2\}$ be a basis of τ.*
   $\tau_{n+2,1}$ *defined as*
   $$[f_1, x_i] = (n-1-i)\, x_i, i = 1, ..., n-1,$$
   $$[f_2, x_i] = x_i, \ i = 1, ..., n-1,$$
   $$[f_1, x_n] = x_n, \ i = 1, ..., n-1.$$

7.1. Index of Lie algebras whose nilradical is $\mathfrak{n}_{n,1}$ (L_n)

Proposition 7.2. *Indexes of Lie algebras whose nilradical is $\mathfrak{n}_{n,1}$ are*

If dim $\tau = n+1$, then $\chi(\tau_{n+1,i}) = n-1, i = 1, 2, 3.$
If dim $\tau = n+2$, then $\chi(\tau_{n+2,1}) = n-2.$

Proof. Set dim $\tau = n+1$. The corresponding matrix to the algebra $\tau_{n+1,1}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -(n-2+\beta)\,x_1 \\ 0 & 0 & \cdots & 0 & 0 & -(n-2+\beta)\,x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -(n-2+\beta)\,x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & -x_n \\ (n-2+\beta)\,x_1 & (n-2+\beta)\,x_2 & \cdots & (n-2+\beta)\,x_{n-1} & x_n & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,1}) = n-1$.
The corresponding matrix of the Lie algebra $\tau_{n+1,2}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -x_1 \\ 0 & 0 & \cdots & 0 & 0 & -x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ x_1 & x_2 & \cdots & x_{n-1} & 0 & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,2}) = n-1$.
The corresponding matrix to the Lie algebra $\tau_{n+1,3}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & -(n-1)\,x_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -(n-2)\,x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & -(n-(n-2))\,x_{n-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 & -x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & -x_n - x_{n-1} \\ (n-1)\,e_1 & (n-2)\,x_2 & \cdots & (n-(n-2))\,x_{n-2} & x_{n-1} & x_n + x_{n-1} & 0 \end{pmatrix}$$

Its rank is 2, then $\chi(\tau_{n+1,3}) = n-1$.
If dim $\tau = n+2$, the corresponding matrix to the Lie algebra $\tau_{n+2,1}$ is of the form:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & (n-2)\,x_1 & x_1 \\ 0 & 0 & \cdots & 0 & 0 & (n-3)\,x_2 & x_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & (n-n)\,x_{n-1} & x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & x_n & 0 \\ -(n-2)\,x_1 & -(n-3)\,x_2 & \cdots & -(n-n)\,x_{n-1} & -x_n & 0 & 0 \\ -x_1 & -x_2 & \cdots & -x_{n-1} & 0 & 0 & 0 \end{pmatrix}$$

Its rank is 4, then $\chi(\tau_{n+2,1}) = n-2$. □

7.1.1. Regular vectors

If dim $\tau = n+1$

1. $\tau_{n+1,1}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_n \neq 0.$$

2. $\tau_{n+1,2}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_{n-1} \neq 0.$$

3. $\tau_{n+1,3}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_n \neq 0.$$

If dim $\tau = n+2$

4. $\tau_{n+2,1}$

$$f = \sum_{i=1}^{n} p_i\, x_i^* \text{ with } p_1, ..., p_n \neq 0.$$

Proof. Straightforward calculations following Remark 2.7. □

7.2. Lie algebras whose nilradical is Q_{2n}

Proposition 7.3. *[11] Any real solvable Lie algebra of dimension $2n+1$ with nilradical Q_{2n} is isomorphic to one of the following Lie algebras:*

Let $\mathcal{B} = \{x_1, ..., x_{2n}, y\}$ be a basis of τ

1. $\tau_{2n+1}(\lambda_2)$
$$\begin{aligned}
&[x_1, x_k] = x_{k+1}, \ 2 \le k \le 2n-2,\\
&[x_k, x_{2n+1-k}] = (-1)^k x_{2n}, \ 2 \le k \le n,\\
&[y, x_1] = x_1,\\
&[y, x_k] = (k-2+\lambda_2)x_k, \ 2 \le k \le 2n-2,\\
&[y, x_{2n}] = (2n-3+2\lambda_2)x_{2n}.
\end{aligned}$$
2. $\tau_{2n+1}(2-n, \varepsilon)$
$$\begin{aligned}
&[x_1, x_k] = x_{k+1}, \ 2 \le k \le 2n-2,\\
&[x_k, x_{2n+1-k}] = (-1)^k x_{2n} \ 2 \le k \le n,\\
&[y, x_1] = x_1 + \varepsilon x_{2n}, \ \varepsilon = -1, 0, 1,\\
&[y, x_k] = (k-n)x_k, \ 2 \le k \le 2n-1,\\
&[y, x_{2n}] = x_{2n}.
\end{aligned}$$
3. $\tau_{2n+1}\left(\lambda_2^5, \cdots, \lambda_2^{2n-1}\right)$
$$\begin{aligned}
&[x_1, x_k] = x_{k+1}, \ 2 \le k \le 2n-2,\\
&[x_k, x_{2n+1-k}] = (-1)^k x_{2n}, \ 2 \le k \le n,\\
&[y, x_{2+t}] = x_{2+t} + \sum_{k=2}^{\left[\frac{2n-3-t}{2}\right]} \lambda_2^{2k+1} x_{2k+1+t}, \ 0 \le t \le 2n-6,\\
&[y, x_{2n-k}] = x_{2n-k}, \ k = 1, 2, 3,\\
&[y, x_{2n}] = 2x_{2n}.
\end{aligned}$$

7.2.1. Index of Lie algebras whose nilradical is Q_{2n}

Proposition 7.4. *Indexes of n-dimensional Lie algebras whose nilradical is Q_{2n} are*

$$\begin{aligned}
&\chi(\tau_{2n+1}(\lambda_2)) = 1,\\
&\chi(\tau_{2n+1}(2-n, \varepsilon)) = 1,\\
&\chi\left(\tau_{2n+1}\left(\lambda_2^5, ..., \lambda_2^{2n-1}\right)\right) = 1.
\end{aligned}$$

Proof. The corresponding matrix of the Lie algebra $\tau_{2n+1}(\lambda_2)$ is of the form:

$$\begin{pmatrix}
0 & x_3 & x_4 & \cdots & 0 & 0 & -x_1\\
-x_3 & 0 & 0 & \cdots & x_{2n} & 0 & -\lambda_2 x_2\\
-x_4 & 0 & 0 & \cdots & 0 & 0 & -(n-(n-1)+\lambda_2)x_3\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & -x_{2n} & 0 & \cdots & 0 & 0 & -(n-1+\lambda_2)x_{2n-1}\\
0 & 0 & 0 & \cdots & 0 & 0 & -(2n-3+2\lambda_2)x_{2n}\\
x_1 & \lambda_2 x_2 & (n-(n-1)+\lambda_2)x_3 & \cdots & (n-1+\lambda_2)x_{2n-1} & (2n-3+2\lambda_2)x_{2n} & 0
\end{pmatrix}$$

Its rank is $2n$, then $\chi(\tau_{2n+1}(\lambda_2)) = 1$.

The corresponding matrix of the algebra $\tau_{2n+1}(2-n, \varepsilon)$ is of the form

$$\begin{pmatrix}
0 & x_3 & x_4 & \cdots & x_{2n-1} & 0 & 0 & -x_1 - \varepsilon x_{2n}\\
-x_3 & 0 & 0 & \cdots & 0 & x_{2n} & 0 & -(n-2)x_2\\
-x_4 & 0 & 0 & \cdots & -x_{2n} & 0 & 0 & -(n-2)x_3\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
-x_{2n-1} & 0 & x_{2n} & \cdots & 0 & 0 & 0 & -(n-(2n-1))x_{2n-1}\\
0 & -x_{2n} & 0 & \cdots & 0 & 0 & 0 & -x_{2n}\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
x_1 + \varepsilon x_{2n} & (n-2)x_2 & (n-3)x_3 & \cdots & (n-(2n-1))x_{2n-1} & x_{2n} & 0 & 0
\end{pmatrix}$$

Its rank is $2n$, then $\chi(\tau_{2n+1}(2-n,\varepsilon)) = 1$.

Since the corresponding matrix of the algebra $\tau_{2n+1}\left(\lambda_2^5, ..., \lambda_2^{2n-1}\right)$ is of rank $2n$ then the index is 1. $\square$

Remark 7.5. *The procedure described in Remark 2.7 could be used to compute the regular vectors of Lie algebras whose nilradical is Q_{2n}.*

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