

Nonnegative generalized inverses in indefinite inner product spaces

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Abstract. The aim of this article is to investigate nonnegativity of the inverse, the Moore-Penrose inverse and other generalized inverses, in the setting of indefinite inner product spaces with respect to the indefinite matrix product. We also propose and investigate generalizations of the corresponding notions of matrix monotonicity, namely, \circ -(rectangular) monotonicity, \circ -semimonotonicity and \circ -weak monotonicity and its interplay with nonnegativity of various generalized inverses in the same setting.

1. Introduction

An indefinite inner product in \mathbb{C}^n is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0 \forall y \in \mathbb{C}^n$ only when $x = 0$. Associated with any indefinite inner product is a unique invertible Hermitian matrix J (called a weight) with complex entries such that $[x, y] = \langle x, Jy \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on \mathbb{C}^n and vice versa. Motivated by the notion of Minkowski space (as studied by physicists), we also make an additional assumption on J , namely, $J^2 = I$. It can be shown that this assumption on J is not really restrictive as the results presented in this manuscript can also be deduced without this assumption on J , with appropriate modifications. It should be remarked that this assumption also allows us to compare our results with the Euclidean case, apart from allowing us to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (See for instance [3]). This causes a problem as there are two different values for the dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [19]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B := AJ_n B$. The adjoint of A , denoted by $A^{[*]}$, is defined to be the matrix $J_n A^* J_m$, where J_m and J_n are weights in the appropriate spaces. Many properties of this product are similar to that of the usual matrix product (refer [19]). Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results in indefinite inner product spaces in a manner analogous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar also established in [19] that in the setting of indefinite inner product spaces, the indefinite

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matrix product is more appropriate than the usual matrix product. Recall that the Moore-Penrose inverse exists if and only if $\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A)$. If we take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $AA^{[*]}$ and $A^{[*]}A$ are both the zero matrix and so $\text{rank}(AA^{[*]}) < \text{rank}(A)$, thereby proving that the Moore-Penrose inverse doesn't exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $\text{rank}(A \circ A^{[*]}) = \text{rank}(A^{[*]} \circ A) = \text{rank}(A)$. Thus, the Moore-Penrose inverse of a matrix with real or complex entries exists over an indefinite inner product space with respect to the indefinite matrix product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinent to extend the study of generalized inverses to the setting of indefinite inner product spaces, with respect to the indefinite matrix product. It should be remarked that the Moore-Penrose inverse in an indefinite inner product space was earlier discussed by Kamaraj and Sivakumar (see [9] and the references cited therein). It was also pointed out in [9] that the matrix A in the above example fails to have a Moore-Penrose inverse with respect to the usual matrix product in indefinite inner product spaces. It should also be pointed out that the existence of the Moore-Penrose inverse in Krein spaces was studied recently by Mary [14]. Let us point out finally that nonnegativity of the Moore-Penrose inverse as well as a generalization of Farkas' alternative in indefinite inner product spaces were also studied by Ramanathan and Sivakumar [17, 18].

The aim of this manuscript is to propose notions of matrix monotonicity in indefinite inner product spaces with respect to the indefinite matrix product with a view to generalize the existing notions and characterizations of generalized inverse nonnegativity to the setting of indefinite inner product spaces. A good source for the theory of generalized inverse nonnegativity in Euclidean spaces is the monograph by Berman and Plemmons [2]. The paper is organized as follows. We recall the basic definitions and facts in the next section. In particular, we recall the definition of an indefinite product of two matrices / vectors (Definition 2.1) and the adjoint with respect to this multiplication in Definition 2.2. The definitions of the range and the null space, denoted by $Ra(\cdot)$ and $Nu(\cdot)$, respectively, invertibility, the Moore-Penrose inverse, all with respect to this indefinite product, and its properties are given next in Definitions 2.3, 2.4, 2.5 and 2.6. The definitions of a cone, its dual (with respect to the Euclidean inner product) and nonnegativity of a matrix with respect to two cones with respect to the indefinite matrix product are given next. Section 3 contains the main results of this paper. We begin with inverse nonnegativity and monotonicity, denoted by \circ -monotonicity (with respect to the indefinite matrix product). One of the main result in this connection is Theorem 3.2. An interesting result in the theory of positive operators on ordered Banach spaces is that a doubly power bounded positive operator on a finite dimensional Banach lattice has a positive inverse [1]. A generalization of this to indefinite inner product spaces is presented next (Theorem 3.4). One of the most well studied notions in the theory of nonnegative matrices is that of splittings. Several notions of splittings of matrices were studied in connection with nonnegativity of various generalized inverses. One such notion is that of B-splittings [16], which was later on generalized to the setting of ordered Banach spaces by Weber [20] (Refer Theorem 3.8 in Section 3). This notion has an obvious generalization to indefinite inner product spaces (Definition 3.7 and this is taken up next in connection with inverse nonnegativity (Theorem 3.11). It turns out that a complete generalization of Weber's theorem to the indefinite setting does not hold. We then discuss \circ -rectangular monotonicity, a natural generalization of rectangular monotonicity in the Euclidean setting. We prove an analogue of Mangasarian's theorem, namely, Theorem 3.16 on the existence of a nonnegative left inverse when the matrix is rectangular monotone (Refer Theorem 1, [13]). The case of nonnegativity of the Moore-Penrose inverse $A^{[+]}$ is taken up next and three characterizations regarding the same are presented (Refer Theorems 3.19, 3.22 and 3.23). A key ingredient in the proof of Theorem 3.23 is that if A is nonnegative with respect to the indefinite matrix product, then $A^{[*]}$ is also nonnegative (as in the Euclidean case), when the cones are self-dual. We prove this result also (see Lemma 3.20). The case of \circ -weak monotonicity is also studied in connection with the existence of a $\{1\}$ -inverse that is nonnegative on the range space of A . A generalization of a structure theorem for weak monotone matrices similar to the Euclidean setting, namely, that a weak monotone matrix A with a nonnegative rank factorization satisfies the equation $XAY = I$ for some nonnegative matrices X and Y is also generalized. Wherever possible, examples are given to illustrate the results and validity of the assumptions made. The manuscript ends

with a few concluding remarks.

2. Notations, Definitions and Preliminaries

We first recall the notion of an indefinite multiplication of matrices. We refer the reader to [19], wherein various properties and also advantages of this product have been discussed in detail.

Definition 2.1. Let A and B be $m \times n$ and $n \times l$ complex matrices, respectively. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. An indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_n B$.

Note that there is only one value for the indefinite product of vectors / matrices. When $J_n = I_n$, the above product becomes the usual product of matrices.

Definition 2.2. Let A be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of A (relative to J_n, J_m) is defined by $A^{[*]} = J_n A^* J_m$.

When the dimensions are equal, the subscripts n, m will be dropped. $A^{[*]}$ satisfies the following identities : $[Ax, y] = [x, A^{[*]}y]$ and $[A \circ x, y] = [x, (I \circ A \circ I)^{[*]} \circ y]$.

Definition 2.3. Let A be an $m \times n$ complex matrix. Then the range space $Ra(A)$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$ and the null space $Nu(A)$ of A is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.

The null and range spaces with respect to the usual product will be denoted by $N(A)$ and $R(A)$, respectively. It follows at once that $Ra(A) = R(A)$ and $Nu(A^{[*]}) = N(A^*)$.

Definition 2.4. Let $A \in \mathbb{C}^{n \times n}$. A is said to be J -invertible if there exists $X \in \mathbb{C}^{n \times n}$ such that $A \circ X = X \circ A = J$.

It can be easily proved that A is J -invertible if and only if A is invertible and in this case the J -inverse is given by $A^{[-1]} = JA^{-1}J$. We now pass on to the notion of the Moore-Penrose inverse in indefinite inner product spaces.

Definition 2.5. For $A \in \mathbb{C}^{m \times n}$, a matrix $X \in \mathbb{C}^{n \times m}$ is called the Moore-Penrose inverse if it satisfies the following equations : $A \circ X \circ A = A, X \circ A \circ X = X, (A \circ X)^{[*]} = A \circ X, (X \circ A)^{[*]} = X \circ A$.

Such an X will be denoted by $A^{[+]}$. As was pointed out in the introduction, it can be shown that $A^{[+]}$ exists if and only if $rank(A) = rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A)$. The Moore-Penrose has the representation $A^{[+]} = J_n A^\dagger J_m$. We also have, $Ra(A \circ A^{[+]}) = Ra(A)$ and $Ra(A^{[+]} \circ A) = Ra(A^{[*]})$ (see for instance Lemma 2.1(v), [18].) One can similarly define the notion of the group inverse in indefinite inner product spaces.

Definition 2.6. For $A \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times n}$ is called the group inverse of A if it satisfies the equations : $A \circ X \circ A = A, X \circ A \circ X = X, A \circ X = X \circ A$.

As in the Euclidean setting, it can be proved that the group inverse exists if and only if $rank(A) = rank(A^{[2]})$ and is denoted by $A^{[\#]}$. In particular, if $A = A^{[*]}$, then $A^{[\#]}$ exists. However, an analogous formula for the group inverse similar to that of the Moore-Penrose inverse does not hold in the indefinite setting. Similarly, one can have a generalization of $\{1\}$ -inverses and left inverses in the above setting : An $n \times m$ matrix X is said to be a $\{1\}$ -inverse if it satisfies the equation $A \circ X \circ A = A$ and a left inverse if it satisfies the equation $X \circ A = J_n$. Throughout this manuscript, we shall work with matrices over the field of real numbers. A subset P of a vector space is called a cone if $\alpha P \subseteq P$ for all $\alpha \geq 0$ and $P + P = P$. P is said to be pointed if $P \cap -P = \{0\}$ and generating if $V = P - P$. For a cone P in a Hilbert space H , the dual cone of P , denoted by P^* , is defined by

$$P^* := \{x \in H : \langle x, y \rangle \geq 0 \forall y \in P\}.$$

A cone P is said to be acute if $P \subseteq P^*$, self-dual if $P^* = P$. The nonnegative orthant \mathbb{R}_+^n of the Euclidean space \mathbb{R}^n , the ice-cream cone $P_I := \{x \in \mathbb{R}^n : x_1 \geq 0, x_1^2 \geq (x_2^2 + \dots + x_n^2)\}$ and the P_0 cone defined by $P_0 := \{x \in \mathbb{R}^3 : x_1 \geq 0, x_3 \geq 0, x_2^2 \leq 2x_1x_3\}$ are examples of pointed, self-dual (closed) generating cones. The P_0 cone was introduced by Ben-Israel and Charnes in connection with interval linear programs. It is to be noted that the ice-cream cone satisfies $x^t J_n x \geq 0$ where, $J_n = \text{diag}(1, -1, \dots, -1)$, whereas the P_0 satisfies $x^t J_3 x$ for $J_3 = \text{diag}(1, -1, 1)$. Finally, given closed cones P_1 and P_2 in \mathbb{R}^n and \mathbb{R}^m , respectively, an $m \times n$ matrix A is said to be nonnegative with respect to the cones P_1 and P_2 with respect to the indefinite matrix product, if $A \circ P_1 \subseteq P_2$.

3. Main Results

We present the main results in this section. This section has four parts, all of them dealing with nonnegativity with respect to the indefinite matrix product as introduced in the previous section. We start with the existence of a nonnegative inverse. The following is the definition of \circ -monotonicity for a square matrix A .

Definition 3.1. A square matrix A is said to be \circ -monotone if $A \circ x \in P_2 \implies x \in P_1$, where P_1 and P_2 are closed convex cones in \mathbb{R}^n and \mathbb{R}^m , respectively.

Let us observe that the above definition is equivalent to saying that AJ is monotone. We now have the following simple characterization of inverse nonnegativity.

Theorem 3.2. Let P_1 and P_2 be closed convex cones in \mathbb{R}^n and \mathbb{R}^m , respectively, with P_1 being pointed and generating. Then, $A^{[-1]} \circ P_2 \subseteq P_1$ if and only if A is \circ -monotone.

Proof. As remarked above, if A is \circ -monotone, then AJ is monotone and hence A is invertible. Thus, $A^{[-1]}$ exists and is given by $A^{[-1]} = JA^{-1}J$. Note that we have used the fact that the cone P_1 is pointed in deducing the above. Now, monotonicity of AJ implies that $(AJ)^{-1}(P_2) \subseteq P_1$. Therefore, for any $x \in P_2, A^{[-1]} \circ x = JA^{-1}x \in P_1$, proving inverse nonnegativity of A in the indefinite setting. The converse can be proved in a similar manner. \square

Example 3.3. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By taking $P_1 = \{x : x_1 \geq 0\}$ and $P_2 = \mathbb{R}_+^2$, we see that AJ is monotone, but not invertible. Therefore, pointedness of the cone P_1 is crucial in the above example.

A bounded linear operator A between Banach spaces is said to be power bounded if $\sup_{n \in \mathbb{N}} \{\|A^n\|\} < \infty$.

An invertible bounded linear operator A with a bounded inverse is said to be doubly power bounded if $\sup_{n \in \mathbb{Z}} \{\|A^n\|\} < \infty$. A well known result in the theory of positive operators on Banach lattices is that a doubly power bounded positive operator on a finite dimensional Banach lattice has a positive inverse. We prove below a result similar to the above stated result, in the setting of indefinite inner product spaces, with respect to the indefinite matrix product. Let us observe that for an invertible A and any integer n , there are $n - 1$ occurrences of J and n occurrences of A in $A^{[n]}$. Therefore, $\|A^{[n]}\| \leq \|J\|^{n-1} \|A\|^n = \|A\|^n$. This means that if $\sup_{n \in \mathbb{Z}} \{\|A^n\|\} < \infty$, then $\sup_{n \in \mathbb{Z}} \{\|A^{[n]}\|\} < \infty$. We then have the following theorem.

Theorem 3.4. Let P_1 and P_2 be self-dual cones in \mathbb{R}^n and let A be an invertible matrix such that $A \circ P_1 \subseteq P_2$. Assume that $\sup_{m \in \mathbb{Z}} \{\|A^m\|\} < \infty$. Then, $A^{[-1]} \circ P_2 \subseteq P_1$.

Proof. It follows from $A \circ P_1 \subseteq P_2$ that $JA^*(P_2) \subseteq P_1$ (recall that if P_1 and P_2 are self-dual cones in \mathbb{R}^n and \mathbb{R}^m , respectively, and if $A(P_1) \subseteq P_2$, then $A^*(P_2) \subseteq P_1$). Notice that $\|(JA^*)^m\| \leq \|J\|^m \|A\|^m = \|A\|^m$. Therefore, $\sup_{m \in \mathbb{Z}} \{\|(JA^*)^m\|\} < \infty$. Thus, JA^* is doubly power bounded and hence $(JA^*)^{-1}(P_1) \subseteq P_2$ which, in turn is equivalent to saying that $(JA^{-1})(P_2) \subseteq P_1$. It is now easy to verify that $A^{[-1]} \circ P_2 \subseteq P_1$. \square

The following example shows that the assumption $\sup_{m \in \mathbb{Z}} \{\|A\|^m\} < \infty$ in the above theorem cannot be dispensed with.

Example 3.5. Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Take $P_1 = P_2 = \mathbb{R}_+^2$. It is obvious that for any $x \in P_1$, $AJx \in P_2$, thereby proving that $A \circ P_1 \subseteq P_2$. Noting that $\|A\| \geq \sup_{i,j} |a_{ij}| = 1$, we see that $\sup_{m \in \mathbb{Z}} \{\|A\|^m\} \not< \infty$. Also, note that $JA^* = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ which implies that $\|JA^*\| \geq 1$. Therefore, JA^* cannot be doubly power bounded. It is easy to see that $JA^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, and hence $JA^{-1}(P_2) \not\subseteq P_1$, proving that $A^{[-1]} \circ P_2 \not\subseteq P_1$.

The following definition is well known in the literature (refer Definition 3.1, [20]).

Definition 3.6. For a square matrix A , a decomposition $A = U - V$ is called positive, if $U \geq 0$, $V \geq 0$ (in this case, the operator A is regular), positive regular, if it is positive, U^{-1} exists and $U^{-1} \geq 0$, and a B-decomposition, if it is positive and satisfies the conditions :

- (a) U^{-1} exists
- (b) $VU^{-1} \geq 0$
- (c) $Ax \geq 0, Ux \geq 0 \implies x \geq 0$.

A natural generalization of the above definition, with respect to the indefinite matrix product is given below. In what follows, we shall assume that P is a pointed generating cone in \mathbb{R}^n (and hence has non-empty interior).

Definition 3.7. Let P be a pointed generating cone in \mathbb{R}^n . For a square matrix A , a decomposition $A = U - V$ is called \circ -positive, if $U \circ P \subseteq P$, $V \circ P \subseteq P$, \circ -positive regular, if it is \circ -positive, $U^{[-1]}$ exists and $U^{[-1]} \circ P \subseteq P$, and a \circ -B-decomposition, if it is \circ -positive and satisfies the conditions :

- (a) $U^{[-1]}$ exists
- (b) $V \circ U^{[-1]} \circ P \subseteq P$
- (c) $A \circ x \in P, U \circ x \in P \implies x \in P$.

The following result on positive invertibility is due to Weber.

Theorem 3.8. (Theorem 3.4, [20]) Let $(X, X_+, \|\cdot\|)$ be an ordered Banach space with a normal cone X_+ that satisfies the condition $\text{int}(X_+) \neq \emptyset$. Let $A : X \rightarrow X$ be a continuous linear operator. Consider the conditions

- (i) A is positively invertible,
- (ii) $X_+ \subseteq A(X_+)$,
- (iii) there exists $x_0 \in X_+$ such that $Ax_0 \in \text{int}(X_+)$,
- (iv) $\rho(VU^{-1}) < 1$ (if $A = U - V$ is a decomposition such that U^{-1} exists).

Then, (i) \implies (ii) \implies (iii). If A possesses a B-decomposition, then (i) - (iv) are equivalent.

Before proving a generalization of the above theorem in indefinite inner product spaces, we state two results in the theory of positive operators on Banach lattices.

Theorem 3.9. (Theorem 25.4, [10]) Let $(X, X_+, \|\cdot\|)$ be an ordered Banach space with a normal cone having non-empty interior. Let C and B be two continuous linear operators on X such that $(B - C)(X_+) \subseteq X_+$ and that B is positively invertible. Then, C is positively invertible if and only if $C(X_+) \cap \text{int}(X_+)$ is non-empty.

Theorem 3.10. (Theorem 2.3 (i), [20]) Let $(X, X_+, \|\cdot\|)$ be an ordered Banach space and C a bounded linear operator on X such that $C(X_+) \subseteq X_+$. If $\rho(C) < 1$, then $I - C$ is positively invertible. The converse is true if the cone X_+ is normal and reproducing.

We now present a partial generalization of Theorem 3.8 to indefinite inner product spaces.

Theorem 3.11. Let P be a pointed cone in \mathbb{R}^n that makes it into a Riesz space (so that the cone P is closed, normal and generating). Let A be an $n \times n$ matrix. Consider the following statements :

- (1) $A^{[-1]} \circ P \subseteq P$
- (2) $P \subseteq A \circ P$
- (3) There exists $x_0 \in P$ such that $A \circ x_0 \in \text{int}(P)$
- (4) $\rho(V \circ U^{[-1]}) < 1$ (if $A = U - V$ is a decomposition such that $U^{[-1]}$ exists).

Then, (1) \implies (2) \implies (3). If $A = U - V$ is a \circ -B-decomposition such that $JVU^{-1} = VU^{-1}J$, then (3) \implies (4).

Proof. (1) \implies (2) : From $A^{[-1]} \circ P \subseteq P$, we see that $A \circ A^{[-1]} \circ P \subseteq A \circ P$, which is equivalent to statement (2).
 (2) \implies (3) : If $u \in \text{int}(P)$, then $u \in A \circ P$ and so $u = A \circ x_0$ for some $x_0 \in P$. Thus, (3) holds.

Suppose $A = U - V$ is a \circ -B-decomposition of A . Then, $U \circ P \subseteq P, V \circ P \subseteq P, U^{[-1]}$ exists, $V \circ U^{[-1]} \circ P \subseteq P$ and $A \circ x \in P, U \circ x \in P \implies x \in P$. Taking $C := V \circ U^{[-1]}$, we see that $A = (J - C) \circ U$. Also note that $C \circ P \subseteq P \iff VU^{-1}(P) \subseteq P$.

(3) \implies (4) : By the above remark, $C \circ P \subseteq P$ if and only if $VU^{-1}(P) \subseteq P$. Since $A = (J - C) \circ U$, we see that $A \circ x_0 \in \text{int}(P)$ for some $x_0 \in P$ is equivalent to $(J - C) \circ y_0 \in \text{int}(P)$ for some $y_0 \in P$. (Here we have used the fact that $A = U - V$ is a \circ -B-decomposition, and hence $y_0 := U \circ x_0 \in P$). After simplification, we see that $A \circ x_0 \in \text{int}(P)$ for some $x_0 \in P$ if and only if there exists $y_0 \in P$ such that $(I - VU^{-1})y_0 \in \text{int}(P)$. Also, nonnegativity of VU^{-1} is equivalent to nonnegativity of $I - (I - VU^{-1})$. It now follows from Theorem 3.9 that $I - VU^{-1}$ is positively invertible; moreover, from Theorem 3.10, we see that $\rho(VU^{-1}) < 1$. Finally, $\rho(V \circ U^{[-1]}) = \rho(VU^{-1}J) \leq \rho(VU^{-1})\rho(J) = \rho(VU^{-1}) < 1$. \square

The following is an example to show that (3) \implies (4) cannot be deduced without the commutativity of J and VU^{-1} .

Example 3.12. Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1/4 & -1 \end{pmatrix}$. Let P be the nonnegative orthant of \mathbb{R}^2 , which is a pointed, generating self-dual cone with non-empty interior. By taking $x = (1, 16)^t \in P$, we see that $A \circ x \in \text{int}(P)$. Consider the splitting $A = U - V$, where $U = \begin{pmatrix} 0 & 1 \\ 1/4 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to see that this is a \circ -B-splitting of A . Also, it can be easily verified that $VU^{-1}J \neq JVU^{-1}$. However, $\rho(V \circ U^{[-1]}) = 1$.

As the following example shows, the implication (4) \implies (1) need not hold good even if A has a \circ -B-decomposition with $\rho(V \circ U^{[-1]}) < 1$.

Example 3.13. Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the matrix $A = \begin{pmatrix} -1/2 & 0 \\ 1 & -1/2 \end{pmatrix}$. Consider the decomposition $U - V$ of A , where $U = J$ and $V = \begin{pmatrix} 1/2 & 1 \\ 0 & 1/2 \end{pmatrix}$. Observe that, $U \circ \mathbb{R}_+^2 \subseteq \mathbb{R}_+^2, V \circ \mathbb{R}_+^2 \subseteq \mathbb{R}_+^2$ and $VU^{-1}(\mathbb{R}_+^2) \subseteq \mathbb{R}_+^2$. Moreover, $A \circ x \in \mathbb{R}_+^2, U \circ x \in \mathbb{R}_+^2 \implies x = 0 \in \mathbb{R}_+^2$. Thus, A has a \circ -B-decomposition. Now, $JA^{-1} = 4 \begin{pmatrix} -1 & -1/2 \\ -1/2 & 0 \end{pmatrix}$, and hence $A^{[-1]} \circ \mathbb{R}_+^2 \not\subseteq \mathbb{R}_+^2$. Finally, observe that $\rho(VU^{-1}J) = \rho(V) = 1/2 < 1$.

We now pass on to the case of a nonnegative left inverse. The notion of monotonicity can be generalized to rectangular matrices with a view to investigate the existence of a one sided inverse. This notion, called rectangular monotonicity, is generalized to the indefinite setting below.

Definition 3.14. An $m \times n$ matrix is said to be \circ -rectangular monotone if $A \circ x \in P_2 \implies x \in P_1$, where P_1 and P_2 are closed convex cones in \mathbb{R}^n and \mathbb{R}^m , respectively.

A well known result of Mangasarian (Refer Theorem 1, [13]) is that, over the nonnegative orthants if an $m \times n$ matrix A is rectangular monotone, then it has a nonnegative left inverse. In an attempt to generalize this to more general cones (possibly) in infinite dimensional real Hilbert spaces, Kulkarni and Sivakumar proved the following :

Theorem 3.15. (Theorem 2.26, [11]) Let H_1 and H_2 be Hilbert spaces with cones P_1 and P_2 , respectively, with P_1 generating and P_2 self-dual. Suppose that there exists an orthonormal basis $\{u^\alpha : \alpha \in I\}$, I an index set, of H_1 with $u^\alpha \in P_1$ for all $\alpha \in I$. Let A be a bounded operator with closed range such that $N(A^*) + P_2$ is closed. Then, A is rectangular monotone if and only if there exists a bounded operator Y such that $YA = I$ with $Y(P_2) \subseteq P_1$.

We now have the following generalization of the above theorem in the setting of indefinite inner product spaces.

Theorem 3.16. Let P_1 and P_2 be closed cones in \mathbb{R}^n and \mathbb{R}^m , respectively, such that P_1 is generating, P_2 is self-dual and that $Nu(A^{[*]}) + P_2$ is closed. Further, assume that there exists an orthonormal basis $\{u^1, \dots, u^n\}$ for \mathbb{R}^n such that $u^i \in P_1$, for $i = 1, \dots, n$. Then, A is \circ -rectangular monotone if and only if there exists an $n \times m$ matrix X such that $X \circ A = J_n$ and such that $X \circ P_2 \subseteq P_1$.

Proof. If A is \circ -rectangular monotone, then AJ_n is rectangular monotone. From Theorem 3.15, we infer that AJ_n has a nonnegative left inverse, say, Y ; that is, $YAJ_n = I_n$ and $Y(P_2) \subseteq P_1$. Note that $Nu(A^{[*]}) + P_2 = N(A^*) + P_2$. Let $X := YJ_m$. Then, $X \circ A = YJ_m J_m A = J_n$. Moreover, for any $x \in P_2$, $X \circ x = YJ_m J_m x = Yx \in P_1$. Thus, there exists an X such that $X \circ A = J_n$ with $X \circ P_2 \subseteq P_1$. The converse is obvious. \square

Remark 3.17. It is clear that if A is \circ -rectangular monotone, then A has a left inverse, and hence $\text{rank}(A) = n$. Remark 2.19 in [11], following Theorem 3.15 shows that the assumption on the closedness of $Nu(A^{[*]}) + P_2$ cannot be dispensed with.

We now take up the case of \circ -semimonotonicity and its interplay with nonnegativity of $A^{[+]}$ in the indefinite setting. An $m \times n$ matrix is said to be semimonotone if $Ax \in P_2 + N(A^*), x \in R(A^*) \implies x \in P_1$. It can then be proved that the following three statements are equivalent :

(1) $A^+(P_2) \subseteq P_1$, (2) A is semimonotone and (3) $Ax \in AA^+(P_2), x \in R(A^*) \implies x \in P_1$ (Refer Theorem 3.2, [11]). We now present a generalization of the above to the indefinite setting.

Definition 3.18. An $m \times n$ matrix is said to be \circ -semimonotone if $A \circ x \in P_2 + Nu(A^{[*]}), x \in Ra(A^{[*]}) \implies x \in P_1$.

It is easy to see that the above definition is equivalent to semimonotonicity of AJ_n . We therefore have the following theorem.

Theorem 3.19. Let P_1 and P_2 be closed cones in \mathbb{R}^n and \mathbb{R}^m , respectively. Then, the following two statements are equivalent :

- (1) A is \circ -semimonotone.
- (2) $A^{[+]} \circ P_2 \subseteq P_1$.

Proof. (1) \implies (2) : If A is \circ -semimonotone, then AJ_n is semimonotone and hence $(AJ_n)^+(P_2) \subseteq P_1$ which, is the same as saying $J_n A^+(P_2) \subseteq P_1$. Therefore, for any $x \in P_2$, $A^{[+]} \circ x = J_n A^+ J_m J_m x = J_n A^+ x \in P_1$, thereby proving nonnegativity of $A^{[+]}$ with respect to the indefinite matrix product.

(2) \implies (1) : Suppose $A^{[+]} \circ P_2 \subseteq P_1$. Now, $A^{[+]} \circ P_2 \subseteq P_1$ is equivalent to AJ_n being semimonotone. This, in turn, is equivalent to $AJ_n x \in P_2 + N(A^*), x \in R((AJ_n)^*) \implies x \in P_1$. Noting that $R(J_n A^*) = Ra(A^{[*]})$ and $N(A^*) = Nu(A^{[*]})$, we see that A is \circ -semimonotone. \square

We now present two more characterizations of nonnegativity of $A^{[+]}$ with respect to the indefinite product. It is a well known result that if P_1 and P_2 are self-dual cones in Hilbert spaces H_1 and H_2 , respectively, and if A is a bounded operator such that $A(P_1) \subseteq P_2$, then $A^*(P_2) \subseteq P_1$. We now generalize this to indefinite inner product spaces and with respect to the indefinite matrix product.

Lemma 3.20. Let P_1 and P_2 be self-dual cones in \mathbb{R}^n and \mathbb{R}^m , respectively. Let A be an $m \times n$ matrix such that $A \circ P_1 \subseteq P_2$. Then, $A^{[*]} \circ P_2 \subseteq P_1$.

Proof. Let $y \in A^{[*]} \circ P_2$. Then $y = A^{[*]} \circ x, x \in P_2$. Therefore, for any $u \in P_1, \langle u, y \rangle = \langle u, A^{[*]} \circ x \rangle$
 $= \langle u, J_n A^* J_m J_m x \rangle$
 $= \langle u, J_n A^* x \rangle$
 $= \langle A J_n u, x \rangle$
 $= \langle v, x \rangle \geq 0$, where $v = A \circ u \in P_2$. (Here we have used the assumption that $A \circ P_1 \subseteq P_2$ and P_2 is self-dual).
 Therefore, $y \in P_1^* = P_1$, as P_1 is self-dual. Thus, $A^{[*]} \circ P_2 \subseteq P_1$. \square

The following example shows that self-duality of the underlying cones is essential in the above lemma.

Example 3.21. Consider \mathbb{R}^2 with the cone $P := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$. The dual of this cone is $P^* := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$. Hence, the cone P is not self-dual. Now, let $P_1 = P_2 = P$ and let A and J be the matrices $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively. Then, $AJ = A$. For any $x \in P_1, A \circ x = (x_1, x_1)^t \in P_2$ and so $A \circ P_1 \subseteq P_2$. On the other hand, $A^{[*]}$, which is given by $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is not nonnegative, since for $x = (1, -2)^t \in P_2, A^{[*]} \circ x = (-1, 0)^t \notin P_1$. Thus, $A^{[*]} \circ P_2 \not\subseteq P_1$.

We now present two characterizations of nonnegativity of $A^{[t]}$ with respect to the indefinite product. The first of these is a generalization of Theorem 3.4 in [7] and the latter is a generalization of Theorem 3.6 in the same.

Theorem 3.22. Let P_1 and P_2 be closed cones in \mathbb{R}^n and \mathbb{R}^m , respectively, and let A be an $m \times n$ matrix such that $A \circ P_1 \subseteq P_2$. If A satisfies the inclusion relation $P_2 \subseteq A \circ P_1 + Nu(A^{[*]})$ and if $A^{[t]} \circ A \circ P_1 \subseteq P_1$, then $A^{[t]} \circ P_2 \subseteq P_1$. The converse is also true.

Proof. Let $x \in P_2$ and $y = A^{[t]} \circ x$. Then, $x = A \circ u + v, u \in P_1, v \in Nu(A^{[*]})$. Notice that $v \in Nu(A^{[*]}) \implies v \in Nu(A^{[t]})$. Therefore, $y = A^{[t]} \circ x = A^{[t]} \circ A \circ u \in P_1$, by assumption.

Conversely, suppose $A^{[t]} \circ P_2 \subseteq P_1$. Since $A \circ P_1 \subseteq P_2$, it follows that $A^{[t]} \circ A \circ P_1 \subseteq P_1$. That $P_2 \subseteq A \circ P_1 + Nu(A^{[*]})$ follows from the lemma on linear equations, namely : For an $m \times n$ matrix A and a $b \in \mathbb{R}^m$, the linear equation $A \circ x = b$ is solvable if and only if $b \in Ra(A)$, in which case the general solution is given by $x = A^{[t]} \circ b + v, v \in Nu(A)$ (Refer Lemma 2.2, [18]). \square

We now present the second characterization of nonnegativity of $A^{[t]}$.

Theorem 3.23. Let P_1 and P_2 be self-dual cones in \mathbb{R}^n and \mathbb{R}^m , respectively. Let A be an $m \times n$ matrix such that $A \circ P_1 \subseteq P_2$. If $(A \circ A^{[*]})^{[t]} \circ P_2 \subseteq P_2$, then $A^{[t]} \circ P_2 \subseteq P_1$. The converse is also true.

Proof. Let $x \in P_1$. Since $A \circ P_1 \subseteq P_2, A^{[t]} \circ A \circ x \in A^{[t]} \circ P_2$. But $(J_n - A^{[t]} \circ A) \circ x \in Nu(A)$. Then, $x = A^{[t]} \circ A \circ x + (J_n - A^{[t]} \circ A) \circ x \in A^{[t]} \circ P_2 + Nu(A)$. Thus, $P_1 \subseteq A^{[t]} \circ P_2 + Nu(A)$. Therefore for $x \in P_1$, we have that $(A^{[t]})^{[*]} \circ x = (A^{[*]})^{[t]} \circ x = (A^{[*]})^{[t]} \circ (A^{[t]} \circ y + z)$, where $y \in P_2, z \in Nu(A)$. Now, $(A^{[*]})^{[t]} \circ A^{[t]} \circ y = (AA^*)^{\dagger} y$. On the other hand, for $y \in P_2, (A \circ A^{[*]})^{[t]} \circ y$ is also equal to $(AA^*)^{\dagger} y$. Note also that, $z \in Nu(A)$ if and only if $J_n z \in N((A^*)^{\dagger})$. Therefore, $(A^{[*]})^{[t]} \circ z = (A^*)^{\dagger} J_n z = 0$. Therefore, for any $x \in P_1$, we have $(A^{[t]})^{[*]} \circ x = (A^{[*]})^{[t]} \circ x = (AA^*)^{\dagger} y = (A \circ A^{[*]})^{[t]} \circ y \in P_2$. It now follows from Lemma 3.20 that $A^{[t]} \circ P_2 \subseteq P_1$, as the cones are self-dual.

Conversely, assume that $A^{[t]} \circ P_2 \subseteq P_1$. Then, $(A \circ A^{[*]})^{[t]} \circ P_2 = (A^{[*]})^{[t]} \circ A^{[t]} \circ P_2 = (A^{[t]})^{[*]} \circ A^{[t]} \circ P_2 \subseteq P_2$, as the cones are self-dual. \square

The following example illustrates Theorems 3.22 and 3.23.

Example 3.24. Let $P_1 = \mathbb{R}_+^2$ and $P_2 = P_0$ (introduced earlier). Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$, $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Clearly, $A \circ P_1 \subseteq P_2$. Note that $N(A^*) = Nu(A^{[*]}) = \text{span}\{e^2\}$. Therefore, any $x = (x_1, x_2, x_3)^t \in P_2$ can be written as $x = (x_1, x_2, x_3) = x_2(0, 1, 0)^t + (x_1, 0, x_3)^t$, thereby proving that $P_2 \subseteq A \circ P_1 + Nu(A^{[*]})$. We also compute $A^{[t]} \circ A$ as $J_2 A^t A$ and so, for any $x \in P_1$, we have $A^{[t]} \circ A \circ x = J_2 A^t A J_2 x = x \in P_1$. Thus, $A^{[t]} \circ P_2 \subseteq P_1$, by Theorem 3.22. Notice also that, $A \circ A^{[*]}$ is the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. One can now easily verify that for any $x \in P_2$, $(A \circ A^{[*]})^{[t]} \circ x \in P_2$. Thus, by Theorem 3.23, we see that $A^{[t]} \circ P_2 \subseteq P_1$.

We finally present a generalization of weak monotonicity and the existence of a nonnegative $\{1\}$ -inverse to the indefinite setting. Recall that an $m \times n$ matrix A is said to be weak monotone if $Ax \in P_2 \implies x \in P_1 + N(A)$. Weak monotonicity is the weakest among all notions of matrix monotonicity and guarantees nonnegative solvability of a consistent system $Ax = b, b \in P_2$. An interesting result concerning weak monotonicity of a matrix A and its connection to nonnegative rank factorization and the existence of a $\{1\}$ -inverse that is nonnegative on the range space of A was brought out by the author and Sivakumar in [5]. We now propose a generalization of weak monotonicity with respect to the indefinite matrix product and investigate similar questions as discussed above.

Definition 3.25. For cones P_1 and P_2 in \mathbb{R}^n and \mathbb{R}^m , respectively, an $m \times n$ matrix A is said to be \circ -weak monotone if $A \circ x \in P_2 \implies x \in P_1 + Nu(A)$

Observe that if A is \circ -weak monotone, then AJ_n is weak monotone. \circ -weak monotonicity is weaker than \circ -monotonicity, as the following matrix illustrates.

$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Let the cones $P_1 = P_2 = P$ be given by $P := \{x = (x_1, x_2, x_3)^t : x_1 \geq 0, x_3 \geq 0, x_2^2 \leq 2x_1x_3\}$. This cone is self-dual. In this case, $AJ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is weak monotone with respect to the cone P , and hence A is \circ -weak monotone. However, A is not \circ -monotone, as it is not invertible. Note that $J(P) = P$.

Suppose P_1 and P_2 are self-dual cones such that $N(A) + P_1$ and $N(A^*) + P_2$ are closed and that $J_n(P_1) = P_1$ (this is equivalent to J_n being monotone). Then, $J_n A^* = (AJ_n)^*$ is weak monotone (Refer Theorem 3.12, [4]). Since J_n is monotone, we have that $A^* = J_n(J_n A^*)$ is weak monotone (Refer Theorem 3.20, [4].) Again, by Theorem 3.12 of [4], we see that A is weak monotone. It should be pointed out that nonnegativity of J_n is crucial in this, as the following example illustrates.

Example 3.26. Let $P_1 = P_2 = P = \{x \in \mathbb{R}^2 : x_2 \geq 0, x_2^2 \geq x_1^2\}$ (the ice-cream cone). Let A and J be the matrices $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively. P is a self-dual generating, polyhedral cone and so, $N(A) + P_1$ and $N(A^*) + P_2$ are closed. Now, $N(AJ) = \text{span}\{(1, 1)^t\}$. If $AJx \in P_2$, then $x_2 - x_1 \geq 0$ and therefore, $x = (x_1, x_2)^t = x_1(1, 1)^t + (0, x_2 - x_1)^t \in N(AJ) + P_1$. Thus, AJ and hence JA^* are weak monotone. However, $A^* = J(JA^*) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is not weak monotone. Note that $N(A^*) = \text{span}\{(1, 1)^t\}$. By taking $x = (1, 0)^t$, we see that $A^*x = (1, 1)^t \in P_1$. However, there is no element $y = (y_1, y_2)^t$ in P_1 such that $x = \alpha(1, 1)^t + (y_1, y_2)^t$. Note that $J(P_1) \not\subseteq P_1$.

We now have the following theorem. Before we proceed, we introduce a notation. Let us say that a Hilbert space H equipped with a closed cone P has property \mathcal{P} if there exists an orthonormal basis $\{u^\alpha : \alpha \in I\}$, I an index set, for H such that $u^\alpha \in P$ for all $\alpha \in I$.

We now have the following theorem. Before we proceed, we introduce a notation. Let us say that a Hilbert space H equipped with a closed cone P has property \mathcal{P} if there exists an orthonormal basis $\{u^\alpha : \alpha \in I\}$, I an index set, for H such that $u^\alpha \in P$ for all $\alpha \in I$.

Theorem 3.27. Let \mathbb{R}^n and \mathbb{R}^m have property \mathcal{P} with respect to self-dual cones P_1 and P_2 , respectively. Assume that $N(A) + P_1$ and $N(A^*) + P_2$ are closed, that $J_n(P_1) = P_1$ and that P_2 is generating. If there exists an $n \times m$ matrix X such that $A \circ X \circ A = A$ with $X \circ R(A) \cap P_2 \subseteq P_1$, then, A is \circ -weak monotone. Conversely, if A is \circ -weak monotone with a rank factorization $A = B \circ C$ such that $B \circ P_3 \subseteq P_2, C \circ P_1 \subseteq P_3$ for some self-dual generating cone P_3 in \mathbb{R}^r . Then, there exists an $n \times m$ matrix X such that $A \circ X \circ A = A$ with $X \circ R(A) \cap P_2 \subseteq P_1$.

Proof. From $X \circ R(A) \cap P_2 \subseteq P_1$ and monotonicity of J_n we infer that for any $x \in R(A) \cap P_2, J_n X J_m x \in P_1$. Therefore, by letting $Y := J_n X J_m$ and using the fact that $A \circ X \circ A = A$, we see that Y is a $\{1\}$ -inverse that is nonnegative with respect to the cones $R(A) \cap P_2$ and P_1 . Thus, A is weak monotone. Since P_1 is self-dual and $N(A^*) + P_2$ is closed, we see that A^* is weak monotone. Therefore, $J_n A^* = (A J_n)^*$ is weak monotone, as J_n is monotone. Again, the assumption on self-duality of P_2 and closedness of $N(A) + P_1$ guarantees weak monotonicity of $A J_n$. Thus, A is \circ -weak monotone.

Conversely, suppose A is \circ -weak monotone with a rank factorization $A = B \circ C$. This means, $A J_n$ is weak monotone. The assumptions now guarantee that A is weak monotone. Thus A is a weak monotone matrix with a rank factorization $A = B J_r C$. From $B \circ P_3 \subseteq P_2$, we see that $B J_r(P_3) \subseteq P_2$. Since $C \circ P_1 \subseteq P_3$ and the cones are self-dual, we see from Lemma 3.20 that $C^{[s]} \circ P_3 \subseteq P_1$, which implies that $J_n C^*(P_3) \subseteq P_1$. From this it follows that $C^*(P_3) \subseteq P_1$ and hence that $C(P_1) \subseteq P_3$ (as the cones are self-dual). Thus, A has a nonnegative rank factorization, namely, $A = B J_r C$. Therefore, by Theorem 3.16 of [4], there exists an $r \times m$ matrix Y , an $n \times r$ matrix Z such that $Y B J_r = I_r$ and $C Z = I_r$. Moreover, $Y(P_2) \subseteq P_3$ and $Z(P_3) \subseteq P_1$. By setting $X := J_n Z Y J_m$, we see that $A \circ X \circ A = A$. Then, for any $x \in R(A) \cap P_2, X \circ x = J_n Z Y x \in P_1$, as $Y(P_2) \subseteq P_3, Z(P_3) \subseteq P_1$ and $J_n(P_1) = P_1$. \square

Note that $X \circ (R(A) \cap P_2) \subseteq P_1$ is equivalent to $X \circ Ra(A) + P_2 \subseteq P_1$. A matrix A having a $\{1\}$ -inverse X that is nonnegative on the range of A is said to be generalized range monotone. This notion was introduced in by the author and Sivakumar (Refer Theorem 3.22, [4]). Thus, the above theorem is a generalization of Theorem 3.22 in [4] to the setting of indefinite inner product spaces with respect to the indefinite matrix product. The above proof depends on the assumption that $A = B \circ C$ with $B \circ P_3 \subseteq P_2, C \circ P_1 \subseteq P_3$ and $J_n(P_1) = P_1$. That these are essential follows from the following example. Let $P_1 = P_2 = \mathbb{R}_+^4$ and $P_3 = \mathbb{R}_+^3$. Consider the

4×4 matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Let $J_3 = I_3$ and $J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. The matrix A is weak monotone,

but not \circ -weak monotone. Note that $J_4(\mathbb{R}_+^4) \not\subseteq \mathbb{R}_+^4$. In this case, $A J_4$ is the matrix $A J_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$,

and so $N(A J_4)$ is spanned by the vector $(-1, 1, 1, -1)^t$. If $x = (-1, 1, 2, -2)^t$, then $A \circ x \in \mathbb{R}_+^4$. However, there is no u, v such that $x = u + v$, with $u \in \mathbb{R}_+^4$ and $v \in N(A J_4)$. Also, it is known that A has no nonnegative rank factorization over the orthants. This matrix A , although weak monotone, has no $\{1\}$ -inverse X that is nonnegative on the range space of A (Refer Remarks 3.2, [5]).

It was proved by the author and Sivakumar that if A is weak monotone with a nonnegative rank factorization $A = BC$, then under some conditions on the underlying cones, there exist nonnegative operators X and Y such that $XAY = I$ (Refer Theorem 3.16, [4]). The following is a generalization of the same to indefinite inner product spaces.

Theorem 3.28. Let \mathbb{R}^n and \mathbb{R}^m have property \mathcal{P} with respect to self-dual cones P_1 and P_2 , respectively. Assume that $N(A) + P_1$ and $N(A^*) + P_2$ are closed and that P_2 is generating. If A is \circ -weak monotone with a rank factorization $A = B \circ C$ such that $B \circ P_3 \subseteq P_2$ and $C \circ P_1 \subseteq P_3$ for some self-dual generating cone P_3 in \mathbb{R}^r , r being the rank of A . Assume further that $J_n(P_1) = P_1$ and $J_r(P_3) = P_3$. Then, there exist matrices \tilde{X} and \tilde{Y} of orders $r \times m$ and $n \times r$, respectively, such that $\tilde{X} \circ A \circ \tilde{Y} = J_r$, with $\tilde{X} \circ P_2 \subseteq P_3$ and $\tilde{Y} \circ P_3 \subseteq P_1$, where J_r is a weight in $\mathbb{R}^{r \times r}$.

Proof. Suppose A is \circ -weak monotone. The assumptions on the closedness of $N(A) + P_1$ and $N(A^*) + P_2$ implies weak monotonicity of A , as $J_n(P_1) = P_1$. Let X and Y be choices of left and right inverses of B and

C respectively, such that $X(P_2) \subseteq P_3$ and $Y(P_3) \subseteq P_1$. Now, define $\tilde{X} := J_r X J_m$ and $\tilde{Y} := J_n Y J_r$. Then, it can be easily seen that $\tilde{X} \circ A \circ \tilde{Y} = J_r$. Moreover, both $\tilde{X} \circ P_2 \subseteq P_3$ and $\tilde{Y} \circ P_3 \subseteq P_1$ hold, as $J_n(P_1) = P_1$ and $J_r(P_3) = P_3$. \square

Let us point out that monotonicity of J_n and J_r are only sufficient conditions in deriving nonnegativity of \tilde{X} and \tilde{Y} , respectively, in the above proof. As the following example shows, there are matrices that are \circ -weak monotone with a nonnegative rank factorization with nonnegative \tilde{X} and \tilde{Y} such that $\tilde{X} \circ A \circ \tilde{Y} = J_r$, but neither J_r nor J_n are monotone.

Example 3.29. Consider the cones P_1, P_2 and the matrices A, J as in Example 3.26. Let $J_1 = (-1)$ and $P_3 = \mathbb{R}_+$. Then, A is \circ -weak monotone with a factorization $A = B J_1 C$ with $B \circ P_3 \subseteq P_2$ and $C \circ P_2 \subseteq P_3$, where $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $C = (-1 - 1)$. All the assumptions of the above theorem are satisfied, except $J(P_1) = P_1$ and $J_1(P_3) = P_3$. It is easy to verify that there is a choice of \tilde{X} and a \tilde{Y} such that $\tilde{X} \circ A \circ \tilde{Y} = J_1$ with $\tilde{X} \circ P_2 \subseteq P_3$ and $\tilde{Y} \circ P_3 \subseteq P_1$.

Note, however, that if $J_1 = (1)$, then in the above example $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $C = (-1 - 1)$, both of which fail to be nonnegative.

It is well known and also easy to prove that if a matrix A has a nonnegative $\{1\}$ -inverse, then A is weak monotone. The converse holds good if A is weak monotone and has a nonnegative rank factorization (see Corollary 3.17, [4]). The matrix $\tilde{Z} := \tilde{Y} \circ \tilde{X}$, where \tilde{X} and \tilde{Y} are as in the above theorem, is a nonnegative $\{1,2\}$ -inverse of A . An interesting characterization of nonnegative weak monotone matrices due to Jeter and Pye is that a nonnegative weak monotone matrix of rank r has a nonnegative rank factorization if and only if it has an $r \times r$ monomial submatrix (Refer Corollary 1, [8]). A generalization of this result can also be obtained and we skip the details.

4. Concluding Remarks

We wind up with a few remarks in this section.

1. Nonnegativity of the group inverse (when it exists) can also be investigated in the indefinite setting. Recall that for a square matrix A , the group inverse (denoted by $A^{\#}$) exists if and only if $Ra(A) = Ra(A^{[2]})$ (as in the Euclidean setting). We then have the following analogue of Theorem 3.10 in [7]. We skip the proof.

Theorem 4.1. Let A be square matrix such that $P \subseteq A \circ P + Nu(A)$, where P is a self-dual generating cone in \mathbb{R}^n . Then, the following two statements are equivalent :

- (1) $A \circ x \in P + Nu(A), x \in Ra(A) \implies x \in P$.
- (2) $A^{\#}$ exists and $A^{\#} \circ P \subseteq P$.

If $A^{\#} \circ A \circ P \subseteq P$, an additional equivalent condition is : $A^{\#} \circ P \subseteq P + Nu(A)$.

2. Suppose P_1 and P_2 are self-dual cones in \mathbb{R}^n and \mathbb{R}^m , respectively. Let A and X be matrices of orders $m \times n$ and $n \times m$, respectively, such that $A \circ X \circ A = A, X \circ A \circ X = X, A \circ P_1 \subseteq P_2, X \circ P_2 \subseteq P_1$. From the first two equations, we infer that the matrix $J_n X J_m$ is a reflexive generalized inverse of A . Suppose that $J_n(P_1) = P_1$. Then, from $A \circ P_1 \subseteq P_2$, self-duality of the cones and monotonicity of J_n , we see that $A(P_1) \subseteq P_2$. Again, monotonicity of J_n and $X \circ P_2 \subseteq P_1$ imply that $J_n X J_m(P_2) \subseteq P_1$. Thus, A is a nonnegative matrix having a nonnegative reflexive generalized inverse $J_n X J_m$ (with respect to self-dual cones P_1, P_2 and P_2, P_1 , respectively). Therefore by Theorem 3.11 of [6], we see that the cones $A(P_1)$ and $J_n X J_m(P_2)$ are self-dual in \mathbb{R}^m and \mathbb{R}^n , respectively. Note, however, that $A(P_1) = A J_n(P_1) = A \circ P_1$. Thus, $A \circ P_1$ is a self-dual cone (with respect to the usual inner product in \mathbb{R}^m).

3. A new type of splitting was introduced recently by Mishra and Sivakumar [15] with a view to study nonnegativity of the Moore-Penrose inverse and the group inverse. These are called B_+ -splitting and $B_{\#}$ splitting. We give the definitions below. Recall that a splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$.

Definition 4.2. A proper splitting $A = U - V$ of A is called a B_+ -splitting if

- (1) $U \geq 0$,
- (2) $V \geq 0$,
- (3) $VU^+ \geq 0$,
- (4) $Ax, Ux \in \mathbb{R}_+^m + N(A^t)$ and $x \in R(A^t) \implies x \in \mathbb{R}_+^n$.

Definition 4.3. A proper splitting $A = U - V$ of A is called a $B_\#$ -splitting if

- (1) $U \geq 0$,
- (2) $V \geq 0$,
- (3) $U^\#$ exists and $VU^\# \geq 0$,
- (4) $Ax, Ux \in \mathbb{R}_+^n + N(A)$ and $x \in R(A) \implies x \in \mathbb{R}_+^n$.

Using these notions, Mishra and Sivakumar characterized nonnegativity of the Moore-Penrose inverse and the group inverse (Refer Theorems 3.8 and 4.4 in [15]). In view of Definition 3.18 and Theorem 3.19, it is quite natural to generalize these two splittings to indefinite inner product spaces. However, we defer this for future investigation.

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