Existence of multiple positive solutions for a nonlocal boundary value problem with sign changing nonlinearities

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Abstract. In this paper, we study the existence of multiple positive solutions for a nonlocal boundary value problem where the nonlinear term f is allowed to change sign. We obtain at least two positive solutions by using a fixed point theorem in double cones.

1. Introduction

The study of multi-point boundary value problems (BVPs for short) for nonlinear second-order ordinary differential equations was initiated by Il'in and Moviseev [8, 9]. Gupta studied certain three-point BVPs [7], and several authors studied nonlinear second order BVPs with integral boundary conditions, see for example [11, 12] and the references therein. We quote also the research of A. Ashyralyev [1, 2] where nonlocal BVPs are considered for parabolic and elliptic differential and difference equations.
The fixed point theorem of Krasnoselskii is one of the methods used in these studies.
Karakostas in [11], by applying the Krasnoselskii fixed point theorem on a suitable cone, proved the existence of multiple positive solutions for a nonlocal BVPs of the form

$$u''(t) + q(t)f(u(t)) = 0, \qquad 0 < t < 1,$$

$$u(0) = 0, \ u(1) = \int_{\alpha}^{\beta} u(r)dg(r),$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, with $f(x) \geq 0$ when $x \geq 0$.

By using the fixed point theorem in double cones, Guo in [6] showed the existence of positive solutions for second-order three point BVP where the nonlinearity is allowed to change sign. And by a theorem similar to the one in [6], Xu [4] considered a nonlinear second-order m-point BVP where the nonlinear term is allowed to change sign.

In [10], Liu studied the existence of positive solutions for BVPs with integral boundary conditions and sign changing nonlinearities of the form:

$$(\varphi_p u')' + f(t, u) = 0 \qquad 0 < t < 1,$$

$$au(0) - bu'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \ u(1) = \int_0^1 g(s)u(s)ds.$$

In this paper we are concerned with existence results for BVP associated to nonautonomous second order differential equations when the non linearities are sign changing and with integral boundary conditions. In particular using the fixed point theorem in double cones, we study the following problem

$$u''(t) + q(t)f(u(t)) = 0, \qquad 0 < t < 1, \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u(1) = \int_\alpha^\beta u(r)dg(r), \tag{3}$$

where q and f verify the assumptions:

(H1) $0 < \alpha < \beta < 1$ and $g : [\alpha, \beta] \to \mathbb{R}$ is an increasing function, left continuous at $t = \beta$, right continuous at $t = \alpha$, and such that $\beta(g(\beta) - g(\alpha)) < 1$.
It is clear that without loss of generality we can assume that $g(\alpha) = 0$.

(H2) $f : [0, \infty) \to \mathbb{R}$ is continuous and $f(0) \geq (\not\equiv 0)$.

(H3) $q : [0, 1] \to [0, \infty)$ is continuous and not identically zero on $[\beta, 1]$.

We start by recalling the fixed point theorem in double cones.
For a cone K in a Banach space $(X, \|.\|)$ and a constant $r > 0$. Let $\theta : K \to \mathbb{R}^+$ a continuous functional such that $\theta(\lambda x) \leq \theta(x)$ for $\lambda \in (0, 1)$. For positive constants a, b we define the following sets:

$$K_r = \{x \in K : \|x\| < r\},$$

$$\partial K_r = \{x \in K : \|x\| = r\},$$

$$K(b) = \{x \in K : \theta(x) < b\},$$

$$\partial K(b) = \{x \in K : \theta(x) = b\},$$

and

$$K_a(b) = \{x \in K : a < \|x\|, \theta(x) < b\}.$$

Theorem 1.1. *Let X be a real Banach space with norm $\|.\|$ and $K, K' \subset X$ two solid cones with $K' \subset K$. Suppose $T : K \to K$ and $T^* : K' \to K'$ are two completely continuous operators and $\theta : K' \to \mathbb{R}^+$ is a continuous functional satisfying $\theta(x) \leq \|x\| \leq M\theta(x)$ for all $x \in K'$, where $M \geq 1$ is a constant. If there exist constants $b > a > 0$ such that*

(C1) $\|Tx\| < a$ for $x \in \partial K_a$;

(C2) $\|T^*x\| < a$ for $x \in \partial K'_a$ and $\theta(T^*x) > b$ for $x \in \partial K'(b)$;

(C3) $Tx = T^*x$, for $x \in K'_a(b) \cap \{u : T^*u = u\}$.

Then T has at least two fixed points y_1 and y_2 in K, such that

$$0 \leq \|y_1\| < a < \|y_2\|, \ \theta(y_2) < b.$$

The paper is organized as follows: Section 2 contains the basic preliminaries. The main result are given in Section 3.

2. Preliminaries

We present some lemmas that are important to prove our main results.

Denote by I the interval $[0, 1]$, and by X the space of all continuous functions $C(I)$. Let $X_0 = \{x \in X : x(0) = 0\}$. The spaces X and X_0 become Banach spaces when they are furnished with the usual sup-norm $\|.\|$.

Lemma 2.1. *If $u \in X_0$ is a concave function satisfying condition* (3) *and g is a function satisdying (H1), then we have*

(i) $u(t) \geq 0$, $t \in [0, 1]$,

(ii) $u(t) \geq \mu\|u\|$, $\qquad t \in [\alpha, 1]$,

where

$$\mu := \min\{\gamma, 1 - \beta, (\beta - \alpha)\gamma g(\beta)\},$$

and

$$\gamma := \min\left\{\alpha, 1 - \beta, \frac{1 - \beta}{1 - \alpha}\right\}.$$

(Notice that $0 < \mu < 1$).

Proof. We prove the lemma in three steps:

1. If $u(1) \geq 0$, then, by the concavity of u and the fact that $u(0) = 0$, we have

$$u(t) \geq 0, \qquad t \in [0, 1].$$

Assume that $u(1) < 0$. From (3), (H1) and the mean value theorem, it follows that there is $\xi_u \in [\alpha, \beta]$ such that $u(1) = u(\xi_u)g(\beta)$ (notice that $g(\alpha) = 0$).

Moreover, since $g(\beta) > 0$ and $u(1) < 0$, we have $u(\xi_u) < 0$. This and $\beta g(\beta) < 1$ lead to

$$u(1) = g(\beta)u(\xi_u) > \frac{1}{\beta}u(\xi_u) \geq \frac{1}{\xi_u}u(\xi_u),$$

which contradicts the concavity of u.

2. Now, we shall prove that, if u is a concave function in X_0, then

$$u(t) \geq \gamma\|u\|, \qquad s \in [\alpha, \beta].$$

Indeed let $t_0 \in [0, 1]$ be such that $\|u\| = u(t_0)$. We distinguish three cases:

- Case(1): $\beta \leq t_0$.

  Then $s \leq t_0$, for every $s \in [\alpha, \beta]$, and, since u is a concave function, we have $su(t_0) \leq t_0 u(s)$. Thus, $\alpha\|u\| \leq u(s)$, and hence
  $$u(s) \geq \gamma\|u\|.$$

- Case(2): $\alpha \leq t_0 \leq \beta$.

  If $s \in [\alpha, t_0]$, then following the same arguments as in case(1), we obtain
  $$u(s) \geq \gamma\|u\|.$$

  Let $s \in (t_0, \beta]$. Then we observe that
  $$\frac{1 - s}{1 - t_0} \leq \frac{u(s) - u(1)}{u(t_0) - u(1)},$$
  because of the concavity of the function u. Thus we have
  $$(1 - s)u(t_0) \leq (1 - t_0)u(s) + (t_0 - s)u(1) \leq (1 - t_0)u(s).$$

This implies that

$$(1 - \beta)u(t_0) \leq (1 - \alpha)u(s),$$

hence,

$$\frac{1 - \beta}{1 - \alpha} \|u\| \leq u(s),$$

and finally,

$$u(s) \geq \gamma \|u\|.$$

- Case(3): $t_0 < \alpha$.

  Then $t_0 < s$ for every $s \in [\alpha, \beta]$, and following the same arguments as in case(2), we obtain

  $$(1 - s)u(t_0) \leq (1 - t_0)u(s),$$

  which implies that

  $$(1 - \beta)\|u\| \leq u(s),$$

  and so,

  $$u(s) \geq \gamma \|u\|.$$

3. In order to show that

$$u(s) \geq \mu \|u\|, \ s \in [\alpha, 1],$$

we distinguish two cases , $u(\beta) < u(1)$ and $u(1) \leq u(\beta)$.

-If $u(\beta) < u(1)$, then by the concavity, for every $s \geq \beta$, we have $u(\beta) \leq u(s)$.

Therefore, by the above first part of the proof of Lemma 2.1 for all $s \in [\alpha, 1]$, we have

$$u(s) \geq \min\left\{\min\left\{u(s) : s \in [\alpha, \beta]\right\}, \min\left\{u(s) : s \in [\beta, 1]\right\}\right\},$$

and so

$$u(s) \geq \min\left\{\gamma\|u\|, u(\beta)\right\} = \gamma\|u\|.$$

-If $u(1) \leq u(\beta)$, then again, by the concavity, we have $u(s) \geq u(1)$, for every $s \in [\beta, 1]$. Therefore, from (3), for any such s we have

$$u(s) \geq \int_\alpha^\beta u(r)dg(r) \geq \gamma\|u\|(\beta - \alpha)g(\beta).$$

Hence in any case it holds $u(s) \geq \mu\|u\|, \ s \in [\alpha, 1]$ and the proof is complete. $\quad\square$

Lemma 2.2. *Let $\delta = 1 - \int_\alpha^\beta rdg(r) > 0$, if $y \in X$. Then the boundary-value problem*

$$u''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{4}$$

$$u(0) = 0, \ u(1) = \int_\alpha^\beta u(r)dg(r), \tag{5}$$

has a unique solution

$$u(t) = \frac{t}{\delta} \int_0^1 (1 - s)y(s)ds - \frac{t}{\delta} \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r) - \int_0^t (t - s)y(s)ds. \tag{6}$$

Proof. From (4), we have

$$u''(t) = -y(t).$$

For $t \in [0, 1)$, integration from 0 to t, gives

$$u'(t) = u'(0) - \int_0^t y(s)ds.$$

For $t \in [0, 1]$, integration from 0 to t, yields

$$u(t) = u'(0)t - \int_0^t (\int_0^r y(s)ds)dr,$$

i.e.,

$$u(t) = u'(0)t - \int_0^t (t - s)y(s)ds. \tag{7}$$

So,

$$u(1) = u'(0) - \int_0^1 (1 - s)y(s)ds.$$

Integrating (7) from α to β, where $0 < \alpha < \beta < 1$ we have

$$\int_\alpha^\beta u(r)dg(r) = - \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r) + u'(0) \int_\alpha^\beta rdg(r).$$

From (5), we obtain

$$u'(0) - \int_0^1 (1 - s)y(s)ds = u'(0) \int_\alpha^\beta rdg(r) - \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r).$$

Thus,

$$u'(0)(1 - \int_\alpha^\beta rdg(r)) = \int_0^1 (1 - s)y(s)ds - \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r),$$

and since $\delta = 1 - \int_\alpha^\beta rdg(r) > 0$,
then

$$u'(0) = \frac{1}{\delta} \left(\int_0^1 (1 - s)y(s)ds - \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r) \right).$$

Therefore, (4)-(5) has a unique solution

$$u(t) = \frac{t}{\delta} \int_0^1 (1 - s)y(s)ds - \frac{t}{\delta} \int_\alpha^\beta \int_0^r (r - s)y(s)dsdg(r) - \int_0^t (t - s)y(s)ds.$$

$\square$

Lemma 2.3. *Suppose that* $\delta = 1 - \int_\alpha^\beta rdg(r) > 0$. *Then the BVP*

$$-u''(t) = 0, \qquad 0 < t < 1,$$

$$u(0) = 0, \ u(1) = \int_\alpha^\beta u(r)dg(r),$$

has the following Green's function

$$G(t,s) = \frac{1}{\delta} \begin{cases} s\int_\alpha^\beta (t-r)dg(r) + s(1-t) & \text{if } 0 \le s < t < r \le \beta \text{ or } 0 \le s < r < t \le 1 \\ (t-s)\int_\alpha^\beta rdg(r) + s(1-t) & \text{if } \alpha \le r \le s \le t \le 1 \\ t\int_\alpha^\beta (s-r)dg(r) + t(1-s) & \text{if } 0 \le t \le s \le r \le \beta \\ (1-s)t & \text{if } \alpha \le r \le t \le s \le 1 \text{ or } 0 \le t < r \le s \le 1 \end{cases}$$

Proof. If $0 \le t \le r$, the unique solution (6) given by Lemma2.2 can be given in the form

$$\begin{aligned} u(t) &= \frac{1}{\delta}\left[\int_0^t \left(s\int_\alpha^\beta (t-r)dg(r) + s(1-t)\right) y(s)ds\right] \\ &+ \frac{1}{\delta}\left[\int_t^r \left(t(1-s) + t\int_\alpha^\beta (s-r)dg(r)\right) y(s)ds + \int_r^1 t(1-s)y(s)ds\right]. \end{aligned}$$

If $r \le t \le 1$, the unique solution (6) can be expressed

$$\begin{aligned} u(t) &= \frac{1}{\delta}\left[\int_0^r \left(s\int_\alpha^\beta (t-r)dg(r) + s(1-t)\right) y(s)ds\right] \\ &+ \frac{1}{\delta}\left[\int_r^t \left((t-s)\int_\alpha^\beta rdg(r) + s(1-t)\right) y(s)ds + \int_t^1 t(1-s)y(s)ds\right]. \end{aligned}$$

Therefore, the unique solution of (4)-(5) can be expressed $u(t) = \int_0^1 G(t,s)y(s)ds$. The proof is completed. $\square$

Consider

$$K = \{u \in X : u(t) \ge 0, t \in [0,1]\},$$

and

$$K' = \{u \in X_0 : u \text{ is concave and (3) holds}\}.$$

Clearly, $K, K' \subset X$ are cones with $K' \subset K$. For all $u \in K$, define

$$\theta(u) = \min_{\alpha \le t \le 1} u(t).$$

Let $(.)^+ = \max\{., 0\}$, we define the operators T, A and T^* by:
$T : K \to K$, $A : K \to X$ and $T^* : K' \to K'$, such that

$$\begin{aligned} Tu(t) &= \left[\int_0^1 G(t,s)q(s)f(u(s))ds\right]^+, & \text{for all } t \in [0,1], \\ Au(t) &= \int_0^1 G(t,s)q(s)f(u(s))ds, & \text{for all } t \in [0,1], \\ T^*u(t) &= \int_0^1 G(t,s)q(s)f^+(u(s))ds, & \text{for all } t \in [0,1], \end{aligned}$$

Remark 1. *If $\psi : X \to K$ is a function such that $(\psi u)(t) = u(t)^+$, then $T = \psi \circ A$.*

Lemma 2.4. *$T^* : K' \to K'$ is completely continuous.*

Proof. Let $u \in K'$, since $f^+(u(t)) \ge 0$ for all $t \in [0,1]$, then

$$(T^*u)''(t) = -q(t)f^+(u(t)) \le 0,$$

this implies that T^*u is concave function. It is clear that T^*u satisfies the boundary conditions (2), (3). Thus $T^* : K' \to K'$. By using the continuity of f and the definition of f^+, we can have that T^* is completely continuous from Ascoli-Arzela theorem. $\square$

Lemma 2.5. *A function $u(t)$ is a solution of BVP (1)-(3) if and only if $u(t)$ is a fixed point of the operator A.*

Lemma 2.6. *If $A : K \to X$ is completely continuous, then $T = \psi \circ A : K \to K$ is also completely continuous.*

Proof. The complete continuity of A implies that A is continuous and applies each bounded subset of K on a relatively compact set of X.
Given a function $h \in X$, for each $\varepsilon > 0$ there is $\sigma > 0$ such that

$$\|Ah - Ak\| < \varepsilon \text{ for } k \in X, \|h - k\| < \sigma.$$

Since

$$\left|(\psi A h)(t) - (\psi A k)(t)\right| = |\max\{(Ah)(t), 0\} - \max\{(Ak)(t), 0\}|$$

$$\leq |(Ah)(t) - (Ak)(t)| < \varepsilon,$$

we have

$$\left\|(\psi A)h - (\psi A)k\right\| < \varepsilon \text{ for } k \in X, \|h - k\| < \sigma,$$

and so ψA is continuous.

For any arbitrary bounded set $D \subset X$ and for all $\varepsilon > 0$, there are $y_i, i = 1, .., m$ such that

$$AD \subset \bigcup_{i=1}^{m} \beta(y_i, \varepsilon),$$

where $\beta(y_i, \varepsilon) = \left\{x \in X : \left\|x - y_i\right\| < \varepsilon\right\}$.
Then, if we denote ψy by $\bar{y}$ for all $\bar{y} \in (\psi o A)(D)$, there is a $y \in AD$ such that $\bar{y}(t) = \max\{y(t), 0\}$. We choose $y_i \in \{y_1, .., y_m\}$ such that

$$\max_{t \in [0,1]} |y(t) - y_i(t)| < \varepsilon.$$

Thus

$$\max_{t \in [0,1]} |\bar{y}(t) - \bar{y}_i(t)| \leq \max_{t \in [0,1]} |y(t) - y_i(t)| < \varepsilon,$$

which implies

$$\bar{y} \in B(\bar{y}_i, \varepsilon),$$

and therefore $(\psi o A)(D)$ is relatively compact. $\square$

By the continuity of f, we have $A : K \to X$ is completely continuous. $T : K \to K$ is completely continuous by using Lemma 2.6.

Lemma 2.7. *If u is a fixed point of the operator T, then u is also a fixed point of the operator A.*

Proof. Let u be a fixed point of the operator T. Obviously, if $Au(t) \geq 0$ for $t \in [0, 1]$, then u is a fixed point of the operator A. So, to prove the lemma, we show that if $Tu(t) = u(t)$, then $Au(t) \geq 0$ for $t \in [0, 1]$. Suppose on the contrary, that there is a $t_0 \in (0, 1)$ such that $Au(t_0) < 0 = u(0)$. Let (t_1, t_2) be the maximal interval which contains t_0 and such that $Au(t) < 0, t \in (t_1, t_2)$. It follows $[t_1, t_2] \neq [0, 1]$ by (H2).
If $t_2 < 1$, we have $u(t) = 0$ for $t \in [t_1, t_2]$, $Au(t) < 0$ for $t \in (t_1, t_2)$ and $Au(t_2) = 0$. Thus $(Au)'(t_2) = 0$. By (H2) we have $(Au)''(t) = -q(t)f(0) \leq 0$ for $t \in [t_1, t_2]$, so, $(Au)'(t) \geq 0$ for $t \in [t_1, t_2]$. We obtain $t_1 = 0$. Therefore, $Au(0) \leq Au(t_0) < 0$, we arrive at a contradiction.

If $t_1 > 0$, we have $u(t) = 0$ for $t \in [t_1, t_2]$, $Au(t) < 0$ for $t \in (t_1, t_2)$ and $(Au)(t_1) = 0$. Thus $(Au)'(t_1) \leq 0$. From (H2) we have $(Au)''(t) = -q(t)f(0) \leq 0$ for $t \in [t_1, t_2]$, so $t_2 = 1$. by the concavity of $Au(t)$ on $[t_1, 1]$ we have

$$\frac{|(Au)(s)|}{s - 1} \leq \frac{|(Au)(1)|}{1 - t_1}.$$

This implies that

$$|(Au)(s)| \leq \frac{s-1}{1-t_1}|(Au)(1)| < s|(Au)(1)|.$$

From the above inequalities, we obtain

$$\int_\alpha^\beta |Au(s)|dg(s) \leq \int_\alpha^\beta s|(Au)(1)|dg(s) < |(Au)(1)|,$$

which contradicts

$$|(Au)(1)| = \left|\int_\alpha^\beta Au(s)dg(s)\right| \leq \int_\alpha^\beta |Au(s)|dg(s).$$

Therefore u is a fixed point of operator A. $\quad\square$

3. Main results

In this section, we show the existence of two positive solutions for BVP (1)-(3) by applying a fixed-point theorem in double cones.
By definition of G, the continuity of G, and the fact that $\delta = 1 - \int_\alpha^\beta rdg(r) > 0$, it is clear that for $s \in [0,1]$:

$$\max_{0 \leq t \leq 1} G(t,s) = \frac{1-s}{\delta}.$$

Theorem 3.1. *Suppose that conditions (H1),(H2), and (H3) hold. Assume that there exist positive numbers a, b, and d such that*

$$0 < (1 + \frac{1}{\mu})\max\left\{1, \frac{1 - \int_\alpha^\beta sdg(s)}{\int_\alpha^\beta (1-s)dg(s)}\right\} d < a < \mu b < b,$$

and that f satisfies the following assumptions:

(H4) $f(u) \geq 0$ for $u \in [d,b]$;

(H5) $\frac{M_a}{\delta} \int_0^1 (1-s)q(s)ds < a$, where

$$M_a = \sup_{\|u\|=a} |f(u)|.$$

(H6) $b(1 - \alpha g(\beta)) \leq m_b \alpha \int_\beta^1 (1-s)q(s)ds$, where

$$m_b = \inf\{f(u) : u \in [\mu b, b]\}.$$

Then, (1)-(3) has at least two positive solutions u_1 and u_2 such that

$$0 \leq \|u_1\| < a < \|u_2\|, \; \theta(u_2) < b.$$

Proof. For all $u \in \partial K_a$, from (H5) we have

$$\|Tu\| = \max_{t\in[0,1]}\left[\int_0^1 G(t,s)q(s)f(u(s))ds\right]^+$$

$$\|Tu\| \leq \max_{t\in[0,1]}\max\left\{\int_0^1 G(t,s)q(s)f(u(s))ds, 0\right\}$$

$$\|Tu\| \leq M_a \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds$$

$$\leq \frac{M_a}{\delta} \int_0^1 (1-s)q(s)ds$$

$$< a.$$

So, (C1) of Theorem 1.1 is satisfied. for $u \in \partial K'_a$; i.e., $\|u\| = a$. from (H5) we have

$$\|T^*u\| = \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)f^+(u(s))ds$$

$$\leq M_a \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds$$

$$\leq \frac{M_a}{\delta} \int_0^1 (1-s)q(s)ds$$

$$< a.$$

Let $u \in \partial K'(\mu b)$; i.e., $u \in K'$ and $\theta(u) = \mu b$. For $t \in [\alpha, 1]$, we have

$$\mu b = \theta(u) = \min_{t\in[\alpha,1]} u(t) \geq \mu\|u\| \qquad \text{(from Lemma 2.1)},$$

hence

$$\|u\| \leq b.$$

On the other hand

$$u(t) \geq \min_{t\in[\alpha,1]} u(t) = \theta(u) = \mu b,$$

so

$$\mu b \leq u(t) \leq \|u\| \leq b,$$

and therefore it holds

$$f(u(s)) \geq m_b, \quad \text{for } s \in [\beta, 1]. \tag{8}$$

Observe that the unique solution of the BVP

$$\begin{cases} x''(t) + q(t)f^+(u(t)) = 0, & 0 < t < 1, \\ x(0) = 0, \ x(1) = \int_\alpha^\beta x(s)dg(s), \end{cases}$$

is the function $x(t)$ given by

$$x(t) = T^*u(t) = \int_0^1 G(t,s)q(s)f^+(u(s))ds$$

$$= \frac{t}{\delta} \int_0^1 (1-s)q(s)f^+(u(s))ds - \frac{t}{\delta} \int_\alpha^\beta \int_0^r (r-s)q(s)f^+(u(s))dsdg(r)$$

$$- \int_0^t (t-s)q(s)f^+(u(s))ds.$$

Let

$$E(x) := \left\{ \xi \in [\alpha, \beta] : \int_\alpha^\beta x(s)dg(s) = x(\xi) \int_\alpha^\beta dg(s) = x(\xi)g(\beta) \right\}$$

be the set of all mean values of x, by the function g. Obviously $E(x)$ is a compact set. Consider the point

$$\xi_x := \min E(x).$$

It is clear that x solves the BVP

$$\begin{cases} y''(t) + q(t)f^+(u(t)) = 0, & 0 < t < 1, \\ y(0) = 0, \; y(1) = y(\xi_x)g(\beta), \end{cases}$$

and so, x is the function given by the closed formula

$$x(t) = \frac{t}{\tau_x} \int_0^1 (1-s)q(s)f^+(u(s))ds - \frac{t}{\tau_x}g(\beta) \int_0^{\xi_x} (\xi_x - s)q(s)f^+(u(s))ds$$

$$- \int_0^t (t-s)q(s)f^+(u(s))ds,$$

for $t \in [0, 1]$, where

$$\tau_x := 1 - \xi_x g(\beta) > 0.$$

Notice that $\alpha \le \xi_x \le \beta$ and, in view of (H1), $\xi_x > 0$. Then we have

$$(T^*u)(\xi_x) = x(\xi_x) = \frac{\xi_x}{\tau_x} \int_0^1 (1-s)q(s)f^+(u(s))ds - \frac{\xi_x}{\tau_x}g(\beta) \int_0^{\xi_x} (\xi_x - s)q(s)f^+(u(s))dsdg(r)$$

$$- \int_0^{\xi_x} (\xi_x - s)q(s)f^+(u(s))ds,$$

$$= \frac{\xi_x}{\tau_x} \int_0^1 (1-s)q(s)f^+(u(s))ds - \frac{1}{\tau_x} \int_0^{\xi_x} (\xi_x - s)q(s)f^+(u(s))ds,$$

$$= \frac{\xi_x}{\tau_x} \int_0^1 q(s)f^+(u(s))ds - \frac{\xi_x}{\tau_x} \int_0^1 sq(s)f^+(u(s))ds$$

$$- \frac{\xi_x}{\tau_x} \int_0^{\xi_x} q(s)f^+(u(s))ds + \frac{1}{\tau_x} \int_0^{\xi_x} sq(s)f^+(u(s))ds,$$

$$= \frac{\xi_x}{\tau_x} \int_{\xi_x}^1 (1-s)q(s)f^+(u(s))ds + \frac{1}{\tau_x}(1-\xi_x) \int_0^{\xi_x} sq(s)f^+(u(s))ds.$$

Taking into account (H6), and (8), we finally obtain that

$$(T^*u)(\xi_x) \ge \frac{\alpha}{\tau_x} \int_\beta^1 (1-s)q(s)f^+(u(s))ds$$

$$(T^*u)(\xi_x) \ge \frac{\alpha m_b}{\tau_x} \int_\beta^1 (1-s)q(s)ds$$

$$\ge \frac{1}{\tau_x}(1 - \alpha g(\beta))b$$

$$\ge \frac{1}{\tau_x}(1 - \xi_x g(\beta))b$$

$$\ge b,$$

hence

$$\|T^*u\| \ge b.$$

On the other hand, from Lemma2.1, we have

$$\theta(T^*u) = \min_{t \in [\alpha, 1]} T^*u(t) \ge \mu \|T^*u\|,$$

thus

$$\theta(T^*u) \geq \mu b.$$

So, (C2) of Theorem 1.1 is satisfied. Finally, we show that (C3) of Theorem 1.1 is also satisfied. Let $u \in K'_a(\mu b) \cap \{u : T^*u = u\}$, then

$$\|u\| > a > (1 + \frac{1}{\mu}) \max \left\{ 1, \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} \right\} d.$$

From (ii) of Lemma 2.1 we have

$$u(\alpha) \geq \min_{t \in [\alpha,1]} u(t) \geq \mu \|u\| > \mu \frac{1}{\mu} \max \left\{ 1, \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} \right\} d,$$

hence,

$$u(\alpha) > \max \left\{ 1, \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} \right\} d. \tag{9}$$

If $u(1) \geq d$, then, by the concavity of u and the fact that $u(0) = 0$, we have $u(t) \geq d$, $t \in [0,1]$. Assume that $u(1) < d$. Since u is a concave function, thus for $s \in [\alpha, 1]$ we have

$$\frac{1}{1-s} \geq \frac{1-\alpha}{1-s} \geq \frac{u(\alpha) - u(1)}{u(s) - u(1)}.$$

This implies that

$$u(\alpha)(1-s) \leq u(s) - su(1).$$

So

$$u(\alpha) \int_\alpha^\beta (1-s) dg(s) \leq \int_\alpha^\beta u(s) dg(s) - \int_\alpha^\beta su(1) dg(s).$$

By $u(1) = \int_\alpha^\beta u(s) dg(s)$ we get

$$u(\alpha) \leq \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} u(1) < \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} d \leq \max \left\{ 1, \frac{1 - \int_\alpha^\beta s dg(s)}{\int_\alpha^\beta (1-s) dg(s)} \right\} d,$$

which contradicts to (9). So, $u(1) \geq d$. Therefore, for $u \in K'_a(\mu b) \cap \{u : T^*u = u\}$ we have

$$d \leq u(t) \leq \|u\| \leq \mu b \leq b.$$

From (H4) we know that

$$f^+(u(s)) = f(u(s)).$$

This implies that

$$Tu = T^*u.$$

So, the conditions of Theorem (1.1) are satisfied. Then T has two fixed points u_1 and u_2 satisfying

$$0 \leq \|u_1\| < a < \|u_2\|, \ \theta(u_2) < b.$$

The proof is complete. $\square$

4. Conclusion

The condition of the positivity of f is not essential for the application of the generalisation of Krasnoselskii's theorem. Indeed, we have established the existence and multiplicity of positive solutions in the case where the nonlinear term f is allowed to change sign.

We obtain the same result in the case where the nonlinearity is not autonomous, and without separated variables. So, we can consider $f(t, u)$ instead of $q(t)f(u)$, we obtain an existence and multiplicity results, if we replace the hypothesis (H2) and (H3) by:

(H2)′ $f : [0, 1] \times [0, \infty[\rightarrow \mathbb{R}$ such that:

(i) f is continuous on $[0, 1]$.

(ii) $f(t, 0) \geq (\not\equiv 0)$ for all t in $[0, 1]$.

(iii) $f(., u) \not\equiv 0$ on $[\beta, 1]$, for every $u \in]0, \infty[$.

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