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Generalizations of the Lagrange mean value theorem and applications

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Abstract. In this paper we give a generalization of the Lagrange mean value theorem via lower and upper derivative, as well as appropriate criteria of monotonicity and convexity for arbitrary function $f:(a,b)\to\mathbf{R}$. Some applications to the neoclassical economic growth model are given (from mathematical point of view).

1. Introduction

Bearing in mind the methodical and pedagogical point of view it seems interesting to consider whether a version of the Lagrange mean value theorem is valid without the assumption of continuity and differentiability of functions as well as what can be obtained using the Cantor principle (the principle of nested sequences) which states: the intersection of a nested sequence of intervals $\{[x_n, y_n]\}_{n\in\mathbf{N}}$ of the number, whose lengths tend to zero, contains a unique point.

For the convenience of the reader we prove some auxiliary results that may exist in some forms in the literature. There is a lot of literature related to the Lagrange mean value theorem, monotonicity and convexity; see for example the monograph [10], the literature cited there and for our purposes the papers [1, 9].

A characterization of monotonicity of an arbitrary real valued function defined on an interval is given in the paper [1].

The Lemma 2 in the paper [9] (see Lemma 2 below) is a possible generalization of the Lagrange mean value theorem. The examples 1 and 2 show that the condition of continuity of function in this lemma, cannot be omitted.

Section 2 contains some definitions and simple lemmas that we use in the paper.

As one of the main results of this paper, we give a generalization of Lemma 2 to arbitrary real valued function defined on an interval (a,b) (see Theorem 1 in Section 3, which we call the generalized Lagrange mean value theorem for arbitrary function).

Using this theorem we also give a characterization of monotonicity of arbitrary real valued function defined on an interval (a,b) both by the lower and the upper derivative. See Propositions 1 and 2 in Section 4. These two propositions also appear in [1], but our approach is different and our proofs are simpler.

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The characterization of convexity of arbitrary real valued function defined on an interval (a, b) both by the lower and the upper derivative is given in Section 5 (see Theorems 2 and 3). Note that Theorem 2 and Theorem 3 together state:

Function $f : (a, b) \to \mathbf{R}$ is convex if and only if the following condition holds:

(I) for $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ it follows that $\overline{D}f(x_1) \leqslant \underline{D}f(x_2)$.

It seems that this is the main result of Section 5.

In Section 6 we give further generalization (which is related to Lemma 2) of the Lagrange mean value theorem for continuous function using supporting lines.

In Section 7 we study the neoclassical economic growth model and in particular the golden rule of the capital accumulation from mathematical point of view.

In 1956 Nobel Prize Laureate Solow published a seminal paper [17]; this model makes great contribution in understanding potential sources of growth and limitation to economic growth in the long-run prosperity.

In the theory of neoclassical economic growth as model the equation

$$k' = sf - mk \tag{1}$$

appears, where k is the capital flow per capita (as the function of time t), f is the production function per capita (as the function of capital flow k), s and m are constants (see [17],[19]).

More precisely, we can write the equation (1) in the form

$$k'(t) = sf(k(t)) - mk(t).$$

Introduced by Phelps (1961, [18]), the golden rule of capital accumulation states the condition under which the stock of capital per worker maximizes steady state consumption.

In other words, we consider maximum of the consumption $c = f - sf$ over the set $\{(s, k) : 0 \leqslant s \leqslant 1, 0 \leqslant k \leqslant k^*(s)\}$, where $k^* = k^*(s)$ is the steady state of capital.

Suppose that maximum quantity of per capital consumption is c_{gold} and the corresponding capital is k_{gold}. Phelps 1961 proved that $f'(k_{\text{gold}}) = m$.

This result appears to have been discovered independently and more or less simultaneously (as often happens in science) by Phelps, AER (1961), Swan in Berrill ed. Economic Development (1964), Desrousseaux, Annales des Mines (1961), Joan Robinson, RES (1962) and von Weizsacker, Wachstum, Zins und Optimale Investitionsquote (1962), and probably the others (see [19]).

It is assumed that the production function is concave in the neoclassical economic growth model. If the production function is only concave the derivative exists except at a countable set of points. In this setting we outline how to use the upper and the lower derivative to get a version of the golden rule of capital accumulation if we drop the concavity hypothesis and suppose that only conditions (A1) and (A2) hold (see below Section 7) .

If we consider the subject from mathematical point of view, it seems appropriate to say that here we indicate how to clarify and generalize the original proof and we also extend some results from [6] using the generalized Lagrange mean value theorem.

2. Preliminaries

We will use the extended set of real number, the set $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$, with obvious ordering. If $X \subset \overline{\mathbf{R}}$ and $X \neq \emptyset$, the supremum and the infimum of X exist in $\overline{\mathbf{R}}$.

Definition 1. *Let $X \subset \mathbf{R}$, $f : X \to \mathbf{R}$ and let x_0 be a limit point of the set X. For $\varepsilon > 0$ we define*

$$m_f(x_0, \varepsilon) = \inf\{f(x) : x \in (X \cap (x_0 - \varepsilon, x_0 + \varepsilon))\backslash\{x_0\}\}$$

and

$$M_f(x_0, \varepsilon) = \sup\{f(x) : x \in (X \cap (x_0 - \varepsilon, x_0 + \varepsilon))\backslash\{x_0\}\}.$$

Using m_f and M_f we define

$$\liminf_{x\to x_0} f(x) = \sup\{m_f(x_0,\varepsilon) : \varepsilon > 0\}$$

and

$$\limsup_{x\to x_0} f(x) = \inf\{M_f(x_0,\varepsilon) : \varepsilon > 0\}.$$

Remark 1. *Note that*

$$\liminf_{x\to x_0} f(x) = \lim_{\varepsilon\to 0+} m_f(x_0,\varepsilon)$$

and

$$\limsup_{x\to x_0} f(x) = \lim_{\varepsilon\to 0+} M_f(x_0,\varepsilon).$$

Hence it is clear that

$$\liminf_{x\to x_0} f(x) \leqslant \limsup_{x\to x_0} f(x).$$

Let $f : (a,b) \to \mathbf{R}$ and $x_0 \in (a,b)$. We introduce

$$D_f(x_0,x) = \frac{f(x)-f(x_0)}{x-x_0},$$

and define

$$\overline{D}f(x_0) = \limsup_{x\to x_0} D_f(x_0,x) \quad \text{and} \quad \underline{D}f(x_0) = \liminf_{x\to x_0} D_f(x_0,x).$$

Note that these limits exist, belong to $\overline{\mathbf{R}}$, and are called *upper* and *lower derivative* of f *at point* x_0, respectively.

By Remark 1 it follows that $\underline{D}f(x_0) \leqslant \overline{D}f(x_0)$.

Note that the function f is differentiable at the point x_0 if and only if $\underline{D}f(x_0)$ and $\overline{D}f(x_0)$ are finite and $\underline{D}f(x_0) = \overline{D}f(x_0)$.

In that case $f'(x_0) = \underline{D}f(x_0) = \overline{D}f(x_0)$.

Lemma 1. *Let $f : (a,b) \to \mathbf{R}$ and $x_0 \in (a,b)$. Then $\underline{D}(-f)(x_0) = -\overline{D}f(x_0)$ and $\overline{D}(-f)(x_0) = -\underline{D}f(x_0)$.*

Proof. The proof follows from the definitions of

$$\underline{D}(-f)(x_0), \quad \overline{D}f(x_0), \quad \overline{D}(-f)(x_0), \quad \underline{D}f(x_0)$$

and the fact that for a non-empty set $A \subset \mathbf{R}$ the equalities $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$ hold. □

For convenience of the reader we first state Lemma 2 [9], which is mentioned in the introduction.

Lemma 2. *Suppose $f : [a,b] \to \mathbf{R}$ is continuous function on $[a,b]$. Then there exists $c \in (a,b)$ such that*

$$\underline{D}f(c) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant \overline{D}f(c).$$

The following examples show that the condition of continuity of function in this lemma, cannot be omitted.

Example 1. *Let* $f : [0,2] \to \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} x, & x \in [0,1) \\ 2, & x = 1 \\ 3 + \frac{x-1}{2}, & x \in (1,2]. \end{cases}$$

Example 2. *Let* $f : [-1,1] \to \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} \frac{1}{x}, & x \in [-1,0) \cup (0,1] \\ 0, & x = 0. \end{cases}$$

3. Main result

Theorem 1. *Let* $f : (a,b) \to \mathbf{R}$ *and let* $x, y \in (a,b)$*, such that* $x < y$*. Then there exist* $x_0, y_0 \in [x,y]$ *such that*

$$\overline{D}f(x_0) \leqslant \frac{f(y) - f(x)}{y - x} \leqslant \underline{D}f(y_0).$$

Proof. Let $z = \frac{x+y}{2}$. We consider the quotients

$$\frac{f(z) - f(x)}{z - x} \tag{2}$$

and

$$\frac{f(y) - f(z)}{y - z}. \tag{3}$$

One of the quotients (2) and (3) is less than or equal to, and the other is greater than or equal to $\frac{f(y)-f(x)}{y-x}$.
If (2) is less than or equal to (3), we denote $x_1 = x$, $y_1 = z$;
If (3) is less than or equal to (2), we denote $x_1 = z$, $y_1 = y$;
Further, let $z_1 = \frac{x_1+y_1}{2}$. Consider the quotients

$$\frac{f(z_1) - f(x_1)}{z_1 - x_1} \tag{4}$$

and

$$\frac{f(y_1) - f(z_1)}{y_1 - z_1}. \tag{5}$$

One of the quotients (4) and (5) is less than or equal to, and the other is greater than or equal to $\frac{f(y_1)-f(x_1)}{y_1-x_1}$.
If (4) is less than or equal to (5), we denote $x_2 = x_1$, $y_2 = z_1$;
If (5) is less than or equal to (4), we denote $x_2 = z_1$, $y_2 = y_1$;
Further, let $z_2 = \frac{x_2+y_2}{2}$. Consider the quotients

$$\frac{f(z_2) - f(x_2)}{z_2 - x_2} \tag{6}$$

and

$$\frac{f(y_2) - f(z_2)}{y_2 - z_2}. \tag{7}$$

One of the quotients (6) and (7) is less than or equal to, and the other is greater than or equal to $\frac{f(y_2)-f(x_2)}{y_2-x_2}$.
If (6) is less than or equal to (7), we denote $x_3 = x_2$, $y_3 = z_2$;
If (7) is less than or equal to (6), we denote $x_3 = z_2$, $y_3 = y_2$;
We continue with this procedure and thus we obtain a sequence of nested segments

$$[x_1, y_1] \supset [x_2, y_2] \supset \ldots \supset [x_n, y_n] \supset \ldots$$

as well as the sequence of inequalities

$$\frac{f(y)-f(x)}{y-x} \geqslant \frac{f(y_1)-f(x_1)}{y_1-x_1} \geqslant \ldots \geqslant \frac{f(y_n)-f(x_n)}{y_n-x_n} \geqslant \ldots$$

Note that $y_n - x_n = \frac{y_1-x_1}{2^{n-1}}$, so the sequence of nested segments $[x_n, y_n]$ has only one common point. Let $\{x_0\} = \cap_{n\in\mathbf{N}}[x_n, y_n]$. We now prove that $\underline{D}f(x_0) \leqslant \frac{f(y)-f(x)}{y-x}$. In order to, we first prove that for every $\varepsilon > 0$, such that $(x_0-\varepsilon, x_0+\varepsilon) \subset (a, b)$, it follows that

$$\inf\left\{\frac{f(x)-f(x_0)}{x-x_0} : x_0-\varepsilon < x < x_0+\varepsilon, x \neq x_0\right\} \leqslant \frac{f(y)-f(x)}{y-x}.$$

Fix any such ε. Then exists $n \in \mathbf{N}$ such that $x_0 \in [x_n, y_n] \subset (x_0-\varepsilon, x_0+\varepsilon)$. Denote by A_ε the set $\left\{\frac{f(x)-f(x_0)}{x-x_0} : x_0-\varepsilon < x < x_0+\varepsilon, x \neq x_0\right\}$.

Assume that $x_0 \neq x_n$ and $x_0 \neq y_n$. By using the same argument as in the forming of the sequence $[x_n, y_n]$, we obtain that $\frac{f(y_n)-f(x_0)}{y_n-x_0}$ or $\frac{f(x_0)-f(x_n)}{x_0-x_n}$ is less than or equal to $\frac{f(y_n)-f(x_n)}{y_n-x_n}$ and therefore from $\frac{f(y)-f(x)}{y-x}$. Hence, since $x_n, y_n \in (x_0-\varepsilon, x_0+\varepsilon)\backslash\{x_0\}$ we obtain at least one element of the set A_ε less than or equal to $\frac{f(y)-f(x)}{y-x}$. Therefore we obtain that $\inf A_\varepsilon$ is less than or equal to $\frac{f(y)-f(x)}{y-x}$.

If $x_0 = x_n$ (similar if $x_0 = y_n$) we obtain that $\frac{f(y_n)-f(x_0)}{y_n-x_0} = \frac{f(y_n)-f(x_n)}{y_n-x_n}$ is an element of the set A_ε, less than or equal to $\frac{f(y)-f(x)}{y-x}$, therefore, this way we also obtain at least one element of the set A_ε less than or equal to $\frac{f(y)-f(x)}{y-x}$, hence $\inf A_\varepsilon$ is less than or equal to $\frac{f(y)-f(x)}{y-x}$.

As ε was arbitrary, we thus obtain that any element of the set $B = \{\inf A_\varepsilon : \varepsilon > 0\}$ is less than or equal to $\frac{f(y)-f(x)}{y-x}$. Finally, it follows that

$$\sup B \leqslant \frac{f(y)-f(x)}{y-x},$$

as well as

$$\underline{D}f(x_0) \leqslant \frac{f(y)-f(x)}{y-x}.$$

This proves the one part of inequality. In order to prove the other inequality, we apply the above proved inequality, but for function $-f$. Namely, there exists $y_0 \in [x, y]$ such that

$$\underline{D}(-f)(y_0) \leqslant \frac{(-f)(y)-(-f)(x)}{y-x}.$$

Based on Lemma 2, we obtain

$$-\overline{D}f(y_0) \leqslant -\frac{f(y)-f(x)}{y-x}$$

i.e.

$$\overline{D}f(y_0) \geqslant \frac{f(y)-f(x)}{y-x}$$

which proves the other inequality. □

4. The Characterization of Monotonicity

For the sake of completeness, we give the definition of monotonicity of function.

Definition 2. *Let $X \subset \mathbf{R}$. A function $f : X \to \mathbf{R}$ is monotonically increasing (respectively, monotonically decreasing) on X, if for all $x_1, x_2 \in X$ such that $x_1 < x_2$ it follows that $f(x_1) \leqslant f(x_2)$ (respectively, $f(x_1) \geqslant f(x_2)$).*

Now we prove two propositions which are consequences of Theorem 1 and which give characterization of the monotonous function by the upper and by the lower derivative.

Proposition 1. *Let $f : (a,b) \to \mathbf{R}$. Function f is increasing if and only if for every $x \in (a,b)$ it follows that $\underline{D}f(x) \geqslant 0$.*

Proof. Suppose that f is increasing and $x_0 \in (a,b)$. It follows that

$$\underline{D}f(x_0) = \sup\left\{\inf\left\{\frac{f(x)-f(x_0)}{x-x_0} : x_0-\varepsilon < x < x_0+\varepsilon, x \neq x_0\right\} : \varepsilon > 0\right\}.$$

Since f is increasing, for $x > x_0$ it follows that $f(x) - f(x_0) \geqslant 0$, i.e. for $x < x_0$ it follows that $f(x) - f(x_0) \leqslant 0$. Hence we get $\frac{f(x)-f(x_0)}{x-x_0} \geqslant 0$. Further, since the infimum of the set whose elements are non-negative is also non-negative, and the same is true for the supremum of the set whose elements are non-negative, we obtain $\underline{D}f(x) \geqslant 0$.

Conversely, suppose that $\underline{D}f(x) \geqslant 0$ for every $x \in (a,b)$ and that the function f is not increasing. Then there exist $x, y \in (a,b)$ such that $x < y$ and $f(x) > f(y)$. By Theorem 1, it follows that there exists $z \in [x,y]$ such that

$$\underline{D}f(z) \leqslant \frac{f(y)-f(x)}{y-x} < 0.$$

However, that contradicts the assumption that for every $x \in (a,b)$ it holds that $\underline{D}f(x) \geqslant 0$. □

Proposition 2. *Let $f : (a,b) \to \mathbf{R}$. Function f is decreasing if and only if for every $x \in (a,b)$ it follows that $\overline{D}f(x) \leqslant 0$.*

Proof. The function f is decreasing if the function $-f$ is increasing. Further, by Proposition 1, $-f$ is increasing if and only if $\underline{D}(-f)(x) \geqslant 0$ for every $x \in (a,b)$. By Lemma 1 $\underline{D}(-f)(x) \geqslant 0$ for every $x \in (a,b)$ if and only if $\overline{D}f(x) \leqslant 0$ for $x \in (a,b)$. □

5. The Characterization of Convexity

For the sake of completeness, we give the definition of convexity of a function. Also, we give a simple well-known lemma that we use.

Definition 3. *Let $X \subset \mathbf{R}$ be an interval. A function $f : X \to \mathbf{R}$ is called convex (respectively concave) on X, if for every $x_1, x_2 \in X$ and every $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 + \alpha_2 = 1$ it follows that*

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$$(\text{respectively, } f(\alpha_1 x_1 + \alpha_2 x_2) \geqslant \alpha_1 f(x_1) + \alpha_2 f(x_2)).$$

Lemma 3. *Let $X \subset \mathbf{R}$ be an interval. A function $f : X \to \mathbf{R}$ is convex on X if and only if for every $x_1, x_2 \in X$ such that $x_1 < x_2$ and for every $x \in (x_1, x_2)$ the following inequality holds*

$$\frac{f(x)-f(x_1)}{x-x_1} \leqslant \frac{f(x_2)-f(x)}{x_2-x}.$$

Theorem 2. *Let $f : (a, b) \to \mathbf{R}$ be convex. Then, the following condition holds:*

(I) *for $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ it follows that $\overline{D}f(x_1) \leqslant \underline{D}f(x_2)$.*

Proof. If f is convex on (a, b), then f'_- and f'_+ exist at every point of the interval (a, b). Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ then $f'_-(x_1) \leqslant f'_+(x_1) \leqslant f'_-(x_2) \leqslant f'_+(x_2)$. Hence we obtain the inequality $\max\{f'_-(x_1), f'_+(x_1)\} \leqslant \min\{f'_-(x_2), f'_+(x_2)\}$ i.e. $\overline{D}f(x_1) \leqslant \underline{D}f(x_2)$. □

It is natural to ask whether or not the converse of Theorem 2 is also true? In other words if the condition (I) is sufficient for the convexity of function f on the corresponding interval? Theorem 3 shows that the answer is positive.

Recall that convexity of function (defined on open interval) implies continuity of function.

In order to prove Theorem 3 we first show that the condition (I) implies the continuity of the corresponding function. Really, Lemma 4 states that the condition (I) implies the condition

(I1) *the lower and the upper derivative are finite on the corresponding interval,*

and Lemma 5 states that the condition (I1) implies the continuity of the function f.

We first prove Lemmas 4 and 5 and then using Lemma 2, Lemma 3, Lemma 4 and Lemma 5 we prove Theorem 3. Note that the proof of Lemma 4 is essentially based on Theorem 1.

Lemma 4. *Let $f : (a, b) \to \mathbf{R}$ and suppose that the condition* (I) *holds. Then $-\infty < \underline{D}f(x) < +\infty$ and $-\infty < \overline{D}f(x) < +\infty$ for every $x \in (a, b)$.*

Proof. On the contrary, suppose that there exists a point $\xi_0 \in (a, b)$ such that at least one of derivatives $\underline{D}f(\xi_0)$ or $\overline{D}f(\xi_0)$ is not finite. For example let $\underline{D}f(\xi_0) = -\infty$. Then from the assumption it follows that $\overline{D}f(\xi) \leqslant \underline{D}f(\xi_0) = -\infty$ for every $\xi \in (a, \xi_0)$. Let $x, y \in (a, \xi_0)$ such that $x < y$. By Theorem 1 there exists $y_0 \in [x, y]$ such that $-\infty = \overline{D}f(y_0) \geqslant \frac{f(y)-f(x)}{y-x}$. Since f is a real- valued function , we have that $\frac{f(y)-f(x)}{y-x} > -\infty$. Hence the hypothesis that $\underline{D}f(\xi_0) = -\infty$ leads to a contradiction. In a similar way we get a contradiction if we assume that $\underline{D}f(\xi_0) = +\infty$, $\overline{D}f(\xi_0) = -\infty$ or $\overline{D}f(\xi_0) = +\infty$. □

Lemma 5. *Let $f : (a, b) \to \mathbf{R}$. Suppose that $-\infty < \underline{D}f(x) < +\infty$ and $-\infty < \overline{D}f(x) < +\infty$ for every $x \in (a, b)$. Then the function f is continuous on the interval (a, b).*

Proof. On the contrary, suppose that there exists a point $x_0 \in (a, b)$ such that the function f has a discontinuity at the point x_0.

We give the proof only in the case if (i1) the function f is discontinuous at x_0 from the right side and (i2) $\lim\limits_{x \to x_0} f(x)$ exists (then it is not equal to $f(x_0)$). (We note that it may happen that the limit in (i2) does not exist, but the proof in that case is similar). Hence, it follows that there exists $\alpha > 0$ such that for all $\beta > 0$ there exists $x \in (x_0, x_0 + \beta) \cap (a, b)$ such that $|f(x) - f(x_0)| \geqslant \alpha$. The condition $|f(x) - f(x_0)| \geqslant \alpha$ means that exactly one of the following two inequalities is true: $f(x) - f(x_0) \geqslant \alpha$ or $f(x) - f(x_0) \leqslant -\alpha$.

If $f(x) - f(x_0) \geqslant \alpha$ then $\frac{f(x)-f(x_0)}{x-x_0} \geqslant \frac{\alpha}{\beta}$ and therefore

$$\sup\left\{\frac{f(x) - f(x_0)}{x - x_0} : x \in (X \cap (x_0, x_0 + \beta))\backslash\{x_0\}\right\} \geqslant \frac{\alpha}{\beta}.$$

Passing to limit, we obtain

$$\lim_{\beta \to 0+} \sup\left\{\frac{f(x) - f(x_0)}{x - x_0} : x \in (X \cap (x_0, x_0 + \beta))\backslash\{x_0\}\right\} \geqslant \lim_{\beta \to 0+} \frac{\alpha}{\beta} = +\infty.$$

Hence $\overline{D}f(x_0) = +\infty$, contrary to hypothesis.

If $f(x) - f(x_0) \leqslant -\alpha$ then $\frac{f(x)-f(x_0)}{x-x_0} \leqslant -\frac{\alpha}{\beta}$ and therefore

$$\inf\left\{\frac{f(x)-f(x_0)}{x-x_0} : x \in (X \cap (x_0, x_0+\beta))\backslash\{x_0\}\right\} \leqslant -\frac{\alpha}{\beta}.$$

Passing to the limit, we obtain

$$\lim_{\beta\to 0+} \inf\left\{\frac{f(x)-f(x_0)}{x-x_0} : x \in (X \cap (x_0, x_0+\beta))\backslash\{x_0\}\right\} \leqslant \lim_{\beta\to 0+} -\frac{\alpha}{\beta} = -\infty.$$

Hence $\underline{D}f(x_0) = -\infty$, contrary to hypothesis.

A similar consideration can be applied to the case when the function f is discontinuous at x_0 from the left side. □

Now we formulate and prove the announced converse of Theorem 2.

Theorem 3. *Let $f : (a, b) \to \mathbf{R}$ and suppose that condition* (I) *holds. Then f is a convex function.*

Proof. By Lemmas 4 and 5 it follows that the function f is continuous on the interval (a, b).

Let $\xi_1, \xi_2 \in (a, b)$ such that $\xi_1 < \xi_2$ and let $\xi \in (\xi_1, \xi_2)$. By Lemma 2 in [9], there exists $x_1 \in (\xi_1, \xi)$ such that

$$\overline{D}f(x_1) \geqslant \frac{f(\xi) - f(\xi_1)}{\xi - \xi_1}.$$

Also, there exists $x_2 \in (\xi, \xi_2)$ such that

$$\frac{f(\xi_2) - f(\xi)}{\xi_2 - \xi} \geqslant \underline{D}f(x_2).$$

¿From the hypothesis of Theorem and the fact that $x_1 < x_2$ it follows that

$$\frac{f(\xi_2) - f(\xi)}{\xi_2 - \xi} \geqslant \underline{D}f(x_2) \geqslant \overline{D}f(x_1) \geqslant \frac{f(\xi) - f(\xi_1)}{\xi - \xi_1},$$

wherefrom, using Lemma 3 we obtain the convexity of f on (a, b). □

Now, we first state a classical proposition related to convexity and as an application of Theorem 3 we prove that converse is also true.

Proposition 3. *Let $f : (a, b) \to \mathbf{R}$ be convex, then f'_- and f'_+ exist at every point of the interval (a, b) and for every $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ hold the inequalities $f'_-(x_1) \leqslant f'_+(x_1) \leqslant f'_-(x_2) \leqslant f'_+(x_2)$. Also, the set of all $x \in (a, b)$ such that $f'_-(x) \neq f'_+(x)$ is at most countable.*

Proposition 4. *Let $f : (a, b) \to \mathbf{R}$ such that f'_- and f'_+ exist at every point of the interval (a, b) and for every $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$ hold the inequalities $f'_-(x_1) \leqslant f'_+(x_1) \leqslant f'_-(x_2) \leqslant f'_+(x_2)$. Then f is convex on (a, b). Therefore, the set of all $x \in (a, b)$ such that $f'_-(x) \neq f'_+(x)$ is at most countable.*

Proof. Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. Since $f'_-(x_1) \leqslant f'_+(x_1) \leqslant f'_-(x_2) \leqslant f'_+(x_2)$ it follows that $\max\{f'_-(x_1), f'_+(x_1)\} \leqslant \min\{f'_-(x_2), f'_+(x_2)\}$. Hence, from the equalities $\overline{D}f(x_1) = \max\{f'_-(x_1), f'_+(x_1)\}$ and $\underline{D}f(x_2) = \min\{f'_-(x_2), f'_+(x_2)\}$ we obtain $\overline{D}f(x_1) \leqslant \underline{D}f(x_2)$. The proof follows immediately from the Theorem 3. □

6. Another generalization of the Lagrange Mean Value Theorem

Now, we give another generalization of the Lagrange mean value theorem (only for continuous functions).

Definition 4. *For a given function $f : [a,b] \to \mathbf{R}$ we denote by $\Gamma_f = \{(x, f(x)) : x \in [a,b]\}$ the graph of the function f. We say that a linear function $l(x) = \alpha x + \beta$ is a upper (respectively, lower) supporting line for f at $M = \Gamma_f \cap \Gamma_l$ (more precisely we also say the graph of the function l is supporting line for the graph of the function f) if for all $x \in [a,b]$ it follows that $f(x) \leqslant l(x)$ (respectively, $f(x) \geqslant l(x)$).*

Theorem 4. *Suppose $f : [a,b] \to \mathbf{R}$ is continuous function on $[a,b]$ and $s(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.Then,*

i) There exist $x_{low}, x_{upp} \in [a,b]$, lower supporting line l_{low} and upper supporting line l_{upp} for the function f such that $l_{low}(x_{low}) = f(x_{low})$, $l_{upp}(x_{upp}) = f(x_{upp})$ and the graphs of l_{low} and l_{upp} are parallel to the graph of s.

ii) If $x_{low} \in (a,b)$ and if there exists $f'(x_{low})$, then l_{low} is a tangent to Γ_f. Analogously for x_{upp}, $f'(x_{upp})$ and l_{upp}.

iii) If f is differentiable convex (respectively concave) function then the point x_{upp} (respectively x_{upp}) is unique or f is a constant.

Proof. i) Let $g = f - s$. Since the function g is continuous on $[a,b]$ by Weierstrass's theorem there exist $x_{low}, x_{upp} \in [a,b]$ such that

$$g(x_{low}) = \min_{x\in[a,b]} g(x) \qquad \text{and} \qquad g(x_{upp}) = \max_{x\in[a,b]} g(x).$$

Define the function l_{low} by

$$l_{low}(x) = \frac{f(b)-f(a)}{b-a}(x - x_{low}) + f(x_{low})$$

and the function l_{upp} by

$$l_{upp}(x) = \frac{f(b)-f(a)}{b-a}(x - x_{upp}) + f(x_{upp}).$$

It is clear that the graphs of l_{low} and l_{upp} are parallel to the graph of s. The rest is yet to prove that the l_{low} and l_{upp} are lower and upper supporting line for the function f, respectively. That is the show that $l_{low}(x) \leqslant f(x)$ and $l_{upp}(x) \geqslant f(x)$ for all $x \in [a,b]$. Let us show that

$$f(x) \leqslant l_{upp}(x).$$

Really,

$$f(x) \leqslant l_{upp}(x) = \frac{f(b)-f(a)}{b-a}(x - x_{upp}) + f(x_{upp})$$

if and only if

$$f(x) - s(x) \leqslant \frac{f(b)-f(a)}{b-a}(x - x_{upp}) + f(x_{upp}) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

i.e.

$$g(x) \leqslant f(x_{upp}) - s(x_{upp}),$$

and since the last inequality is true it follows that the l_{upp} is upper supporting line for the function f. Completely analogously proves that the l_{low} is lower supporting line for the function f. We leave to the reader to verify ii) and iii). □

6.1. Comments

Before indicating application of convexity to the neoclassical economic growth model we will note difference between approach to monotony and convexity in classical textbooks and in this paper.

The characterizations of monotonicity and convexity of function $f : (a, b) \to \mathbf{R}$ via derivative (second derivative) of f are well-known. We follow some comments concerning the characterizations of monotonicity given in [1]. In textbooks exposing foundations of mathematical analysis the connection between monotony (convexity) and sign of derivative (second derivative) is usually established in the form of a statement which supposes the continuity of the function and existence of derivative (second derivative). Recall, the differentiable function $f : (a, b) \to \mathbf{R}$ is monotonically increasing (respectively, convex) if and only if the function f' is nonnegative (respectively, monotonically increasing) on (a, b). In the proof of these theorems in classical textbooks the use of the Lagrange mean value theorem has an essential role. For the proof of the Lagrange mean value theorem it is used the Rolle theorem, and the proof of the Rolle theorem is based on the Fermat theorem (if differentiable function at any point has a local extremum then derivative is equal zero at that point) and the Weierstrass theorem (the continuous function attain the global minimum and global maximum on compact). The proof of the Weierstrass theorem is based on the Bolzano theorem (every sequence of real number has at least one limit point). On the other hand, in this paper we give the criteria which do not request differentiability or any other property of function defined on interval (a, b). In the cases when a function f is differentiable (respectively, has second derivative) on (a, b) the well-known characterizations of monotony (respectively, convexity) via derivative (respectively, second derivative) immediately follow from our results.

7. Applications

Production of goods requires resources or inputs as land, labor, capital and organization. We suppose that production process is efficient and that firms get the maximum amount of output of goods from the set of resources or inputs. A production function shows the relationship between the quantity of the product Y which can be produced by the given quantities of inputs (lands labor, capital) that are used in the process of production.

The analysis of production function is generally carried with reference to time period which is called short period and long period. In the short run, production function is explained with one variable factor and other factors of productions are held constant. We have called this production function as the Law of Variable Proportions or the Law of Diminishing returns. In the long run, production function is explained by assuming all the factors of production as variable. There are no fixed inputs in the long run. Here the production function is called the Law of Returns according to the scale of production.

Below, we consider neoclassical economic growth model, also known as the Solow-Swan growth model. In the neoclassical economic growth theory the production function is concave.

7.1. The Neoclassical Model and Golden Rule of Capital Accumulation

In this subsection we outline how to refine neoclassical model from mathematical point of view.

We denote by $K = K(t) = K_t$ the capital flow (shortly, the capital) as function of time t. Also, let us denote by $Y = Y(K) = Y(K(t)) = Y_t$ the production function as function of capital K. More precisely, Y is total production i.e. the monetary value of all goods produced in a year (more generally in fixed interval of time). Production function Y is determined by decreasing incomes on capital accumulation.

If we introduce the assumption that people save constant saving rate $s \in (0, 1)$ from its gross incomes Y, and amortization rate $\delta \in (0, 1)$ represents part of expended capital, then net rate of increasing of physical capital per time unit i.e. net investment is:

$$I = sY - \delta K.$$

In economic theory usually it is assume that the net investment is growth speed of K, i.e. the change in capital, the derivative $\dot{K}$, so

$$\dot{K} = sY - \delta K. \tag{8}$$

Equation (8) is fundamental differential equation for theories of neoclassical growth.

As in the neoclassical model, there is a fixed saving rate s and a fixed depreciation rate δ, so the aggregate capital stock K_t will evolve according to

$$K_{t+1} - K_t = sY_t - \delta K_t \tag{9}$$

which states that net investment equals gross investment sY_t minus depreciation δK_t. Hence $K_{t+1} = sY_t + (1-\delta)K_t$. Here K_t is capital in moment t and from mathematical point of view it seems logical that Y_t is the value of production in time interval $[t, t+1]$.

We consider time as running discretely like the integers $0, 1, 2, 3, \cdots$; in a discrete sequence of periods indexed by t. A period should be thought of as one year. Assume that time runs smoothly, or continuously, in an uninterrupted stream like the real numbers. The neoclassical economic growth model is often formulated in the continuous time. By the Lagrange mean value theorem, $K_{t+1} - K_t = \dot{K}_x$, where $t < x < t+1$. So if we approximately write $\dot{K}_t$ instead of $\dot{K}_x$ we get (8).

In the economic theory usually the equation (9) is replaced by equation (8). Note that, strictly speaking, these equations are not equivalent from mathematical point of view.

Further, we will use the following notation. By $A = A(t) > 0$ we denote the endogenous variable, the total factor productivity that grows exponentially over time, i.e. $\frac{\dot{A}}{A} = g$, where $g > 0$ is a constant, and by $L = L(t) > 0$ we denote the number of labor that grows exponentially over time, i.e. $\frac{\dot{L}}{L} = n$, where $n > 0$ is a constant.

Example 3. *The Cobb-Douglas production function is the simple model of production function. According to [15] it has the following form:* $Y = K^{\alpha}(AL)^{1-\alpha}$, *where* $0 < \alpha < 1$.

Also, we can consider the capital per capita ($k = \frac{K}{AL}$) and the production function per capita ($f(k) = \frac{Y(kAL)}{AL}$). For example the Cobb-Douglas production function per capita has the form: $f(k) = k^{\alpha}$.

Since $\dot{A} = gA$ and $\dot{L} = nL$ we obtain $\dot{K} = (\dot{k} + (g+n)k)AL$. Hence, using that $Y(K) = f(k)AL$ we transform the equation (8) in the following manner:

$$(\dot{k} + (g+n)k)AL = sf(k)AL - \delta kAL.$$

Divide last equation by AL we obtain the equation:

$$\dot{k} + (g+n)k = sf(k) - \delta k. \tag{10}$$

Let $m = \delta + g + n$ (note that m is a constant). Then the equation (10) has the form:

$$\dot{k} = sf(k) - mk. \tag{11}$$

Note that from assumption that the function Y is concave it follows that the function f is also concave. Further, we also suppose that the function f satisfies Inada conditions:

(In1) $f'(k) > 0$ for every $k \in (0, +\infty)$,

(In2) $f''(k) < 0$ for every $k \in (0, +\infty)$,

(In3) $\lim_{k \to 0+} f'(k) = +\infty$

(In4) $\lim_{k\to+\infty} f'(k) = 0$.

By definition, level of per capita consumption is

$$c = f - sf = (1 - s)f.$$

If t^* is time for which $\dot{k}(t^*) = 0$ we call $k^* = k(t^*)$ the steady state of capital. It is clear that at the steady state of capital $0 = \dot{k} = sf - mk$ and therefore $sf(k^*) = mk^*$. As we assumed that the function f satisfies Inada conditions it is clear that there exists a unique steady state of capital.

For given a fixed saving rate s we find $k^* = k^*(s)$ the steady state of capital and consider the capital consumption $c^*(s) = c(k^*(s))$ in the steady state and then maximum of $c^* = c^*(s)$ with respect to s.

We leave to the reader to check that $k^* = k^*(s)$ increases in relation to s.

Recall that at the steady state $0 = \dot{k} = sf - mk$ for each value of saving rate s. Hence $0 = sf(k^*(s)) - mk^*(s)$ i.e. $sf(k^*(s)) = mk^*(s)$.

We now consider maximum of $c^*(s)$ with respect to s. For this purpose we first find $c^{*\prime}(s)$. Since $k^*(s)$ is steady state of capital then $c^*(s) = f(k^*(s)) - mk^*(s)$ and therefore $c^{*\prime}(s) = f'(k^*(s))k^{*\prime}(s) - mk^{*\prime}(s)$. Hence $c^{*\prime}(s) = 0$ if and only if $f'(k^*(s)) = m$. Further, since $f''(k) < 0$ it follows f' is strictly decreasing function and therefore c^* has a maximum at point s for which $c^{*\prime}(s) = 0$. Denote that value of saving rate by s_{gold}. Now, using s_{gold} we define $c_{\text{gold}} = c^*(s_{\text{gold}})$ and $k_{\text{gold}} = k^*(s_{\text{gold}})$.

Note that k^* has derivative (see Proposition 5, below). So we can justify the above procedure.

In particular, by simple geometric consideration concerning supporting line and an application of the Lagrange mean value theorem, we can conclude directly that k_{gold} satisfies $f'(k_{\text{gold}}) = m$. We leave it to the reader to verify this.

The Golden Rule says: ,,Do unto others, as you would have others do unto you". In economic sense can be interpreted as if we do not provide to future generation less then we have provided to ourselves, then maximum quantity of per capital consumption is c_{gold}.

7.2. Generalization of Neoclassical Model and Golden Rule of Capital Accumulation

For $a > 0$ define line ℓ_a by $\ell_a(k) = ak$.

We suppose that

(A1) $f : [0, +\infty) \to [0, +\infty)$ is continuous function.

(A2) For every $a > 0$, there is a point k_a such that $f > \ell_a$ on $(0, k_a)$ and that $f < \ell_a$ on $(k_a, +\infty)$.

Proposition 5. *Assume that the function f satisfies (A1) and (A2). Let $m > 0$. Then,*

a) for every $s \in [0, 1]$, the equation $0 = sf(k) - mk$ has a unique solution respect to k. We denote this solution by $k^* = k^*(s)$,

b) k^* increases respect to s,

c) if function f has derivative then k^* has it on $[0, 1]$,

d) maximum of function $c^*(s) = f(k^*(s)) - sf(k^*(s))$ on $[0, 1]$ is the same as maximum of h defined by $h(k) = f(k) - mk$ on $[0, k^*(1)]$,

e) if f has derivative on $[0, k^*(1)]$ and, h has maximum at k_0, then $f'(k_0) = m$.

Proof.

a) It follows from property (A2) of function f.

b) Let $0 \leq s_1 < s_2 \leq 1$. Since $s_1 f(k^*(s_2)) < s_2 f(k^*(s_2))$ and $s_2 f(k^*(s_2)) = mk^*(s_2)$ it follows that $s_1 f(k^*(s_2)) < mk^*(s_2)$ i.e. $\frac{s_1}{m} f(k^*(s_2)) < k^*(s_2)$. Using it and the property (A2) of function f it follows that $k^*(s_1) < k^*(s_2)$.

c) Since $s = s(k^*) = m\frac{k^*}{f(k^*)}$, if f has derivative different from 0, then $s = s(k^*)$ has derivative and therefore $k^*(s)$ has derivative.

d) It follows from a) and b).

e) Using $k = k^*(s)$ we can consider the function $h(k) = f(k) - mk$, which has maximum at k_0 such that $h'(k_0) = f'(k_0) - m = 0$, that is $f'(k_0) = m$.

□

Now, we give a heuristic consideration. Let $\Delta t > 0$. If $Y_{t,t+\Delta t}$ is total production i.e. the monetary value of all goods produced in interval $[t, t + \Delta t]$, we can modify equation (9) in following manner:

$$K_{t+\Delta t} - K_t = sY_{t,t+\Delta t} - \delta \Delta t K_t. \tag{12}$$

We can define $\hat{Y} = \hat{Y}_t = \lim\limits_{\Delta t \to 0+} \frac{Y_{t,t+\Delta t}}{\Delta t}$ which we call growth of change of production function. Divide equation (12) by Δt and taking the limit when $\Delta t \to 0+$ we get

$$\dot{K} = s\hat{Y} - \delta K. \tag{13}$$

Using it, we modify equation (11) by $\hat{f}$ instead of f:

$$\dot{k} = s\hat{f} - mk. \tag{14}$$

In a similar way as we considered equation (11) in previous subsection, we now consider equation (14) and give the corresponding definitions. By $\hat{k}(s)$ we denote the steady state of capital and we define consumption $\hat{c}(s) = \hat{f}(\hat{k}(s)) - m\hat{k}(s)$ at steady state of capital. Also, we give the corresponding assertions.

a1) Maximum of function $\hat{c}(s) = \hat{f}(\hat{k}(s)) - m\hat{k}(s)$ on $[0, 1]$ is the same as maximum of $\hat{h}$ defined by $\hat{h}(k) = \hat{f}(k) - mk$ on $[0, \hat{k}(1)]$.

b1) If $\hat{h}$ has maximum at k_0 then $\underline{D}\hat{f}(k_0) \leq m \leq \overline{D}\hat{f}(k_0)$.

c1) If $\hat{f}$ has the derivative on $[0, \hat{k}(1)]$ and $\hat{h}$ has maximum at k_0 , then $\hat{f}'(k_0) = m$.

7.3. Final remark

We start to investigate characterization of monotone and convex functions in connection with subject related to teaching of mathematics; cf. [7] for a visual characterization of convex functions. Since we consider very old subject it is possible that some of our results exist in some forms in the literature. Even in those cases, we hope that there is something new in our approach. We plan to investigate further the subject in connection to the neoclassical economic growth model and this work could be viewed as a starting point for deriving more substantial results on the subject.

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