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Bicrossproduct Revisited

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Abstract. We give the factorization of a class of bialgebras into bicrossproduct introduced by Kim-Park-Yoon in [7], which generalizes Radford's known results in [10].

1. Introduction and Preliminaries

Crossed products introduced independently by Blatter-Cohen-Montgomery [4] and by Doi-Takeuchi [6] play a fundamental role in the theory of extensions of Hopf algebras. Moreover, let H be a Hopf algebra and B an H-comodule algebra, set $A = B^{coH}$, then an H-extension $A \subset B$ is H-cleft if and only if $B \cong A\#_\sigma H$ (see [9, 7.2.2]). And subsequently crossed products and crossed coproducts were studied by many authors (see [1, 2, 5, 7, 13]). Especially in [1, 2], Y. Bespalov and B. Drabant investigated all sorts of cross product bialgebras deeply and for which a fully-fledged formulation in terms of (co-)modular and (co-)cyclic conditions (called Hopf Data) was developed. In 1999, E. S. Kim, Y. S. Park and S. B. Yoon gave the necessary and sufficient conditions for crossed product and crossed coproduct to be a bialgebra named *bicrossproduct* in [7] generalizing Radford's biproduct (see [8, 10]).

In 1985, Radford gave a result that a bialgebra with a projection had a factorization of biproduct. It is a natural question: Under which conditions an (co)algebra can be factorized into crossed (co)product and when does a bialgebra have a factorization of bicrossproduct?

On the other hand, Bespalov and Drabant obtained an equivalent description of cross product bialgebras in terms of projections and injections or splittings of idempotents (see [2, Proposition 2.3, 2.9 et al]).

In this paper, we will give the answers to the above questions by *a concrete construction based on the fundamental theorem of Hopf modules* [9, 1.9.4], which also provides a class of explicit examples for the above mentioned Bespalov and Drabant's results.

The paper is organized as follows.

Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and (B, ρ) is a right H-comodule algebra. Set $A = B^{coH} = \{a \in B | \rho(a) = a \otimes 1_H\}$. In section 2, we obtain that B can be factorized into crossed product $A\#_\sigma H$ under an extra condition (see Theorem 2.2). As a corollary, we can get that if $A \subset B$ is an H-cleft extension, then

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$B \cong A\#_\sigma H$ as an algebra (see Corollary 2.4). The duality of Section 2 is given in Section 3. Section 4 is devoted to the situation of bialgebra. We consider the case of bialgebras (see Theorem 4.1) by combining section 2 with section 3 and we can conclude Radford's known results in [10] by using our approach (see Corollary 4.4).

Throughout the paper, we freely use the definitions and terminologies of [7, 9, 12] and all algebraic systems are supposed to be over the field k. Unless specifically stated, H denotes an Hopf algebra with antipode S. Let C be a coalgebra. Then we use Sweedler's notation for the comultiplication: $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ for any $c \in C$. We denote by ${}^C\mathcal{M}$ ($\mathcal{M}^C$) the category of left (right) C-comodules and for any $V \in {}^C\mathcal{M}$ ($\mathcal{M}^C$), we still use a simple Sweedler's notations: $\tilde{\rho}(v) = \sum v_{<-1>} \otimes v_{<0>}$ ($\rho(v) = \sum v_0 \otimes v_1$) for all $v \in V$. If $B \in \mathcal{M}^H$, then we denote $\{a \in B | \rho(a) = a \otimes 1_H\}$ by B^{coH}. Given a k-space M, we write id_M for the identity map on M.

The Fundamental Theorem of Hopf Modules Let M be a right H-Hopf module. Then $M \cong M^{coH} \otimes H$ as right H-Hopf modules (see [9, 1.9.4] or [12, Theorem 4.1.1]).

Let H be Hopf algebra and (B, ρ) be a right H-comodule algebra. Set $A = B^{coH}$. If there exists a right H-colinear map $\gamma : H \to B$ which is convolution invertible, then we call $A \subset B$ is an *H-cleft extension and denoted by* (B, ρ, γ).

In the following we recall from [7, 13] or [1, 2] about crossed (co)product and bicrossproduct.

Let H be a Hopf algebra and A an algebra. A *weak action* of H on A means a bilinear map $(h, a) \mapsto h \cdot a$ of $H \times A \longrightarrow A$ such that, for $h \in H$ and $a, b \in A$,

$$h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b),\ h \cdot 1 = \varepsilon(h)1 \text{ and } 1 \cdot a = a.$$

If the extra condition $h \cdot (g \cdot a) = (hg) \cdot a$ holds for all $h, g \in H$ and $a \in A$, then A is called an *H-module algebra*.

Let H be a Hopf algebra with a weak action on the algebra A, and let $\sigma : H \otimes H \longrightarrow A$ be a k-linear map. Let $A\#_\sigma H$ be the (in general nonassociative) algebra whose underlying space is $A \otimes H$ and whose multiplication is given by

$$(a \otimes h)(b \otimes g) = \sum a(h_1 \cdot b)\sigma(h_2, g_1) \otimes h_3 g_2$$

for all $a, b \in A$ and $h, g \in H$. The algebra $A\#_\sigma H$ is called a *crossed product* if it is associative and with unit $1_A \otimes 1_H$.

$A\#_\sigma H$ is an associative algebra with unit $1_A \otimes 1_H$ if and only if σ satisfy the following conditions:

(C1) σ is normal. i.e., $\sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon_H(h)1_A$,

(C2) $\sum (h_1 \cdot \sigma(l_1, g_1))\sigma(h_2, l_2 g_2) = \sum \sigma(h_1, l_1)\sigma(h_2 l_2, g)$,

(C3) $\sum (h_1 \cdot (l_1 \cdot b))\sigma(h_2, l_2) = \sum \sigma(h_1, l_1)(h_2 l_2 \cdot b)$

for all $b \in A$ and $h, g, l \in H$.

Dually, let H be a Hopf algebra and C a coalgebra. We call H *weakly coacts on* C if there exists a k-linear map $\rho : C \longrightarrow H \otimes C$, $c \mapsto \sum c_{<-1>} \otimes c_{<0>}$ such that the conditions

$$(W1)\quad \sum c_{<-1>} \otimes c_{<0>(1)} \otimes c_{<0>(2)} = \sum c_{(1)<-1>} c_{(2)<-1>} \otimes c_{(1)<0>} \otimes c_{(2)<0>},$$

$$(W2)\quad \sum \varepsilon(c_{<0>}) c_{<-1>} = \varepsilon(c) 1_H,$$

$$(W3)\quad \sum \varepsilon(c_{<-1>}) c_{<0>} = c$$

hold for all $c \in C$. Moreover if the following condition

$$\sum c_{<-1>(1)} \otimes c_{<-1>(2)} \otimes c_{<0>} = \sum c_{<-1>} \otimes c_{<0><-1>} \otimes c_{<0><0>}$$

holds for all $c \in C$, then we call C a *left H-comodule coalgebra*.

Let $\alpha : C \longrightarrow H \otimes H$ be a linear map, write $\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c)$. Let $C \rtimes_\alpha H$ be the (in general noncoassociative) coalgebra (in general without a counit), whose underlying vector space is $C \otimes H$ with comultiplication given by

$$\Delta_\alpha(c \otimes h) = \sum c_{(1)} \otimes c_{(2)<-1>}\alpha_1(c_{(3)})h_{(1)} \otimes c_{(2)<0>} \otimes \alpha_2(c_{(3)})h_{(2)}.$$

If $(C \otimes H, \Delta_\alpha, \varepsilon_C \otimes \varepsilon_H)$ is coassociative counitary coalgebra, then we call it *crossed coproduct* denoted by $C \rtimes_\alpha H$.

Moreover, $C \rtimes_\alpha H$ is crossed coproduct if and only if for all $c \in C$ the following conditions hold:

$$(CC1)\quad (id_H \otimes \varepsilon_H) \circ \alpha = (\varepsilon_H \otimes id_H) \circ \alpha = \mu_{H\otimes H}\varepsilon_C,$$

$$\begin{aligned}(CC2)\quad &\sum c_{(1)<-1>}\alpha_1(c_{(2)}) \otimes \alpha_1(c_{(1)<0>})\alpha_2(c_{(2)})_{(1)} \otimes \alpha_2(c_{(1)<0>})\alpha_2(c_{(2)})_{(2)}\\ &= \sum \alpha_1(c_{(1)})\alpha_1(c_{(2)})_{(1)} \otimes \alpha_2(c_{(1)})\alpha_1(c_{(2)})_{(2)} \otimes \alpha_2(c_{(2)}),\end{aligned}$$

$$\begin{aligned}(CC3)\quad &\sum c_{(1)<-1>}\alpha_1(c_{(2)}) \otimes c_{(1)<0><-1>}\alpha_2(c_{(2)}) \otimes c_{(1)<0><0>}\\ &= \sum \alpha_1(c_{(1)})c_{(2)<-1>(1)} \otimes \alpha_2(c_{(1)})c_{(2)<-1>(2)} \otimes c_{(2)<0>}.\end{aligned}$$

Bicrossproduct Let A be both an algebra and a coalgebra. Assume that $A\#_\sigma H$ is a crossed product and $A \rtimes_\alpha H$ is a crossed coproduct. Then $(A_\sigma \bowtie_\alpha H, m_{A\#_\sigma H}, \mu_{A\#_\sigma H}, \Delta_{A\rtimes_\alpha H}, \varepsilon_{A\rtimes_\alpha H})$ is a bialgebra if and only if $(B1) \sim (B10)$ as in [7, Theorem 2.1] are satisfied. In this case we call $A_\sigma \bowtie_\alpha H$ a *bicrossproduct*.

From now on, we employ the techniques of Hopf modules and the projections and injections into and from the (co-)invariants B^{coH} deeply investigated in [11] and [3].

Proposition 1.1. Let H be Hopf algebra and $(B, \cdot)$ is a right H-module coalgebra. Suppose that there exists a right H-linear map $\phi : B \to H$ which is invertible with convolution inverse ϕ^{-1}. Define a linear map $\rho : B \to B \otimes H$ by

$$\rho(b) = \sum b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}) \otimes \phi(b_{(3)})_{(2)}.$$

Then (B, ρ) is a right H-comodule.

Proof. Since ϕ is right H-linear, then for all $h \in H$ and $b \in B$ we have

$$\sum h_{(1)}\phi^{-1}(b \cdot h_{(2)}) = \varepsilon(h)\phi^{-1}(b). \tag{1.1}$$

For all $b \in B$, we compute as follows:

$$\begin{aligned}
&(\rho \otimes id_H)\rho(b)\\
=\ &\sum (b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}))_{(1)} \cdot (\phi^{-1}((b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}))_{(2)})\\
&\phi((b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}))_{(3)})_{(1)}) \otimes \phi((b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}))_{(3)})_{(2)}\\
&\otimes\phi(b_{(3)})_{(2)}\\
=\ &\sum b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)} \cdot (\phi^{-1}(b_{(4)})\phi(b_{(5)})_{(1)}))_{(1)}) \otimes \phi(b_{(3)} \cdot (\phi^{-1}(b_{(4)})\phi(b_{(5)})_{(1)}))_{(2)}\\
&\otimes\phi(b_{(5)})_{(2)} \quad \text{(by Eq. (1.1))}\\
=\ &\sum b_{(1)} \cdot (\phi^{-1}(b_{(2)})(\phi(b_{(3)})\phi^{-1}(b_{(4)})\phi(b_{(5)})_{(1)})_{(1)}) \otimes (\phi(b_{(3)})\phi^{-1}(b_{(4)})\phi(b_{(5)})_{(1)})_{(2)}\\
&\otimes\phi(b_{(5)})_{(2)}\\
=\ &(id_B \otimes \Delta)\rho(b).
\end{aligned}$$

While $(id_B \otimes \varepsilon_H)\rho(b) = \sum b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})) = b.$ ■

Definition 1.2. Let H be Hopf algebra and $(B,\cdot)$ a right H-module coalgebra. Suppose that there exists a right H-linear map $\phi : B \to H$ which is convolution invertible. Let ρ be given as in Proposition 1.1. Set $A = B^{coH}$, then we call $A \subset B$ *is a quasi-H-cleft coextension* and denoted by $(B,\cdot,\phi)$.

Similarly, we have

Proposition 1.3. Let H be Hopf algebra with antipode S. Assume that $(B,\cdot,\rho)$ is a right H-Hopf module and $(B,\cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. Define the map $\nu : B \to A \otimes B$ by

$$\nu(b) = \sum b_{(1)0} \cdot S(b_{(1)1}) \otimes b_{(2)},$$

then (B,ν) is a left A-comodule.

2. The Factorization of a Class of Comodule Algebras

In this section, we give the factorization of a class of comodule algebras into crossed product based on the fundamental theorem of Hopf modules.

Lemma 2.1. Let H be Hopf algebra with antipode S. Assume that $(B,\cdot,\rho)$ is a right H-Hopf module and (B,ρ) is a right H-comodule algebra. Set $A = B^{coH}$. Define the map $\rightharpoonup : H \otimes A \longrightarrow A$ by $h \rightharpoonup a = \sum((1\cdot h_{(1)})a)\cdot S(h_{(2)})$ and an element $\sigma \in Hom_k(H\otimes H, A)$ by

$$\sigma(h,g) = \sum((1\cdot h_{(1)})(1\cdot g_{(1)}))\cdot S(h_{(2)}g_{(2)}).$$

If for all $a \in A$, $b \in B$ and $h \in H$, the following condition hold:

$$(ab)\cdot h = a(b\cdot h). \tag{2.1}$$

Then (1) $\rightharpoonup$ is a weak action of H on A.

(2) $(C1)$, $(C2)$ and $(C3)$ are satisfied.

Proof. (1) First we note that

$$\rho(\sum((1\cdot h_{(1)})a)\cdot S(h_{(2)})) = \sum((1\cdot h_{(1)})a)\cdot S(h_{(2)})\otimes 1$$

for all $a \in A$ and $h \in H$, so $\rightharpoonup$ is well-defined.

For all $h \in H$ and $a, b \in A$, we can obtain

$$\begin{aligned} h \rightharpoonup (ab) &= \sum((1\cdot h_{(1)})ab)\cdot S(h_{(2)}) \\ &= \sum(((h_{(1)} \rightharpoonup a)\cdot h_{(2)})b)\cdot S(h_{(3)}) \\ &= \sum(h_{(1)} \rightharpoonup a)(((1\cdot h_{(2)})b)\cdot S(h_{(3)})) \quad \text{(by Eq.(2.1))} \\ &= \sum(h_{(1)} \rightharpoonup a)(h_{(2)} \rightharpoonup b) \end{aligned}$$

and $h \rightharpoonup 1_A = \varepsilon_H(h)1_A$.

Hence $\rightharpoonup$ is a weak action of H on A.

(2) It is obvious that σ is normal. Next we will verify that (C2) and (C3) hold. First by Eq.(2.1) we have

$$\sum(h_{(1)} \rightharpoonup a)(1\cdot h_{(2)}) = (1_A\cdot h)a. \tag{2.2}$$

For all $a \in A$ and $h, g, l \in H$, we compute as follows.

$$\begin{aligned}
&\sum (h_{(1)} \rightharpoonup (g_{(1)} \rightharpoonup a))\sigma(h_{(2)}, g_{(2)})\\
= &\sum (h_{(1)} \rightharpoonup (g_{(1)} \rightharpoonup a))((1 \cdot h_{(2)})(1 \cdot g_{(2)})) \cdot S(h_{(3)}g_{(3)})\\
= &\sum [(h_{(1)} \rightharpoonup (g_{(1)} \rightharpoonup a)(1 \cdot h_{(2)})(1 \cdot g_{(2)})] \cdot S(h_{(2)}g_{(3)}) \quad \text{(by Eq.(2.1))}\\
= &\sum ((1 \cdot h_{(1)})(1 \cdot g_{(1)})a) \cdot S(h_{(2)}g_{(2)}) \quad \text{(by Eq.(2.2))}\\
= &\sum [((1 \cdot h_{(1)})(1 \cdot g_{(1)})) \cdot S(h_{(2)}g_{(2)})][((1 \cdot h_{(3)}g_{(3)})a) \cdot S(h_{(4)}g_{(4)})] \quad \text{(by Eq.(2.1))}\\
= &\sum \sigma(h_{(1)}, g_{(1)})(h_{(2)}g_{(2)} \rightharpoonup a)
\end{aligned}$$

and

$$\begin{aligned}
&\sum (h_{(1)} \rightharpoonup \sigma(g_{(1)}, l_{(1)}))\sigma(h_{(2)}, g_{(2)}l_{(2)})\\
= &\sum (h_{(1)} \rightharpoonup \sigma(g_{(1)}, l_{(1)}))[(1 \cdot h_{(2)})(1 \cdot (g_{(2)}l_{(2)})) \cdot S(h_{(3)}g_{(3)}l_{(3)})]\\
= &\sum [(h_{(1)} \rightharpoonup \sigma(g_{(1)}, l_{(1)}))(1 \cdot h_{(2)})(1 \cdot (g_{(2)}l_{(2)}))] \cdot S(h_{(3)}g_{(3)}l_{(3)}) \quad \text{(by Eq.(2.1))}\\
= &\sum [(1 \cdot h_{(1)})\sigma(g_{(1)}, l_{(1)})(1 \cdot (g_{(2)}l_{(2)}))] \cdot S(h_{(2)}g_{(3)}l_{(3)}) \quad \text{(by Eq.(2.2))}\\
= &\sum [((1 \cdot h_{(1)})(1 \cdot g_{(1)})) \cdot S(h_{(2)}g_{(2)})][((1 \cdot h_{(3)}g_{(3)})(1 \cdot l_{(1)})) \cdot S(h_{(4)}g_{(4)}l_{(2)})]\\
&\qquad \text{(by Eq.(2.1))}\\
= &\sum \sigma(h_{(1)}, g_{(1)})\sigma(h_{(2)}g_{(2)}, l).
\end{aligned}$$

■

Remark. Eq.(2.1) manifests that B is an A-H-bimodule.

Theorem 2.2. Let H be Hopf algebra with antipode S. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and (B, ρ) is a right H-comodule algebra. Set $A = B^{coH}$. If for all $a \in A$, $b \in B$ and $h \in H$, Eq.(2.1) holds. Then we get a crossed product $A\#_\sigma H$ and $B \cong A\#_\sigma H$ as an algebra.

Proof. By Lemma 2.1 we have a crossed product $A\#_\sigma H$ with multiplication

$$(a \otimes h)(a' \otimes h') = \sum a(h_{(1)} \rightharpoonup a')\sigma(h_{(2)}, h'_{(1)}) \otimes h_{(3)}h'_{(2)}$$

for all $a, a' \in A$ and $h, h' \in H$.

We note that $B \cong A \otimes H$ as right H-Hopf module and the isomorphic map is given by

$$\varphi : B \longrightarrow A \otimes H, \varphi(b) = \sum b_0 \cdot S(b_1) \otimes b_2.$$

Then we only need to prove that φ is an algebra map. In fact, for all $b, b' \in B$, we have

$$\begin{aligned}
\varphi(b)\varphi(b') &= \sum (b_0 \cdot S(b_1))(b_2 \rightharpoonup (b'_0 \cdot S(b'_1)))\sigma(b_3, b'_2) \otimes b_4b'_3\\
&= \sum (b_0 \cdot S(b_1))[((b_2 \rightharpoonup (b'_0 \cdot S(b'_1)))(1 \cdot b_3)(1 \cdot b'_2)) \cdot S(b_4b'_3)] \otimes b_5b'_4\\
&\qquad \text{(by Eq.(2.1))}\\
&= \sum (b_0 \cdot S(b_1))[((1 \cdot b_2)(b'_0 \cdot S(b'_1))(1 \cdot b'_2)) \cdot S(b_3b'_3)] \otimes b_4b'_4 \quad \text{(by Eq.(2.2))}\\
&= \sum (b_0b'_0) \cdot S(b_1b'_1) \otimes b_2b'_2 \quad \text{(by Eq.(2.1))}\\
&= \varphi(bb').
\end{aligned}$$

■

Proposition 2.3. Assume that (B, ρ, γ) is an H-cleft extension. Define:

$$\cdot\ : B \otimes H \longrightarrow B,\ b \cdot h = \sum b_0\gamma^{-1}(b_1)\gamma(b_2h).$$

Set $A = B^{coH}$. Then

(1) $(B, \cdot)$ is a right H-module and Eq.(2.1) holds for all $a \in A$, $b \in B$ and $h \in H$.

(2) $(B, \cdot, \rho)$ is a right H-Hopf module.

Proof. First, since γ is invertible colinear map, we have

$$\sum \gamma^{-1}(h_{(2)})_0 \otimes h_{(1)}\gamma^{-1}(h_{(2)})_1 = \gamma^{-1}(h) \otimes 1_H \tag{2.3}$$

By Eq.(2.3), Part (1) is straightforward.

So we only check that $(B, \cdot, \rho)$ is a right H-Hopf module as follow: for all $b \in B$ and $h \in H$,

$$\begin{aligned}
\rho(b \cdot h) &= \sum b_{00}\gamma^{-1}(b_{1(1)})_0\gamma(b_{1(2)}h)_0 \otimes b_{01}\gamma^{-1}(b_{1(1)})_1\gamma(b_{1(2)}h)_1 \\
&= \sum b_0\gamma^{-1}(b_{1(2)})_0\gamma(b_{1(3)}h)_0 \otimes b_{1(1)}\gamma^{-1}(b_{1(2)})_1\gamma(b_{1(3)}h)_1 \\
&= \sum b_0\gamma^{-1}(b_{1(1)})\gamma(b_{1(2)}h)_0 \otimes \gamma(b_{1(2)}h)_1 \quad \text{(by Eq.(2.3))} \\
&= \sum b_0\gamma^{-1}(b_{1(1)})\gamma(b_{1(2)}h_{(1)}) \otimes b_{1(3)}h_{(2)} \quad \text{(by } \gamma \text{ is colinear)} \\
&= \sum b_{00}\gamma^{-1}(b_{01(1)})\gamma(b_{01(2)}h_{(1)}) \otimes b_1h_{(2)} \\
&= \sum b_0 \cdot h_{(1)} \otimes b_1h_{(2)}.
\end{aligned}$$

■

By Theorem 2.2 and Proposition 2.3, the following corollary is obvious.

Corollary 2.4. ([4]) Let (B, ρ, γ) is an H-cleft extension. Set $A = B^{coH}$. Then we get a crossed product $A\#_\sigma H$ and $B \cong A\#_\sigma H$ as algebra.

Below we give the factorization of a class of algebras into smash product that we should note that the condition (2.1) is changed.

Lemma 2.5. Let H be Hopf algebra with antipode S. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and (B, ρ) is a right H-comodule algebra such that the following condition

$$(bb') \cdot h = b(b' \cdot h) \tag{2.4}$$

holds for all $b, b' \in B$ and $h \in H$. Set $A = B^{coH}$. Define the left action of H on A by

$$\rightharpoonup: H \otimes A \longrightarrow A, \ \ h \rightharpoonup a = \sum((1 \cdot h_{(1)})a) \cdot S(h_{(2)}).$$

Then $(A, \rightharpoonup)$ is a left H-module algebra.

Proof. First we have

$$\begin{aligned}
h \rightharpoonup (g \rightharpoonup a) &= \sum[(1 \cdot h_{(1)})(((1 \cdot g_{(1)})a) \cdot S(g_{(2)}))] \cdot S(h_{(2)}) \\
&= \sum[(1 \cdot h_{(1)})(1 \cdot g_{(1)})a] \cdot S(h_{(2)}g_{(2)}) \quad \text{(by Eq.(2.4))} \\
&= \sum[(1 \cdot (h_{(1)}g_{(1)}))a] \cdot S(h_{(2)}g_{(2)}) \\
&= (hg) \rightharpoonup a
\end{aligned}$$

for all $a \in A$ and $h, g \in H$ and $1_H \rightharpoonup a = a$, so $(A, \rightharpoonup)$ is a left H-module.

On the other hand, for all $a, a' \in A$ and $h \in H$

$$\begin{aligned}
\sum(h_{(1)} \rightharpoonup a)(h_{(2)} \rightharpoonup a') &= \sum[((1 \cdot h_{(1)})a) \cdot S(h_{(2)})][((1 \cdot h_{(3)})a') \cdot S(h_{(4)})] \\
&= \sum[((1 \cdot h_{(1)})a) \cdot S(h_{(2)}))(1 \cdot h_{(3)})a'] \cdot S(h_{(4)}) \quad \text{(by Eq.(2.4))} \\
&= \sum((1 \cdot h_{(1)})aa') \cdot S(h_{(2)}) \quad \text{(by Eq.(2.4))} \\
&= h \rightharpoonup (aa').
\end{aligned}$$

Thus $(A, \rightharpoonup)$ is a left H-module algebra. ■

Theorem 2.6. Let H be Hopf algebra. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and (B, ρ) is a right H-comodule algebra such that Eq.(2.4) holds for all $b, b' \in B$ and $h \in H$. Set $A = B^{coH}$. Then we obtain Molnar's smash product $A\sharp H$ and $B \cong A\sharp H$ as an algebra.

Proof. It straightforward by Lemma 2.5. ■

3. The Factorization of a Class of Module Coalgebras

In this section, we mainly discuss the dual cases of Section 2 and obtain the factorization of a class of module coalgebras into crossed coproduct. For the sake of the technique and the convenience to read, we also sketch the proof of some results.

Proposition 3.1. Let H be Hopf algebra. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$, then $(A, \underline{\Delta}, \varepsilon_A)$ is a coalgebra where

$$\underline{\Delta}(a) = \sum a^{(1)} \otimes a^{(2)} = \sum a_{(1)0} \cdot S(a_{(1)1}) \otimes a_{(2)0} \cdot S(a_{(2)1})$$

and $\varepsilon_A(a) = \varepsilon_B(a)$.

Proof. Straightforward. ■

Theorem 3.2. Let H be Hopf algebra with antipode S. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. Define the weak coaction ω of H on A as:

$$\omega : A \longrightarrow H \otimes A,\ \omega(a) = \sum a_{<-1>} \otimes a_{<0>} = \sum \varepsilon(a_{(1)0})a_{(1)1} \otimes a_{(2)0} \cdot S(a_{(2)1}). \tag{3.1}$$

and the map $\alpha : A \longrightarrow H \otimes H$ by

$$\alpha(a) = \sum \alpha_1(a) \otimes \alpha_2(a) = \sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon(a_{(2)0})a_{(2)1}. \tag{3.2}$$

If the extra condition

$$\sum b_{(1)0} \cdot S(b_{(1)1}) \otimes b_{(2)0} \otimes b_{(2)1} = \sum b_{0(1)0} \cdot S(b_{0(1)1}) \otimes b_{0(2)} \otimes b_1 \tag{3.3}$$

holds for all $b \in B$. Then $A \rtimes_\alpha H$ is a crossed coproduct with comultiplication

$$\begin{aligned}\Delta_\alpha(a \otimes h) &= \sum a^{(1)} \otimes a^{(2)}{}_{<-1>}\alpha_1(a^{(3)})h_{(1)} \otimes a^{(2)}{}_{<0>} \otimes \alpha_2(a^{(3)})h_{(2)} \\ &= \sum a_{(1)0} \cdot S(a_{(1)1}) \otimes a_{(1)2}h_{(1)} \otimes a_{(2)0} \cdot S(a_{(2)1}) \otimes a_{(2)2}h_{(2)}.\end{aligned}$$

Proof. Firstly by Eq.(3.3) we have

$$\sum b_{(1)0} \cdot S(b_{(1)1}) \otimes \varepsilon(b_{(2)0})b_{(2)1} = \sum b_0 \cdot S(b_1) \otimes b_2 \tag{3.4}$$

for all $b \in B$ and

$$\sum a_{(1)0} \cdot S(a_{(1)1}) \otimes a_{(2)0} \otimes a_{(2)1} = \sum a_{(1)0} \cdot S(a_{(1)1}) \otimes a_2 \otimes 1 \tag{3.5}$$

for all $a \in A$.

Secondly, it is obvious that $(W2)$, $(W3)$ and $(CC1)$ are satisfied. We check that $(W1)$ holds as follows:

$$\sum a_{<-1>} \otimes a_{<0>}{}^{(1)} \otimes a_{<0>}{}^{(2)} = \sum \varepsilon(a_{(1)0})a_{(1)1} \otimes a_{(2)0} \cdot S(a_{(2)1}) \otimes a_{(3)0} \cdot S(a_{(3)1})$$

On the other hand,

$$\begin{aligned}
&\sum a^{(1)}{}_{<-1>}a^{(2)}{}_{<-1>} \otimes a^{(1)}{}_{<0>} \otimes a^{(2)}{}_{<0>}\\
=&\sum (a_{(1)0}\cdot S(a_{(1)1}))_{<-1>}a_{(2)<-1>} \otimes (a_{(1)0}\cdot S(a_{(1)1}))_{<0>} \otimes a_{(2)<0>} \quad (\text{ by Eq.(3.5) })\\
=&\sum \varepsilon((a_{(1)0}\cdot S(a_{(1)1}))_{(1)0})(a_{(1)0}\cdot S(a_{(1)1}))_{(1)1}a_{(1)2}\\
&\otimes(a_{(1)0}\cdot S(a_{(1)1}))_{(2)0}\cdot S((a_{(1)0}\cdot S(a_{(1)1}))_{(2)1}) \otimes a_{(2)0}S(a_{(2)1}) \quad (\text{ by Eq.(3.4) })\\
=&\sum \varepsilon(a_{(1)0(1)0})a_{(1)0(1)1}S(a_{(1)1})a_{(1)2} \otimes a_{(1)0(2)0}\cdot S(a_{(1)0(2)1}) \otimes a_{(2)0}S(a_{(2)1})\\
=&\sum \varepsilon(a_{(1)0})a_{(1)1} \otimes a_{(2)0}\cdot S(a_{(2)1}) \otimes a_{(3)0}\cdot S(a_{(3)1}).
\end{aligned}$$

Finally, we compute the condition $(CC2)$ as follows:

$$\begin{aligned}
&\sum a^{(1)}{}_{<-1>}\alpha_1(a^{(2)}) \otimes \alpha_1(a^{(1)}{}_{<0>})\alpha_2(a^{(2)})_{(1)} \otimes \alpha_2(a^{(1)}{}_{<0>})\alpha_2(a^{(2)})_{(2)}\\
=&\sum \varepsilon(a_{(1)0(1)0})a_{(1)0(1)1}S(a_{(1)1})a_{(1)2} \otimes \varepsilon[(a_{(1)0(2)0}\cdot S(a_{(1)0(2)1}))_{(1)0}]\\
&(a_{(1)0(2)0}\cdot S(a_{(1)0(2)1}))_{(1)1}\varepsilon(a_{(2)0})a_{(2)1}\\
&\otimes\varepsilon[(a_{(1)0(2)0}\cdot S(a_{(1)0(2)1}))_{(2)0}](a_{(1)0(2)0}\cdot S(a_{(1)0(2)1}))_{(2)1}a_{(2)2} \quad (\text{ by Eq.(3.5) })\\
=&\sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon[(a_{(2)0}\cdot S(a_{(2)1}))_{(1)0}](a_{(2)0}\cdot S(a_{(2)1}))_{(1)1}a_{(2)2}\\
&\otimes\varepsilon[(a_{(2)0}\cdot S(a_{(2)1}))_{(2)0}](a_{(2)0}\cdot S(a_{(2)1}))_{(2)1}a_{(2)3} \quad (\text{ by Eq.(3.4) })\\
=&\sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon(a_{(2)0(1)0})a_{(2)1(1)1}S(a_{(2)1})_{(1)}a_{(2)2}\\
&\otimes\varepsilon(a_{(2)0(2)0})a_{(2)0(2)1}S(a_{(2)1})_{(2)}a_{(2)3}\\
=&\sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon(a_{(2)0})a_{(2)1} \otimes \varepsilon(a_{(3)0})a_{(3)1}
\end{aligned}$$

while

$$\begin{aligned}
&\sum \alpha_{(1)}(a^{(1)})\alpha_1(a^{(2)})_{(1)} \otimes \alpha_2(a^{(1)})\alpha_1(a^{(2)})_{(2)} \otimes \alpha_2(a^{(2)})\\
=&\sum \alpha_{(1)}(a_{(1)0}\cdot S(a_{(1)1}))\alpha_1(a_{(2)})_{(1)} \otimes \alpha_2(a_{(1)0}\cdot S(a_{(1)1}))\alpha_1(a_{(2)})_{(2)} \otimes \alpha_2(a_{(2)})\\
&\qquad (\text{ by Eq.(3.5) })\\
=&\sum \varepsilon(a_{(1)0(1)0})(a_{(1)0(1)1})S(a_{(1)1})_{(1)}a_{(1)2} \otimes \varepsilon(a_{(1)0(2)0})a_{(1)0(2)1}S(a_{(1)1})_{(2)}a_{(1)3}\\
&\otimes\varepsilon(a_{(2)0})a_{(2)1} \quad (\text{ by Eq.(3.4) })\\
=&\sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon(a_{(2)0})a_{(2)1} \otimes \varepsilon(a_{(3)0})a_{(3)1}
\end{aligned}$$

Similarly $(CC3)$ holds.

Thus we get a crossed coproduct $A \rtimes_\alpha H$. A straight calculation shows that

$$\Delta_\alpha(a\otimes h) = \sum a_{(1)0}\cdot S(a_{(1)1}) \otimes a_{(1)2}h_{(1)} \otimes a_{(2)0}\cdot S(a_{(2)1}) \otimes a_{(2)2}h_{(2)}. \qquad \blacksquare$$

Remark. The Eq.(3.3) manifests that B is an A-H-bicomodule.

Theorem 3.3. Let H be Hopf algebra with antipode S. Assume that $(B,\cdot,\rho)$ is a right H-Hopf module and $(B,\cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. If Eq.(3.3) holds for all $b \in B$. Then $B \cong A \rtimes_\alpha H$ as coalgebras.

Proof. First $B \cong A\otimes H$ as right H-Hopf module and the isomorphic map $\varphi : B \longrightarrow A\otimes H$ is given by

$$\varphi(b) = \sum b_0 \cdot S(b_1) \otimes b_2.$$

Next we check that φ is a coalgebra map. For any $b \in B$, we obtain

$$\begin{aligned}
\Delta_\alpha\varphi(b) &= \sum \Delta_\alpha(b_0 \cdot S(b_1) \otimes b_2)\\
&= \sum (b_0 \cdot S(b_1))_{(1)0} \cdot S(b_0 \cdot S(b_1))_{(1)1} \otimes (b_0 \cdot S(b_1))_{(1)2} b_2\\
&\quad \otimes (b_0 \cdot S(b_1))_{(2)0} \cdot S(b_0 \cdot S(b_1))_{(2)1} \otimes (b_0 \cdot S(b_1))_{(2)2} b_3\\
&= \sum b_{0(1)0} \cdot S(b_{0(1)1}) \otimes b_{0(1)2} S(b_1)_{(1)} b_{(2)} \otimes b_{0(2)0} \cdot S(b_{0(2)1}) \otimes b_{0(2)2} S(b_1)_{(2)} b_{(3)}\\
&= \sum b_{(1)0} \cdot S(b_{(1)1}) \otimes b_{(1)2} \otimes b_{(2)0} \cdot S(b_{(2)1}) \otimes b_{(2)2}\\
&= (\varphi \otimes \varphi)\Delta(b).
\end{aligned}$$
■

Lemma 3.4. Let $(B, \cdot, \phi)$ is a quasi-H-cleft coextension. Define the right coaction ρ of H on B by

$$\rho(b) = \sum b_{(1)} \cdot (\phi^{-1}(b_{(2)})\phi(b_{(3)})_{(1)}) \otimes \phi(b_{(3)})_{(2)}.$$

Then (1) $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra.
(2) The Eq.(3.3) holds.

The following result is obvious by Theorem 3.3 and Lemma 3.4.

Corollary 3.5. Let $(B, \cdot, \phi)$ is a quasi-H-cleft coextension. Set $A = B^{coH}$. Then we have a crossed coproduct $A \rtimes_\alpha H$ with coproduct

$$\Delta(a \otimes h) = \sum a_{(1)} \cdot \phi^{-1}(a_{(2)}) \otimes \phi(a_{(3)})h_{(1)} \otimes a_{(4)} \cdot \phi^{-1}(a_{(5)}) \otimes \phi(a_{(6)})h_{(2)}$$

and counit $\varepsilon(a \otimes h) = \varepsilon_A(a)\varepsilon_H(h)$ and $B \cong A \rtimes_\alpha H$ as a coalgebra.

In the following we factorize a class of module coalgebras to smash coproduct.

Proposition 3.6. Let H be Hopf algebra. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. If the extra condition

$$\sum b_{(1)} \otimes b_{(2)0} \otimes b_{(2)1} = \sum b_{0(1)} \otimes b_{0(2)} \otimes b_1 \tag{3.6}$$

for all $b \in B$ holds. Then $\Delta_B(a) \subset B \otimes A$ for all $a \in A$.

Proof. By Eq.(3.6) we have

$$\sum a_{(1)} \otimes a_{(2)0} \otimes a_{(2)1} = \sum a_{(1)} \otimes a_{(2)} \otimes 1 \tag{3.7}$$

for all $a \in A$. i.e., $\Delta_B(a) \subset B \otimes A, \forall\, a \in A$. ■

Theorem 3.7. Let H be Hopf algebra with antipode S. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. Define $\omega : A \longrightarrow H \otimes A$ as Eq.(3.1) and suppose that the Eq.(3.6) is satisfied for all $b \in B$. Then we have smash coproduct $A \times H$ with comultiplication

$$\Delta_{A\times H}(a \otimes h) = \sum a_{(1)0} \cdot S(a_{(1)1}) \otimes \varepsilon(a_{(2)0})a_{(2)1}h_{(1)} \otimes a_{(3)} \otimes h_{(2)}.$$

Theorem 3.8. Let H be Hopf algebra. Assume that $(B, \cdot, \rho)$ is a right H-Hopf module and $(B, \cdot)$ is a right H-module coalgebra. Set $A = B^{coH}$. If the Eq.(3.6) holds for all $b \in B$. Then $B \cong A \times H$ as a coalgebra.

Proof. Similarly to the proof of Theorem 3.3. ■

4. The Factorization of Bialgebra Into Bicrossproduct

In this section, we are devoted to the factorization of bialgebra and arrive at Radford's known result.

Theorem 4.1. Let H be Hopf algebra and B a bialgebra. Suppose that $(B, \cdot, \rho)$ is a right H-Hopf module, $(B, \cdot)$ is a right H-module coalgebra and (B, ρ) is a right H-comodule algebra. Set $A = B^{coH}$. If the Eq.(2.1) and Eq.(3.3) are satisfied for all $a \in A$, $b \in B$ and $h \in H$. Then $A_\sigma \bowtie_\alpha H$ is a bialgebra whose multiplication and comultiplication are given by

$$(a \otimes h)(a' \otimes h') = \sum a(h_{(1)} \rightharpoonup a')\sigma(h_{(2)}, h'_{(1)}) \otimes h_{(3)}h'_{(2)}$$

and

$$\Delta_\alpha(a \otimes h) = \sum a^{(1)} \otimes a^{(2)}{}_{<-1>}\alpha_1(a^{(3)})h_{(1)} \otimes a^{(2)}{}_{<0>} \otimes \alpha_2(a^{(3)})h_{(2)}$$

respectively, where

$$\omega : A \to H \otimes A,\ \omega(a) = \sum a_{<-1>} \otimes a_{<0>} = \sum \varepsilon(a_{(1)0})a_{(1)1} \otimes a_{(2)0} \cdot S(a_{(2)1}),$$

$$\alpha : A \to H \otimes H,\ \alpha(a) = \alpha_1(a) \otimes \alpha_2(a) = \sum \varepsilon(a_{(1)0})a_{(1)1} \otimes \varepsilon(a_{(2)0})a_{(2)1}$$

and

$$\rightharpoonup : H \otimes A \longrightarrow A, h \rightharpoonup a = \sum((1 \cdot h_{(1)})a) \cdot S(h_{(2)}).$$

Furthermore, $B \cong A_\sigma \bowtie_\alpha H$ as bialgebra.

Proof. Since the map φ is isomorphic in Theorem 2.5 and Theorem 3.3 and B is a bialgebra, the proof is direct. ■

Theorem 4.2. Let H be Hopf algebra with antipode S and B a bialgebra. Suppose that $(B, \cdot, \rho)$ is a right H-Hopf module, $(B, \cdot)$ is a right H-module coalgebra and (B, ρ) is a right H-comodule algebra. Set $A = B^{coH}$. If the Eq.(2.4) and Eq.(3.6) are satisfied for all $b, b' \in B$. Then we can obtain a Radford's biproduct bialgebra $A^\#_\times H$, its multiplication and comultiplication are given as follows:

$$(a \otimes h)(a' \otimes g) = \sum a[((1 \cdot h_{(1)})a') \cdot S(h_{(2)})] \otimes h_{(3)}g_{(3)}$$

and

$$\Delta_\alpha(a \otimes h) = \sum a_{(1)0} \cdot S(a_{(1)1}) \otimes \varepsilon(a_{(2)0})a_{(2)1}h_{(1)} \otimes \otimes a_{(3)} \otimes h_{(2)}.$$

Furthermore $B \cong A^\#_\times H$ as bialgebra.

Proof. It follows from Theorem 2.6 and Theorem 3.8. ■

Lemma 4.3. Let B be a bialgebra and H be Hopf algebra. Then the following conditions are equivalent:
(1) $H \leftrightarrows^{\pi}_{i} B$ is bialgebra projective map. i.e. π and i are bialgebra map satisfying $\pi \circ i = id$.
(2) $(B, \cdot, \rho)$ is a right H-Hopf module, $(B, \cdot)$ is a right H-module coalgebra, (B, ρ) is a right H-comodule algebra and the equalities below hold for all $h \in H$ and $b, b' \in B$:

1) $(bb') \cdot h = b(b' \cdot h)$,
2) $\sum b_{(1)} \otimes b_{(2)0} \otimes b_{(2)1} = \sum b_{0(1)} \otimes b_{0(2)} \otimes b_1$.

Proof. (1) ⇒ (2) Define two maps:

$$\rho : B \longrightarrow B \otimes H, \rho(b) = \sum\ b_{(1)} \otimes \pi(b_{(2)})$$

for all $b \in B$ and

$$\cdot : B \otimes H \longrightarrow B, b \cdot h = bi(h)$$

for all $b \in B$ and $h \in H$.

Then $(B, \cdot, \rho)$ is a right H-Hopf module bialgebra and 1), 2) in Part 2 hold.

(2) ⇒ (1) Define: $i : H \to B$ by $i(h) = 1_B \cdot h$ and $\pi : B \to H$ by $\pi(b) = \sum \varepsilon(b_0)b_1$. It is straightforward that π and i are bialgebra map satisfying $\pi \circ i = id$. ■

Corollary 4.4. (see [12]) Let H be Hopf algebra and B a bialgebra. Let $H \leftrightarrows^{\pi}_{i} B$ be bialgebra projective map. i.e. π and i are bialgebra map satisfying $\pi \circ i = id$. Set $A = \{a \in B \mid \sum a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1\}$. Then $B \cong A^\#_\times H$ as bialgebra.

Proof. It follows from Theorem 4.2 and Lemma 4.3. ■

Next we give a concrete example.

Example 4.5. Let $H_4 = k\{1, g, x, gx\}$ be Sweelder's 4-dimensional Hopf algebra with $Chark \neq 2$. Its product and coproduct are given below:

$$g^2 = 1,\ x^2 = 0,\ xg = -gx$$

and

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x.$$

Its counit and antipode are given by $\varepsilon(g) = 1, \varepsilon(x) = 0$ and $S(g) = g = g^{-1}, S(x) = -gx$ respectively.

Set $H = k\{1, g\}$. Define the right action $\cdot$ of H on H_4 by its multiplication and the right coaction ρ of H on H_4 by

$$\rho : H_4 \longrightarrow H_4 \otimes H,\ \rho(1) = 1 \otimes 1,\ \rho(g) = g \otimes g,\ \rho(x) = x \otimes 1,\ \rho(gx) = gx \otimes g.$$

Then $(H_4, \cdot)$ is a right H-module coalgebra, (H_4, ρ) is a right H-comodule algebra, and $(H_4, \cdot, \rho)$ is a right H-Hopf module. Set $A = {H_4}^{coH} = \{1, x\}$. Furthermore the following conditions hold:

$$(bb') \cdot h = b(b' \cdot h),$$

$$\sum b_{(1)} \otimes b_{(2)0} \otimes b_{(2)1} = \sum b_{0(1)} \otimes b_{0(2)} \otimes b_1$$

for all $b, b' \in H_4$ and $H_4 \cong A^{\#}_{\times}H$ as bialgebra.

Next we give the concrete stucture of $A^{\#}_{\times}H$: (Multiplication $\bullet$ and comultiplication Δ)

$\bullet$	$1\otimes 1$	$1\otimes g$	$x\otimes 1$	$x\otimes g$
$1\otimes 1$	$1\otimes 1$	$1\otimes g$	$x\otimes 1$	$x\otimes g$
$1\otimes g$	$1\otimes g$	$1\otimes 1$	$-x\otimes g$	$-x\otimes 1$
$x\otimes 1$	$x\otimes 1$	$x\otimes g$	0	0
$x\otimes g$	$x\otimes g$	$x\otimes 1$	0	0

and

$\Delta(1 \otimes 1) = (1 \otimes 1) \otimes (1 \otimes 1),\ \ \epsilon(1 \otimes 1) = 1;$
$\Delta(1 \otimes g) = (1 \otimes g) \otimes (1 \otimes g),\ \ \epsilon(1 \otimes g) = 1;$
$\Delta(x \otimes 1) = (x \otimes 1) \otimes (1 \otimes 1) + (1 \otimes g) \otimes (x \otimes 1),\ \ \epsilon(x \otimes 1) = 0;$
$\Delta(x \otimes g) = (x \otimes g) \otimes (1 \otimes g) + (1 \otimes 1) \otimes (x \otimes g),\ \ \epsilon(x \otimes g) = 0.$

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