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CLOSED EXPRESSIONS FOR COEFFICIENTS IN WEIGHTED NEWTON-COTES QUADRATURES

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Abstract. In this short note, we derive closed expressions for Cotes numbers in the weighted Newton-Cotes quadrature formulae with equidistant nodes in terms of moments and Stirling numbers of the first kind. Three types of equidistant nodes are considered. The corresponding program codes in Mathematica Package are presented. Finally, in order to illustrate the application of the obtained quadrature formulas a few numerical examples are included.

1. Introduction

We consider the weighted quadrature formulas on the finite interval $[a, b]$,

$$\int_a^b f(x)w(x)\,dx = \sum_{k\in S_n} W_k f(x_k) + R_n(f), \tag{1}$$

where w is a given weight function on (a, b) and the nodes x_k are equidistantly distributed with the step $h = (b - a)/n$ in the following three cases:

1° $x_k = a + kh$, $S_n = \{0, 1, \ldots, n\}$;

2° $x_k = a + kh$, $S_n = \{1, \ldots, n - 1\}$;

3° $x_k = a + (k - \frac{1}{2})h$, $S_n = \{1, \ldots, n\}$.

In the first case the formula (1) is of the closed type, and in other ones we have formulas of the open type. Such quadrature formulas are called the weighted Newton-Cotes formulas if they are interpolatory, i.e., $R_n(f) = 0$ whenever $f \in \mathcal{P}_d$, where $\mathcal{P}_d$ is the space of all algebraic polynomials of degree at most

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$d = d_n = \text{card}\, S_n - 1$. Thus, the formulae (1) are exact on the linear space $\mathcal{P}_d$. Taking any basis in this space, e.g., $\{u_0, u_1, \ldots, u_d\}$, the coefficients W_k in (1) must satisfy the following system of linear equations

$$\sum_{k\in S_n} u_\nu(x_k)W_k = \int_a^b u_\nu(x)w(x)\,dx, \quad \nu = 0, 1, \ldots, d_n.$$

For example, for $u_\nu(x) = x^\nu$, it reduces to

$$\sum_{k\in S_n} x_k^\nu W_k = \int_a^b x^\nu w(x)\,dx, \quad \nu = 0, 1, \ldots, d_n.$$

The weight coefficients, known as the Cotes numbers, can be also expressed, using the Lagrange interpolation, in the form

$$W_k = \frac{1}{\Psi'(x_k)} \int_a^b \frac{\Psi(x)w(x)}{x - x_k}\,dx, \quad k \in S_n, \tag{2}$$

where Ψ is the node polynomial defined by

$$\Psi(x) = \prod_{k\in S_n}(x - x_k). \tag{3}$$

In [2], Gautschi considered the numerical construction of these coefficients, associated by the weight function w and the nodes x_k, in two ways: (a) using (2) and the barycentric formula for the elementary Lagrange interpolation polynomials

$$\ell_k(x) = \frac{\Psi(x)}{(x - x_k)\Psi'(x_k)} = \prod_{\nu\in S_n\setminus\{k\}} \frac{x - x_\nu}{x_k - x_\nu} = \frac{\dfrac{\lambda_k}{x - x_k}}{\displaystyle\sum_{\nu\in S_n\setminus\{k\}} \frac{\lambda_\nu}{x - x_\nu}} \quad (x \neq x_k),$$

where λ_k are the auxiliary quantities

$$\lambda_k = \prod_{\nu\in S_n\setminus\{k\}} \frac{1}{x_k - x_\nu}, \qquad k \in S_n;$$

(b) using moment-based moments, taking certain orthogonal polynomials as basis functions in $\{u_0, u_1, \ldots, u_d\}$. For some remarks on Newton-Cotes rules with Jacobi weight functions see [5, §5.1.2]. An interesting connection between closed Newton–Cotes differential methods and symplectic integrators has been considered in [3].

As we know, numerical integration begins by Newton's idea from 1676. In modern terminology, for given distinct points x_k and corresponding values $f(x_k)$, Newton constructs the unique polynomial which at the points x_k assumes the same values as f, expressing this interpolation polynomial in terms of divided differences. However today, in almost all applications Cotes numbers are written in the Lagrange form (2).

In this paper we directly follow Newton's approach in order to obtain appropriate closed-form expressions for Cotes numbers in each of cases 1°–3° (Section 2). A similar approach with geometric distributed nodes has been recently obtained in [4]. The corresponding program codes in Mathematica Package for calculating nodes and weights are also included in Section 2. In Section 3 we give a few numerical examples in order to illustrate the application of these quadrature formulas. We think that this approach may be useful in applications that require explicit expressions for the Cotes numbers.

2. Closed-form expressions for weighted Cotes numbers

Let the nodes x_k be given as in 1°, i.e., $x_k = a + kh$, $k = 0, 1, \dots, n$, and $h = (b - a)/n$. We start this section with the Newton interpolation formula [1, pp. 96–101]

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) + r_{n+1}(f; x), \tag{4}$$

where

$$b_0 = f[x_0], \quad b_1 = f[x_0, x_1], \quad b_2 = f[x_0, x_1, x_2], \ \dots, \ b_n = f[x_0, x_1, \dots, x_n]$$

respectively denote divided differences, and $r_{n+1}(f; x)$ is the corresponding interpolation error

$$r_{n+1}(f; x) = f[x_0, x_1, \dots, x_n, x]\Psi_{n+1}(x),$$

where the node polynomial $\Psi_{n+1}(x)$ is defined as in (3), i.e.,

$$\Psi_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n).$$

In the sequel we use the Stirling numbers of the first kind $s(m, \nu)$, which are defined by the coefficients in the expansion

$$(x)_m = x(x - 1)\cdots(x - m + 1) = \sum_{\nu=0}^{m} s(m, \nu)x^{\nu}. \tag{5}$$

For $m = 0$ we have $(x)_0 = 1$ and $s(0, 0) = 1$. In general, the following recurrence relation

$$s(m + 1, \nu) = s(m, \nu - 1) - ms(m, \nu), \quad 1 \le \nu < m,$$

holds, with the following initial conditions $s(m, 0) = 0$ and $s(1, 1) = 1$.

Theorem 2.1. *Let $n \in \mathbb{N}$, $h = (b - a)/n$, $x_k = a + kh$, $k \in S_n = \{0, 1, \dots, n\}$, and*

$$\mu_\nu(a, h) = \int_a^b \left(\frac{x - a}{h}\right)^{\nu} w(x)\, dx, \quad \nu = 0, 1, \dots, \tag{6}$$

Then the coefficients W_k in the quadrature formula (1) *are given by*

$$W_k = (-1)^k \sum_{m=k}^{n} \binom{m}{k} A_m(a, h), \quad k \in S_n, \tag{7}$$

where

$$A_m(a, h) = \frac{(-1)^m}{m!} \sum_{\nu=0}^{m} s(m, \nu)\mu_\nu(a, h), \quad m = 0, 1, \dots, n, \tag{8}$$

and $s(m, \nu)$ are Stirling numbers of the first kind defined in (5).

Proof. Let $\Psi_0(x) = 1$ and $\Psi_{m+1}(x) = \Psi_m(x)(x - x_m)$, $0 \le m \le n$, and $x = a + th$, where $h = (b - a)/n$. Since $x - x_\nu = h(t - \nu)$, we have

$$\int_a^b \Psi_m(x)w(x)\, dx = h^{m+1} \int_0^n (t)_m w(a + th)\, dt = h^{m+1} \int_0^n \sum_{\nu=0}^{m} s(m, \nu)t^{\nu}\, dt,$$

where the Stirling numbers of the first kind are defined in (5). Further, it gives

$$\begin{aligned}
\int_a^b \Psi_m(x)w(x)\,dx &= h^{m+1}\sum_{\nu=0}^{m} s(m,\nu)\int_0^n t^\nu w(a+th)\,dt \\
&= h^m \sum_{\nu=0}^{m} s(m,\nu)\int_a^b \left(\frac{x-a}{h}\right)^\nu w(x)\,dx \\
&= h^m \sum_{\nu=0}^{m} s(m,\nu)\mu_\nu(a,h) \\
&= (-1)^m m!\,h^m A_m(a,h),
\end{aligned}$$

where we introduced the notations (6) and (8).

Now, integrating (4) with respect to the weight $w(x)$ over (a,b) we obtain

$$\int_a^b f(x)w(x)\,dx = \sum_{m=0}^{n} b_m \int_a^b \Psi_m(x)w(x)\,dx + \int_a^b r_{n+1}(f;x)w(x)\,dx. \tag{9}$$

According to the general identity

$$f[x_0,x_1,\ldots,x_m] = \sum_{k=0}^{m} \frac{f(x_k)}{\Psi'_{m+1}(x_k)},$$

in which $\Psi'_{m+1}(x_k)$ is the derivative of the polynomial $\Psi_{m+1}(x)$ at $x = x_k$, we get

$$b_m = \sum_{k=0}^{m} (-1)^{m-k}\frac{f(x_k)}{k!(m-k)!h^m} = \frac{(-1)^m}{m!h^m}\sum_{k=0}^{m}(-1)^k\binom{m}{k}f(x_k),$$

because $\Psi'_{m+1}(x_k) = \prod_{\nu=0,\nu\neq k}^{m}(x_k - x_\nu) = (-1)^{m-k}k!(m-k)!h^m$. Therefore, (9) reduces to

$$\begin{aligned}
\int_a^b f(x)w(x)\,dx &= \sum_{m=0}^{n} A_m(a,h)\sum_{k=0}^{m}(-1)^k\binom{m}{k}f(x_k) + R_n(f) \\
&= \sum_{k=0}^{n}\left((-1)^k\sum_{m=k}^{n}\binom{m}{k}A_m(a,h)\right)f(x_k) + R_n(f) \\
&= \sum_{k=0}^{n} W_k f(x_k) + R_n(f),
\end{aligned}$$

where the coefficients W_k are given by (7) and $R_n(f) = \int_a^b r_{n+1}(f;x)w(x)\,dx$. □

A program code in the Mathematica Package for the nodes and weights (Cotes numbers) from Theorem 2.1 can be done by the following procedure:

```
NC1[n_,a_,b_,w_]:= Module[{h =(b-a)/n,mu,x,k,m,nu,A,nodes,weights},
mu=Table[Integrate[((x-a)/h)^nu w[x],{x,a,b}],{nu,0,n}];
A=Table[(-1)^m/m! Sum[StirlingS1[m,nu] mu[[nu+1]],{nu,0,m}],{m,0,n}];
nodes = Table[a+k h,{k,0,n}];
weights = Table[(-1)^k Sum[Binomial[m,k] A[[m+1]],
                      {m,k,n}],{k,0,n}]//Simplify;
Return[{nodes,weights}];]
```

We take the following seven weight functions

```
w1[x_]:= 1;
w2[x_]:= x^2;
w3[x_]:= Abs[x];
w4[x_]:= Exp[x];
w5[x_]:= Cos[Pi x/2];
w6[x_]:= x^(-1/2)Log[1/x];
w7[x_]:= Cos[100 Pi x];
```

where the last of them is not a standard (nonnegative) weight function.

Remark. Alternatively, in the cases when a symbolic integration of the moments $\mu_\nu(a,h)$ is not possible, then a numerical calculation must be included in the previous subprogram.

Using the previous procedure we obtain the following results for some selected intervals, weights, and number of nodes:

In[3]:= **NC1[8, -1, 1, w1]**

Out[3]= $\left\{\left\{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}, \left\{\frac{989}{14\,175}, \frac{5888}{14\,175}, -\frac{928}{14\,175}, \frac{10\,496}{14\,175}, -\frac{908}{2835}, \frac{10\,496}{14\,175}, -\frac{928}{14\,175}, \frac{5888}{14\,175}, \frac{989}{14\,175}\right\}\right\}$

In[4]:= **NC1[8, -1, 1, w2][[2]]**

Out[4]= $\left\{\frac{9769}{155\,925}, \frac{15\,104}{51\,975}, -\frac{33\,632}{155\,925}, \frac{69\,376}{155\,925}, -\frac{148}{297}, \frac{69\,376}{155\,925}, -\frac{33\,632}{155\,925}, \frac{15\,104}{51\,975}, \frac{9769}{155\,925}\right\}$

In[5]:= **NC1[8, -1, 1, w3][[2]]**

Out[5]= $\left\{\frac{1249}{18\,900}, \frac{544}{1575}, -\frac{116}{675}, \frac{352}{675}, -\frac{47}{90}, \frac{352}{675}, -\frac{116}{675}, \frac{544}{1575}, \frac{1249}{18\,900}\right\}$

In[6]:= **NC1[5, -1, 1, w4][[2]]**

Out[6]= $\left\{-\frac{14\,947}{48\,e} + \frac{253\,e}{6}, \frac{67\,450 - 9125\,e^2}{48\,e}, \frac{25\left(-2474 + 335\,e^2\right)}{24\,e}, \frac{57\,275 - 7750\,e^2}{24\,e}, \frac{25\left(-2137 + 290\,e^2\right)}{48\,e}, \frac{1253}{6\,e} - \frac{1351\,e}{48}\right\}$

In[7]:= **NC1[5, -1, 1, w5][[2]]**

Out[7]= $\left\{\frac{7500 - 875\,\pi^2 + 12\,\pi^4}{6\,\pi^5}, \frac{25\left(-300 + 31\,\pi^2\right)}{2\,\pi^5}, \frac{7500 - 725\,\pi^2}{3\,\pi^5}, \frac{7500 - 725\,\pi^2}{3\,\pi^5}, \frac{25\left(-300 + 31\,\pi^2\right)}{2\,\pi^5}, \frac{7500 - 875\,\pi^2 + 12\,\pi^4}{6\,\pi^5}\right\}$

In[8]:= **NC1[5, 0, 1, w6]**

Out[8]= $\left\{\left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}, \left\{\frac{1\,054\,232}{480\,249}, \frac{2\,783\,252}{1\,440\,747}, -\frac{1\,134\,032}{1\,440\,747}, \frac{8024}{9801}, -\frac{290\,168}{1\,440\,747}, \frac{8816}{205\,821}\right\}\right\}$

In[9]:= **NC1[5, -1, 1, w7][[2]]**

Out[9]= $\left\{\frac{-3 + 4000\,\pi^2}{7\,680\,000\,\pi^4}, \frac{3 - 2400\,\pi^2}{2\,560\,000\,\pi^4}, \frac{-3 + 1600\,\pi^2}{3\,840\,000\,\pi^4}, \frac{-3 + 1600\,\pi^2}{3\,840\,000\,\pi^4}, \frac{3 - 2400\,\pi^2}{2\,560\,000\,\pi^4}, \frac{-3 + 4000\,\pi^2}{7\,680\,000\,\pi^4}\right\}$

Now, we consider the open Newton-Cotes quadrature formula (1) with nodes $x_k = a+kh$, $k = 1, \dots, n-1$, given as in 2°. In this case, the corresponding Newton interpolation formula is

$$f(x) = c_1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2) + \cdots + c_{n-1}(x-x_1)(x-x_2)\cdots(x-x_{n-2}) + \hat{r}_{n-1}(f;x), \tag{10}$$

where $c_1 = f[x_1]$, $c_2 = f[x_1, x_2]$, $c_2 = f[x_1, x_2, x_3]$, ..., $c_{n-1} = f[x_1, x_2, \dots, x_{n-1}]$ and the corresponding error is $\hat{r}_{n-1}(f;x) = f[x_1, x_2, \dots, x_{n-1}, x]\,\Omega_{n-1}(x)$, where the node polynomial $\omega_{n-1}(x)$ is defined now as

$$\Omega_{n-1}(x) = (x-x_1)(x-x_2)\cdots(x-x_{n-1}).$$

Theorem 2.2. *Let $n \geq 2$, $h = (b-a)/n$, $x_k = a+kh$, $k \in S_n = \{1, \dots, n-1\}$, and*

$$\mu_\nu^*(a,h) = \int_a^b \left(\frac{x-a}{h} - 1\right)^\nu w(x)\,dx, \quad \nu = 0, 1, \dots, \tag{11}$$

Then the coefficients W_k in the quadrature formula (1) *are given by*

$$W_k = k(-1)^k \sum_{m=k}^{n-1} \binom{m}{k} A_m^*(a,h), \quad k \in S_n, \tag{12}$$

where

$$A_m^*(a,h) = \frac{(-1)^m}{m!} \sum_{\nu=0}^{m-1} s(m-1,\nu)\mu_\nu^*(a,h), \quad m = 1, \dots, n-1, \tag{13}$$

and $s(m-1,\nu)$ are Stirling numbers of the first kind defined in (5)*.*

Proof. Taking $\Omega_m(x) = (x-x_1)\cdots(x-x_m)$, we get

$$\begin{aligned}\int_a^b \Omega_{m-1}(x)w(x)\,dx &= h^{m-1}\sum_{\nu=0}^{m-1} s(m-1,\nu)\int_a^b \left(\frac{x-a}{h}-1\right)w(x)\,dx \\ &= h^{m-1}\sum_{\nu=0}^{m-1} s(m-1,\nu)\mu_\nu^*(a,h) \\ &= (-1)^m m!\, h^{m-1} A_m^*(a,h),\end{aligned}$$

where $\mu_\nu^*(a,h)$ and $A_m^*(a,h)$ are defined by (11) and (13), respectively. Similarly as before we have

$$c_m = f[x_1, x_2, \dots, x_m] = \sum_{k=1}^m \frac{f(x_k)}{\Omega_m'(x_k)} = \frac{(-1)^m}{m! h^{m-1}} \sum_{k=1}^m k(-1)^k \binom{m}{k} f(x_k),$$

so that (12) follows immediately. □

Mathematica code of the corresponding procedure is as follows:

```
NC2[n_,a_,b_,w_]:= Module[{h =(b-a)/n,mu,x,k,m,nu,A,nodes,weights},
mu=Table[Integrate[((x-a)/h-1)^nu w[x],{x,a,b}],{nu,0,n-1}];
A=Table[(-1)^m/m! Sum[StirlingS1[m-1,nu] mu[[nu+1]],
                    {nu,0,m-1}],{m,1,n-1}];
nodes = Table[a+k h,{k,1,n-1}];
weights = Table[k(-1)^k Sum[Binomial[m,k] A[[m]],
              {m,k,n-1}],{k,1,n-1}]//Simplify;
Return[{nodes,weights}];]
```

Taking the same previous weight functions we get the following results:

```
In[3]:= NC2[8, -1, 1, w1]
```

Out[3]= $\left\{\left\{-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}, \left\{\frac{184}{189}, -\frac{212}{105}, \frac{488}{105}, -\frac{4918}{945}, \frac{488}{105}, -\frac{212}{105}, \frac{184}{189}\right\}\right\}$

```
In[4]:= NC2[8, -1, 1, w2][[2]]
```

Out[4]= $\left\{\frac{11\,224}{14\,175}, -\frac{9308}{4725}, \frac{3736}{945}, -\frac{1978}{405}, \frac{3736}{945}, -\frac{9308}{4725}, \frac{11\,224}{14\,175}\right\}$

```
In[5]:= NC2[8, -1, 1, w3][[2]]
```

Out[5]= $\left\{\frac{118}{135}, -\frac{91}{45}, \frac{38}{9}, -\frac{139}{27}, \frac{38}{9}, -\frac{91}{45}, \frac{118}{135}\right\}$

```
In[6]:= NC2[5, -1, 1, w4][[2]]
```

Out[6]= $\left\{\frac{-1181 + 161\, e^2}{24\, e}, \frac{1013}{8\, e} - \frac{133\, e}{8}, \frac{39\left(-23 + 3\, e^2\right)}{8\, e}, \frac{809 - 89\, e^2}{24\, e}\right\}$

```
In[7]:= NC2[5, -1, 1, w5][[2]]
```

Out[7]= $\left\{\frac{-50 + 6\pi^2}{\pi^3}, \frac{50 - 4\pi^2}{\pi^3}, \frac{50 - 4\pi^2}{\pi^3}, \frac{-50 + 6\pi^2}{\pi^3}\right\}$

```
In[8]:= NC2[5, 0, 1, w6]
```

Out[8]= $\left\{\left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}, \left\{\frac{14\,116}{1323}, -\frac{6080}{441}, \frac{4120}{441}, -\frac{2944}{1323}\right\}\right\}$

```
In[9]:= NC2[5, -1, 1, w7][[2]]
```

Out[9]= $\left\{\frac{1}{1600\,\pi^2}, -\frac{1}{1600\,\pi^2}, -\frac{1}{1600\,\pi^2}, \frac{1}{1600\,\pi^2}\right\}$

Finally, for an open quadrature formula with nodes given as in 3° we can prove the following statement:

Theorem 2.3. *Let* $n \in \mathbb{N}$, $h = (b-a)/n$, $x_k = a + \left(k - \frac{1}{2}\right)h$, $k \in S_n = \{1, \ldots, n\}$, *and*

$$\tilde{\mu}_\nu^*(a,h) = \int_a^b \left(\frac{x-a}{h} - \frac{1}{2}\right)^\nu w(x)\,dx, \quad \nu = 0, 1, \ldots, \tag{14}$$

Then the coefficients W_k *in the quadrature formula* (1) *are given by*

$$W_k = k(-1)^k \sum_{m=k}^{n} \binom{m}{k} \tilde{A}_m^*(a,h), \quad k \in S_n, \tag{15}$$

where

$$\tilde{A}_m^*(a,h) = \frac{(-1)^m}{m!} \sum_{\nu=0}^{m-1} s(m-1,\nu)\tilde{\mu}_\nu^*(a,h), \quad m = 1, \ldots, n, \tag{16}$$

and $s(m-1,\nu)$ *are Stirling numbers of the first kind defined in* (5)*.*

Mathematica code of the corresponding procedure is as follows:

```
NC3[n_,a_,b_,w_]:= Module[{h =(b-a)/n,mu,x,k,m,nu,A,nodes,weights},
mu=Table[Integrate[((x-a)/h-1/2)^nu w[x],{x,a,b}],{nu,0,n}];
A=Table[(-1)^m/m! Sum[StirlingS1[m-1,nu] mu[[nu+1]],
                         {nu,0,m-1}],{m,1,n}];
nodes = Table[a+(k-1/2)h,{k,1,n}];
weights = Table[k(-1)^k Sum[Binomial[m,k] A[[m]],
                    {m,k,n}],{k,1,n}]//Simplify;
Return[{nodes,weights}];]
```

For the same previous weight functions we get the following results:

In[3]:= **NC3[8, -1, 1, w1]**

Out[3]= $\left\{\left\{-\frac{7}{8}, -\frac{5}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}, \left\{\frac{295\,627}{967\,680}, \frac{71\,329}{967\,680}, \frac{17\,473}{35\,840}, \frac{128\,953}{967\,680}, \frac{128\,953}{967\,680}, \frac{17\,473}{35\,840}, \frac{71\,329}{967\,680}, \frac{295\,627}{967\,680}\right\}\right\}$

In[4]:= **NC3[8, -1, 1, w2][[2]]**

Out[4]= $\left\{\frac{534\,929}{2\,073\,600}, -\frac{265\,823}{2\,903\,040}, \frac{459\,983}{1\,612\,800}, -\frac{343\,367}{2\,903\,040}, -\frac{343\,367}{2\,903\,040}, \frac{459\,983}{1\,612\,800}, -\frac{265\,823}{2\,903\,040}, \frac{534\,929}{2\,073\,600}\right\}$

In[5]:= **NC3[8, -1, 1, w3][[2]]**

Out[5]= $\left\{\frac{77\,437}{276\,480}, -\frac{1525}{55\,296}, \frac{3479}{10\,240}, -\frac{5101}{55\,296}, -\frac{5101}{55\,296}, \frac{3479}{10\,240}, -\frac{1525}{55\,296}, \frac{77\,437}{276\,480}\right\}$

In[6]:= **NC3[5, -1, 1, w4][[2]]**

Out[6]= $\left\{\frac{5\left(-9593+1305\,e^2\right)}{384\,e}, \frac{42\,305-5725\,e^2}{96\,e}, \frac{-38\,189+5189\,e^2}{64\,e}, -\frac{5\left(-6989+945\,e^2\right)}{96\,e}, \frac{5\left(-6457+905\,e^2\right)}{384\,e}\right\}$

In[7]:= **NC3[5, -1, 1, w5][[2]]**

Out[7]= $\left\{\frac{25\left(9600-1168\,\pi^2+21\,\pi^4\right)}{96\,\pi^5}, -\frac{25\left(9600-1072\,\pi^2+9\,\pi^4\right)}{24\,\pi^5}, \frac{15\,000}{\pi^5}-\frac{1625}{\pi^3}+\frac{189}{16\,\pi}, -\frac{25\left(9600-1072\,\pi^2+9\,\pi^4\right)}{24\,\pi^5}, \frac{25\left(9600-1168\,\pi^2+21\,\pi^4\right)}{96\,\pi^5}\right\}$

In[8]:= **NC3[5, 0, 1, w6]**

Out[8]= $\left\{\left\{\frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}\right\}, \left\{\frac{2\,286\,121}{381\,024}, -\frac{542\,119}{95\,256}, \frac{361\,021}{63\,504}, -\frac{239\,899}{95\,256}, \frac{199\,921}{381\,024}\right\}\right\}$

In[9]:= **NC3[5, -1, 1, w7][[2]]**

Out[9]= $\left\{\frac{-3+4600\,\pi^2}{3\,840\,000\,\pi^4}, \frac{3-3400\,\pi^2}{960\,000\,\pi^4}, \frac{3\left(-1+1000\,\pi^2\right)}{640\,000\,\pi^4}, \frac{3-3400\,\pi^2}{960\,000\,\pi^4}, \frac{-3+4600\,\pi^2}{3\,840\,000\,\pi^4}\right\}$

3. Numerical examples

As a first example we consider an integral, which value can be expressed in terms of the generalized hypergeometric function ${}_pF_q(a;b;z)$,

$$\int_0^1 \frac{\sin \pi x}{\sqrt{x}} \log \frac{1}{x}\, dx = 4\,\mathrm{Im}\left\{{}_2F_2\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \mathrm{i}\pi\right)\right\} \approx 1.0489155915263696930987897861188534461 54.$$

In this case we take quadrature rules with respect to the weight function $w(x) = w_6(x) = x^{-1/2}\log(1/x)$. Then, for $f(x) = \sin \pi x$, the relative errors in the corresponding quadratures with equidistant nodes given in the cases 1°–3° (Theorems 2.1–2.3), are presented in Table 1. Numbers in parentheses indicate decimal exponents. Notice that the corresponding number of quadrature nodes in these cases are $n+1$, $n-1$, and n, respectively.

Table 1: Relative errors of quadrature sums for $n = 5(5)30$

$w(x)$	$w_6(x) = x^{-1/2}\log(1/x)$			$w_5(x) = \cos(\pi x/2)$	
n	Case 1°	Case 2°	Case 3°	Case 2°	Case 3°
5	1.69(−3)	2.98(−1)	1.01(−2)	1.21(−1)	1.70(−2)
10	4.26(−9)	7.14(−6)	2.14(−6)	1.67(−2)	4.46(−3)
15	9.08(−14)	4.14(−10)	1.05(−12)	6.54(−3)	1.99(−3)
20	4.03(−21)	4.92(−17)	1.07(−17)	3.03(−3)	1.10(−3)
25	1.21(−26)	2.60(−22)	1.91(−25)	1.82(−3)	6.82(−4)
30	4.90(−35)	1.99(−30)	3.56(−31)	1.13(−3)	4.67(−4)

Evidently, the closed rule (Case 1°) converges faster than other two open rules (Cases 2° and 3°) with a smaller number of nodes.

As a second example we consider the following integral

$$\int_{-1}^{1} \log(1-x^2)\cos\frac{\pi x}{2}\,dx = -\frac{4}{\pi}\Big(\gamma - \mathrm{Ci}(\pi) + \log(\pi/4)\Big) = -0.333567469\ldots$$

and an application of the previous quadrature rules, with respect to the weight function $w(x) = w_5(x) = \cos(\pi x/2)$, to the function $f(x) = \log(1-x^2)$.

Regarding the logarithmic singularities at ±1, the first rule cannot be applied, and the other ones show a very slow convergence (of course, because of the influece of these singularities). The corresponding relative errors in quadrature sums are presented in the second part of the same Table 1.

Finally, we consider an integral of a highly oscillatory function,

$$\int_{-1}^{1} e^x \cos(100\pi x)\,dx = \frac{e^2-1}{e(1+10^4\pi^2)} \approx 2.38143139021284126073282 \times 10^{-5},$$

which integrand is displayed in Fig. 1.

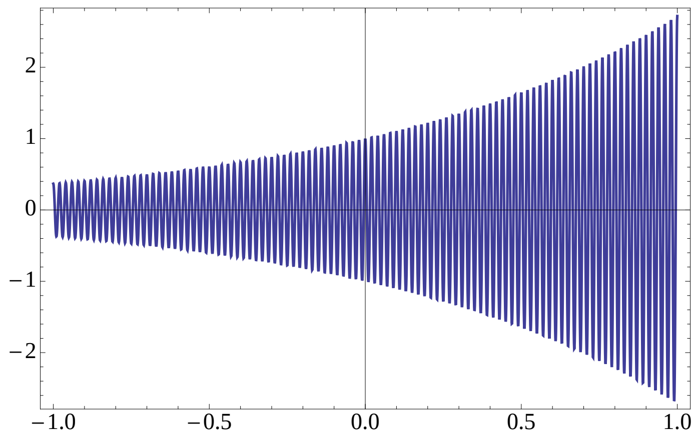

Figure 1: Integrand $x \mapsto e^x \cos(100\pi x)$ on $[-1, 1]$

Table 2: Relative errors of quadrature sums for $n = 5(5)20$, in three different cases for $w_7(x) = \cos(100\pi x)$ and $f(x) = e^x$

n	Case 1°	Case 2°	Case 3°
5	1.51(−3)	1.20(−1)	3.68(−3)
10	6.68(−10)	6.71(−7)	3.34(−7)
15	3.97(−15)	1.18(−11)	2.08(−14)
20	1.79(−23)	1.55(−19)	5.27(−20)

In Table 2 we present relative errors in quadrature sums for each of the obtained quadrature rules (Cases 1°−3°) with respect to the oscillatory weight function $w_7(x) = \cos(100\pi x)$. As we can see the convergence of these rules is very fast.

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