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Some Invariants of Quarter-Symmetric Metric Connections Under the Projective Transformation

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Abstract. This paper studies the characteristics of quarter-symmetric metric connections. Some invariants with respect to the projective transformation are obtained.

1. Introduction

For the study of connection transformations, one of main topics is to consider the various manifolds endowed with special connections, such as S. Fueki and Hiroshi Endo [10] investigated the contact metric structure with Chern connection; I. E. Hirica [14] considered the pseudo-symmetric Riemannian space, she indicated that any semi-symmetric manifold ($R \cdot R = 0$) is of Ricci semi-symmetric ($R \cdot S = 0$). M. M. Tripathi [27] studied $\xi-$Ricci-semi-symmetric $(\kappa, \mu)-$manifolds. Another topic is to consider some special transformations corresponding to certain posed connections, for instance, N. S. Sinyukov [24] considered the geodesic mapping of Riemannian spaces; P. Venzi [29] studied the celebrated geodesic mapping in pseudo-Riemannian manifolds; J. V. Kiosak Mikeš and A. Vanžurová [16] also studied the geodesic mapping in manifolds with affine connections; Liang [18] investigated the semi-symmetric connection; C. Udrişte and I. E. Hirică [28] obtained the family of projective projections on tensors and connections; I. E. Hirică and L. Nicolescu [15] gave an algebraic characterization of the case when the conformal Weyl and conformal Lyra connections have the same curvature tensor; G. Mărgulescu in [19] studied the conformal transformation of Minskowski spaces; F. Y. Fu, X. P. Yang and P. B. Zhao [9] consider a class of conformal mappings between two semi-Riemannian manifolds and obtain the corresponding characteristics of geometries and physics for this mapping. In particular, they proved that this type of conformal mapping keeps a generalized quasi-Einstein manifold unchanged. Later, F. Y. Fu and P. B. Zhao [8] discussed the semi-symmetric projective mapping in pseudo-symmetric Riemannian manifolds and proved that a semi-symmetric projective connection mapping could change a pseudo-symmetric manifold $(M; g)$ into a locally pseudo-symmetric manifold.

As we know a linear connection $\tilde{\nabla}$ is symmetric if its torsion tensor $\tilde{T}$ vanishes, otherwise it is non-symmetric. We all know that a manifold with a symmetric linear connection is projectively flat if and only

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if the projective curvature tensor with respect to it vanishes identically. A linear connection $\tilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\tilde{\nabla}g = 0$, otherwise it is non-metric. It is well known that a linear connection $\tilde{\nabla}$ is symmetric and metric if and only if it is the Levi-Civita connection. In 1973, B. G. Schmidt [23] proved that if the holonomy group of $\tilde{\nabla}$ is a subgroup of the orthogonal group, then $\tilde{\nabla}$ is the Levi-Civita connection of a Riemannian metric. In 1932, H. A. Hayden [13] has induced the idea of metric connection with torsion on a Riemannian manifold. Such a connection is called Hayden connection. On the other hand, for a given 1-form λ in a Riemannian manifold, the Weyl connection constructed with λ and its associated vector B [6] is a symmetric non-metric connection. In fact, the Riemannian metric of the manifold is recurrent with respect to Weyl connection with the recurrence factor λ, that is, $\tilde{\nabla}g = \lambda \otimes g$. Another symmetric non-metric connection is projectively related to the Levi-Civita connection (see [3], [30] for details).

A. Friedmann and J. A. Schouten [7] introduced the concept of the semi-symmetric linear connection in a differential manifold in 1924. The linear connection $\tilde{\nabla}$ is said to be a semi-symmetric connection if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y, \ \ \forall X, Y \in \chi(M),$$

where π is of 1-form associated with a vector P on M, and P is defined by $g(X,P) = \pi(X)$. In 1970, K. Yano [30] considered a semi-symmetric metric connection (that means a linear connection is both metric and semi-symmetric) on a Riemannian manifold and studied some of its properties. He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically. He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric metric connection for which the manifold is a group manifold, where a group manifold is a differential manifold admitting a linear connection $\tilde{\nabla}$ such that its curvature tensor $\tilde{R}$ vanishes and its torsion tensor $\tilde{T}$ is covariantly constant with respect to $\tilde{\nabla}$. Liang in his paper [18] discussed some properties of semi-symmetric metric connections and proved that the projective curvature tensor with respect to semi-symmetric metric connections coincides with the projective curvature tensor with respect to the Levi-Civita connection if and only if the characteristic vector is proportional to the Riemannian metric. P. B. Zhao, H. Z. Song and X. P. Yang [38] introduced the concept of the projective semi-symmetric metric connection, and found an invariant under the transformation of projective semi-symmetric connections and indicated that this invariant could degenerate into the Weyl projective curvature tensor under certain conditions, so the Weyl projective curvature tensor is an invariant as for the transformation of the special projective semi-symmetric connection. To the study of semi-symmetric metric connections, the authors propose other interesting results [32–34, 36, 37]. Recently, the authors in paper [35] even studied the theory of transformations on Carnot Caratheodory spaces, and obtained the conformal invariants and projective invariants on Carnot-Caratheodory spaces with the view of Felix Klein.

Then N. S. Agache and M. R. Chafle[1], U. C. De and S. C. Biswas [4] discussed a semi-symmetric non-metric connection on a Riemannian manifold. If the semi-symmetric connection $\tilde{\nabla}$ satisfies the condition: for any $X, Y, Z \in \chi(M)$, there holds

$$\tilde{\nabla}_X Y = \nabla_X Y + \pi(Y)X + g(X,Y)P,$$

$$\tilde{\nabla}_Z g(X,Y) = -2\pi(X)g(Y,Z) - 2\pi(Y)g(X,Z),$$

where ∇ is the Levi-Civita connection, then $\tilde{\nabla}$ is called the semi-symmetric non-metric connection. This was further developed by Agashe and Chafle [1], U. C. De and Kamily [5]. Agashe and Chafle [1] defined the curvature tensor with respect to semi-symmetric non-metric connections. They proved the Weyl projective curvature tensor with respect to semi-symmetric non-metric connections is equal to the Weyl projective curvature tensor with respect to Riemannian connection, then they derived that a Riemannian manifold with vanishing Ricci tensor with respect to semi-symmetric non-metric connections was projectively flat if and only if the curvature tensor with respect to this semi-symmetric non-metric connection was vanished. U. C. De and S. C. Biswas [4] discussed the semi-symmetric non-metric connection on Riemannian manifolds by using these concepts and similar approaches, they studied some properties of the curvature tensor with respect to the semi-symmetric non-metric connection and proved that two semi-symmetric non-metric connections would be equal under certain conditions.

The notion of the pseudo-symmetry [17] is a natural generalization of the semi-symmetric geometry [2] along the line of spaces of constant sectional curvatures and locally symmetric spaces as follows:

$$R_0 \subset R_1 \subset R_2 \subset R_3,$$

where R_0 is the class of constant sectional curvature Riemann spaces, R_1 is the class of locally symmetric Riemann spaces (i.e. $\nabla R = 0$), R_2 is the class of semi-symmetric Riemann spaces (i.e. $R \cdot R = 0$), R_3 is the class of pseudo-symmetric Riemann spaces (i.e. $R \cdot R = LQ(g; R)$). The class R_2 of semi-symmetric spaces was introduced by E. Cartan, where R_2-spaces were classified by Szabó [26]. It is trivial that all semi-symmetric manifolds are Ricci-semisymmetric ($R \cdot S = 0$), but, in general, the converse criterion is not true unless the Ricci semi-symmetric hypersurfaces of Euclidean spaces ($n > 3$) have positive scalar curvatures. However, the problem that whether these notions are equivalent for hyper-surfaces in Euclidean spaces is still open.

In 1987, Gong [12] investigated the projective s-semi-symmetric connection on a Riemannian manifold and proved that a Riemannian manifold was projectively flat if and only if there existed a s-semi-symmetric connection with vanishing curvature tensors. He also proved that a Riemannian manifold M admitting s-semi-symmetric connection was recurrent manifold, if the recurrence factor λ is closed, then M is projectively flat.

In 1980, R. S. Mishra and S. N. Pardey [20] investigated quarter-symmetric metric connections in Riemannian, Kaehlerian and Sasakian manifolds, they, in particular, studied Ricci quarter-symmetric metric connections and obtained some properties of curvature tensors of these connections. They proved that if an Einstein manifold M admits a quarter-symmetric metric connection whose curvature tensor vanishes, then M is projectively flat, they got an necessary and sufficient condition that an Einstein manifold M associated with a quarter-symmetric metric connection is a group manifold, and so on. While S. Golab [11] derived Schouten's and Struik's, by using the second Bianchi identity, curvature tensor with respect to quarter-symmetric connections. In 1982, K. Yano and T. Imai [31] studied quarter-symmetric metric connections and gave some examples of these connections. They applied quarter-symmetric metric connections into Hermitian manifold and proved that the covariant derivative of almost complex structure tensor F_i^h with respect to the Levi-Civita connection coincides with that of F_i^h with respect to quarter-symmetric metric connections; they also proved that a Kaehlerian manifold endowed with the quarter-symmetric metric connection is flat when the curvature tensor vanishes. For the study of various types of quarter-symmetric metric connections and applications, one can also see [11, 20, 22] for details.

Taking into account that the quarter-symmetric metric connection is a natural generalization of a semi-symmetric metric connection, we would ask whether we can consider the invariants of quarter-symmetric metric connections under some connection transformations just as the case of semi-symmetric metric connections. In fact, there were few results about quarter-symmetric metric connections because of its formal complexity and computational difficulty.

In this paper, we first consider the general form of quarter-symmetric metric connections and find a semi-symmetric metric connection is indeed a special case with $\varphi_i^j = \delta_i^j$; Then, we compute the curvature tensor of quarter-symmetric metric connections, and study the properties of the projective transformation, and give a sufficient condition that a linear connection is exactly a projective transformation of quarter-symmetric metric connections, and find out some invariants under this connection transformation; At last, we define and discuss the mutual connection of quarter-symmetric connections just as the mutual connection of semi-symmetric connections and find the condition, under the connection transformation, that keeps the curvature tensor unchanged.

The organization of this paper is as follows. In section 2, we will recall and give some necessary notations and terminologies. Section 3 is devoted to the main theorems and their proofs. Some examples will appear in the fourth section.

2. Preliminaries

Let (M^n, g), $n = dimM > 2$, be an n-dimensional Riemannian manifold equipped with a Riemannian metric g, and ∇ be the Levi-Civita connection associated with g. Let $\chi(M)$ denote the set of all tangent vector fields on M.

Let D be a linear connection on M, if it satisfies

$$(D_X g)(Y, Z) = 0, \forall X, Y \in \chi(M), \tag{2.1}$$

then D is called a metric connection.

We define the torsion tensor T of D by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \forall X, Y, Z \in \chi(M). \tag{2.2}$$

A metric connection D is called a quarter-symmetric metric connection if there holds

$$T(X, Y) = \varphi(X)\pi(Y) - \varphi(Y)\pi(X), \forall X, Y \in \chi(M), \tag{2.3}$$

where φ is a tensor field of type $(1, 1)$, and π is a 1-form, called an associated 1-form.

Taking a local coordinate system in M such that g, ∇, D, π, φ, T have the local expression, respectively, g_{ji}, $\{^h_{ji}\}$, Γ^h_{ji}, π_i, φ^h_j, T^h_{ji}, then, by a direct computation, we have

$$T^h_{ji} = \pi_j \varphi^h_i - \pi_i \varphi^h_j. \tag{2.4}$$

Theorem 2.1. *For a quarter-symmetric metric connection, in a local coordinate, there holds*

$$\Gamma^h_{ji} = \{^h_{ji}\} + \frac{1}{2}\pi_j(\varphi_{ki} + \varphi_{ik})g^{kh} - \frac{1}{2}\pi_i(\varphi_{jk} - \varphi_{kj})g^{kh} - \frac{1}{2}\pi^h(\varphi_{ji} + \varphi_{ij}), \tag{2.5}$$

where $\pi^j = \pi_i g^{ij}$.

Proof. Since D is a metric connection, then, $\forall X, Y, Z \in \chi(M)$, we have

$$(D_X g)(Y, Z) = X(g(Y, Z)) - g(D_X Y, Z) - g(Y, D_X Z) = 0, \tag{2.6}$$

$$(D_Y g)(Z, X) = Y(g(Z, X)) - g(D_Y Z, X) - g(Z, D_Y X) = 0, \tag{2.7}$$

$$(D_Z g)(X, Y) = Z(g(X, Y)) - g(D_Z X, Y) - g(X, D_Z Y) = 0. \tag{2.8}$$

By using (2.6), (2.7) and (2.8), we get

$$\begin{aligned} 2g(D_X Y, Z) &= 2g(\nabla_X Y, Z) + \pi(Y)\{g(\varphi(Z), X) + g(\varphi(X), Z)\} \\ &- \pi(X)\{g(\varphi(Y), Z) - g(\varphi(Z), Y)\} \\ &- \pi(Z)\{g(\varphi(X), Y) + g(\varphi(Y), X)\}. \end{aligned} \tag{2.9}$$

In a local coordinate (U, x^i), we can choose

$$X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}. \tag{2.10}$$

Substituting (2.10) into (2.9) above, we have locally the following

$$\begin{aligned} 2\Gamma^h_{ji} g_{hk} &= 2\{^h_{ji}\}g_{hk} + \pi_j(\varphi^l_k g_{li} + \varphi^l_i g_{lk}) - \pi_i(\varphi^l_j g_{li} - \varphi^l_k g_{lj}) - \pi_k(\varphi^l_i g_{lj} + \varphi^l_j g_{li}) \\ &= 2\{^h_{ji}\}g_{hk} + \pi_j(\varphi_{ki} + \varphi_{ik}) - \pi_i(\varphi_{jk} - \varphi_{kj}) - \pi_k(\varphi_{ij} + \varphi_{ji}), \end{aligned} \tag{2.11}$$

where $\varphi_{ij} = \varphi^h_i g_{hj}$.

Contracting the above equality (2.11) with g^{kp}, then we get

$$2\Gamma^h_{ji}\delta^p_h = 2\{^k_{ji}\}\delta^p_h + \pi_j(\varphi_{ki} + \varphi_{ik})g^{kp} - \pi_i(\varphi_{jk} - \varphi_{kj})g^{kp} - \pi^p(\varphi_{ij} + \varphi_{ji}), \tag{2.12}$$

where $\pi^p = \pi_k g^{kp}$. Thus, we know that the equation (2.5) is tenable. This ends the proof of Theorem 2.1. □

Remark 2.2. *We write*

$$U_{ij} = \frac{1}{2}(\varphi_{ij} + \varphi_{ij}), V_{ij} = \frac{1}{2}(\varphi_{ij} - \varphi_{ij}), \tag{2.13}$$

then it is obvious that there exists the following

$$U_{ij} = U_{ji}, V_{ij} = -V_{ji}, \tag{2.14}$$

which means V_{ij}, U_{ij} are of symmetric and of skew-symmetric, respectively, with respect to i, j, and

$$U_{ij} + V_{ij} = U_{ij} - V_{ij} = \varphi_{ij}.$$

Equation (2.5) *is equivalent to*

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + \pi_j U^h_i - \pi_i V^h_j - \pi^h U_{ij}, \tag{2.15}$$

where $U^j_i = U_{ik} g^{kj}$, $V^j_i = V_{ik} g^{kj}$.

Remark 2.3. *If φ^j_i is proportional to the identity tensor δ^j_i, then the quarter-symmetric metric connection is reduced into a semi-symmetric connection, and the coefficient is given as*

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + \pi_j \delta^h_i - \pi_i \delta^h_j - \pi^h g_{ji}.$$

3. Main Theorems and Proofs

Using (2.15) and the identity

$$R^h_{kji} = \frac{\partial \Gamma^h_{ji}}{\partial x_k} - \frac{\partial \Gamma^h_{ki}}{\partial x_j} + \Gamma^\alpha_{ji}\Gamma^h_{k\alpha} - \Gamma^\alpha_{ki}\Gamma^h_{j\alpha}. \tag{3.1}$$

By a straightforward calculation, we find

$$\begin{aligned} R^h_{kji} &= K^h_{kji} - U^h_k \pi_{ji} + U^h_j \pi_{ki} - \pi^h_k \varphi_{ji} + \pi^h_j \varphi_{ji} + (\nabla_k U^h_j - \nabla_j U^h_k)\pi_i \\ &- (\nabla_k U_{ji} - \nabla_j U_{ki})\pi^h - (\nabla_k \pi^h_j - \nabla_j \pi^h_k) V^h_i + (\pi_j \nabla_k V^h_i - \pi_k \nabla_j V^h_i) \\ &+ V^h_i (U^t_k \pi_j - U^t_j \pi_k)\pi_i - (\pi_k U_{ji} - \pi_j U_{ki}) V^t_i \pi^h \\ &- V^h_i (U^h_k \pi_j - U^h_j \pi_k) V^t_i \pi_t + (\pi_k U_{ji} - \pi_j U_{ki}) V^h_t \pi^t, \end{aligned}$$

where K^h_{kji} is the Riemanninan curvature tensor of ∇, $\pi^j_i = \pi_{ih} g^{hj}$, and

$$\pi_{ji} = \nabla_j \pi_i - U_{j\alpha}\pi^\alpha \pi_i + \frac{1}{2}\pi^\alpha \pi_\alpha U_{ji}, \tag{3.2}$$

where π^i is the contravariant component of π_j.

Now, let φ_{ij} be symmetric, that is, $V_{ij} = 0$, then the curvature tensor of the quarter-symmetric metric connection D becomes

$$\begin{aligned} R^h_{kji} &= K^h_{kji} - U^h_k \pi_{ji} + U^h_j \pi_{ki} - \pi^h_k \varphi_{ji} + \pi^h_j \varphi_{ji} \\ &+ (\nabla_k U^h_j - \nabla_j U^h_k)\pi_i - (\nabla_k U_{ji} - \nabla_j U_{ki})\pi^h. \end{aligned}$$

Let $\bar{D}$ be another quarter-symmetric metric connection, and the torsion tensor, the connection coefficient, the associated 1-form and the tensor field of type (1, 1) be denoted, respectively, by $\bar{T}^h_{ji}$, $\bar{\Gamma}^h_{ji}$, $\bar{\pi}_i$, $\bar{\varphi}^j_i$, we have the similar identities as D:

$$\bar{T}^h_{ji} = \bar{\pi}_j\bar{\varphi}^h_i - \bar{\pi}_i\bar{\varphi}^h_j. \tag{3.3}$$

$$\bar{\Gamma}^h_{ji} = \{^h_{ji}\} + \bar{\pi}_j\bar{U}^h_i - \bar{\pi}_i\bar{V}^h_j - \bar{\pi}^h\bar{U}_{ij}, \tag{3.4}$$

where $\bar{U}_{ij} = \frac{1}{2}(\bar{\varphi}_{ij} + \bar{\varphi}_{ji})$, $\bar{V}_{ij} = \frac{1}{2}(\bar{\varphi}_{ij} - \bar{\varphi}_{ji})$, $\bar{U}^j_i = \bar{U}_{ik}g^{kj}$, $\bar{V}^j_i = \bar{V}_{ik}g^{kj}$.

Definition 3.1. *If the geodesics with respect to $\bar{D}$ are always consistent with those of D, then $\bar{D}$ is called the projective transformation of D.*

Lemma 3.2. *When $\varphi_{ij} = \varphi_{ji}, \bar{\varphi}_{ij} = \bar{\varphi}_{ji}$, if there holds*

$$\varphi^h_j - \bar{\varphi}^h_j = f\delta^h_j, \pi_j = \bar{\pi}_j, \tag{3.5}$$

then $\bar{D}$ is the projective transformation of D.

Proof. Let the connection coefficients of the quarter-symmetric metric connections D and $\bar{D}$ be, respectively, Γ^k_{ji} and $\bar{\Gamma}^k_{ji}$. When $\varphi_{ij} = \varphi_{ji}, \bar{\varphi}_{ij} = \bar{\varphi}_{ji}$, then there holds

$$\Gamma^k_{ji} = \{^k_{ji}\} + \varphi^k_j\pi_i - \varphi_{ji}\pi^k. \tag{3.6}$$

$$\bar{\Gamma}^k_{ji} = \{^k_{ji}\} + \bar{\varphi}^k_j\bar{\pi}_i - \bar{\varphi}_{ji}\bar{\pi}^k. \tag{3.7}$$

Denote the symmetric part of $\bar{\Gamma}^k_{ji}$ and Γ^k_{ji} by A^k_{ji} and $\bar{A}^k_{ji}$, then we have

$$A^k_{ji} = \frac{1}{2}(T^k_{ji} + T^k_{ij}) = \{^k_{ji}\} + \frac{1}{2}(\varphi^k_j\pi_i + \varphi^k_i\pi_j).$$

$$\bar{A}^k_{ji} = \frac{1}{2}(\bar{T}^k_{ji} + \bar{T}^k_{ij}) = \{^k_{ji}\} + \frac{1}{2}(\bar{\varphi}^k_j\bar{\pi}_i + \bar{\varphi}^k_i\bar{\pi}_j).$$

As we all know, the linear connection D^* with the connection coefficient A^k_{ji} has the same geodesic as D, and the linear connection $\bar{D}^*$ with the connection coefficient $\bar{A}^k_{ji}$ has the same geodesic as $\bar{D}$, so we need only to prove the linear connection with the connection coefficient A^k_{ji} has the same geodesic as $\bar{A}^k_{ji}$.

Substituting (3.6) and (3.7) into the geodesic equation below

$$\frac{dx^l}{dt}(\frac{d^2x^k}{dt^2} + \Gamma^k_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}) = \frac{dx^k}{dt}(\frac{d^2x^l}{dt^2} + \Gamma^l_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}), \tag{3.8}$$

then we have

$$\begin{aligned} & \frac{dx^l}{dt}[\frac{d^2x^k}{dt^2} + (\{^k_{ji}\} + \frac{1}{2}(\varphi^k_j\pi_i + \varphi^k_j\pi_i) - \varphi_{ji}\pi^k)\frac{dx^j}{dt}\frac{dx^i}{dt}] \\ = \quad & \frac{dx^k}{dt}[\frac{d^2x^l}{dt^2} + (\{^l_{ji}\} + \frac{1}{2}(\varphi^l_j\pi_i + \varphi^l_j\pi_i) - \varphi_{ji}\pi^l)\frac{dx^j}{dt}\frac{dx^i}{dt}]. \end{aligned} \tag{3.9}$$

$$\begin{aligned} & \frac{dx^l}{dt}[\frac{d^2x^k}{dt^2} + (\{^k_{ji}\} + \frac{1}{2}(\bar{\varphi}^k_j\bar{\pi}_i + \bar{\varphi}^k_j\bar{\pi}_i) - \bar{\varphi}_{ji}\bar{\pi}^k)\frac{dx^j}{dt}\frac{dx^i}{dt}] \\ = \quad & \frac{dx^k}{dt}[\frac{d^2x^l}{dt^2} + (\{^l_{ji}\} + \frac{1}{2}(\bar{\varphi}^l_j\bar{\pi}_i + \bar{\varphi}^l_j\bar{\pi}_i) - \bar{\varphi}_{ji}\bar{\pi}^l)\frac{dx^j}{dt}\frac{dx^i}{dt}]. \end{aligned} \tag{3.10}$$

From (3.9) minus (3.10), we get

$$\begin{aligned}
&\frac{dx^l}{dt}[\frac{1}{2}(\varphi_j^k - \bar{\varphi}_j^k)\pi_i + \frac{1}{2}(\varphi_i^k - \bar{\varphi}_i^k)\pi_j - (\varphi_{ji} - \bar{\varphi}_{ji})\pi^k)]\frac{dx^j}{dt}\frac{dx^i}{dt} \\
= \ &\frac{dx^k}{dt}[\frac{1}{2}(\varphi_j^l - \bar{\varphi}_j^l)\pi_i + \frac{1}{2}(\varphi_i^l - \bar{\varphi}_i^l)\pi_j - (\varphi_{ji} - \bar{\varphi}_{ji})\pi^l)]\frac{dx^j}{dt}\frac{dx^i}{dt}. \\
&\frac{dx^l}{dt}(\frac{1}{2}f\delta_j^k\pi_i + \frac{1}{2}f\delta_j^i\pi_k - fg_{ji}\pi^k)\frac{dx^j}{dt}\frac{dx^i}{dt} \\
= \ &\frac{dx^k}{dt}(\frac{1}{2}f\delta_j^l\pi_i + \frac{1}{2}f\delta_j^i\pi_l - fg_{ji}\pi^l)\frac{dx^j}{dt}\frac{dx^i}{dt}.
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{1}{2}f\pi_i\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} + \frac{1}{2}f\pi_j\frac{dx^k}{dt}\frac{dx^i}{dt}\frac{dx^l}{dt} - f\pi_j\frac{dx^k}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} - f\pi_i\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} \\
= &\frac{1}{2}f\pi_i\frac{dx^k}{dt}\frac{dx^l}{dt}\frac{dx^l}{dt} + \frac{1}{2}f\pi_j\frac{dx^i}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} - f\pi_j\frac{dx^k}{dt}\frac{dx^i}{dt}\frac{dx^l}{dt} - f\pi_i\frac{dx^k}{dt}\frac{dx^l}{dt}\frac{dx^l}{dt}.
\end{aligned}$$

This ends the proof of Lemma 3.2 □

Remark 3.3. *For the quarter-symmetric connection $\bar{D}$, when $\bar{\varphi}_{ij}$ is symmetric, $\bar{U}_{ij} = \bar{\varphi}_{ij}$, we also have the curvature tensor as follows*

$$\begin{aligned}
\bar{R}_{kji}^h &= K_{kji}^h - \bar{\varphi}_k^h\bar{\pi}_{ji} + \bar{\varphi}_j^h\bar{\pi}_{ki} - \bar{\pi}_k^h\bar{\varphi}_{ji} + \bar{\pi}_j^h\bar{\varphi}_{ji} \\
&+ (\nabla_k\bar{\varphi}_j^h - \nabla_j\bar{\varphi}_k^h)\bar{\pi}_i - (\nabla_k\bar{\varphi}_{ji} - \nabla_j\bar{\varphi}_{ki})\bar{\pi}^h.
\end{aligned}$$

Theorem 3.4. *If 1-form π is closed, then the tensor as below*

$$\begin{aligned}
S_{kji}^h &= R_{kji}^h - \frac{1}{n-2}(\delta_k^h R_{ji} - \delta_j^h R_{ki} + R_k^h g_{ji} - R_j^h g_{ki}) \\
&+ \frac{R}{(n-1)(n-2)}(\delta_k^h g_{ji} - \delta_j^h g_{ki}).
\end{aligned}$$

is an invariant under the projective transformation of quarter-symmetric metric connections.

Proof. By virtue of Lemma 3.2 and that 1-form π is closed, then we have

$$\begin{aligned}
\bar{R}_{kji}^h - R_{kji}^h &= \pi_{ji}(\varphi_k^h - \bar{\varphi}_k^h) - \pi_{ki}(\varphi_j^h - \bar{\varphi}_j^h) + \pi_k^h(\varphi_{ji} - \bar{\varphi}_{ji}) - \pi_j^h(\varphi_{ki} - \bar{\varphi}_{ki}) \\
&\quad -\pi_i\nabla_k(\varphi_j^h - \bar{\varphi}_j^h) + \pi_i\nabla_j(\varphi_k^h - \bar{\varphi}_k^h) + \pi^h\nabla_k(\varphi_{ji} - \bar{\varphi}_{ji}) - \pi^h\nabla_j(\varphi_k^h - \bar{\varphi}_k^h) \\
&= f\pi_{ji}\delta_k^h - f\pi_{ki}\delta_j^h + f\pi_k^h g_{ji} - f\pi_j^h g_{ki} - \pi_i\nabla_k(f\delta_j^h) + \pi_i\nabla_j(f\delta_k^h) \\
&\quad +\pi^h\nabla_k(fg_{ji}) - \pi^h\nabla_j(fg_{ki}) \\
&= f\pi_{ji}\delta_k^h - f\pi_{ki}\delta_j^h + f\pi_k^h g_{ji} - f\pi_j^h g_{ki} - \pi_i\delta_j^h\nabla_k f + \pi_i\delta_k^h\nabla_j f \\
&\quad +\pi^h g_{ji}\nabla_k f - \pi^h g_{ki}\nabla_j f.
\end{aligned}$$

Let $k = h = \alpha$, we obtain

$$\begin{aligned}
\bar{R}_{ji} - R_{ji} &= nf\pi_{ji} - f\pi_{ji} + f\pi_\alpha^a g_{ji} - f\pi_{ji} - \pi_i\nabla_j f + n\pi_i\nabla_j f \\
&\quad +g_{ji}\pi^\alpha\nabla_\alpha f - \pi_i\nabla_j f \\
&= (n-2)f\pi_{ji} - f\pi_\alpha^\alpha + (n-2)\pi_i\nabla_j f - g_{ji}\pi^\alpha\nabla_\alpha f. \qquad (3.11)
\end{aligned}$$

Contracting the above equation with g^{ji}, then we obtain

$$\bar{R} - R = 2(n-1)f\pi_\alpha^\alpha + 2(n-1)\pi^\alpha\nabla_\alpha f.$$

We choose f with $\pi^\alpha \nabla_\alpha f = 0$, so we arrive at

$$f\pi_\alpha^\alpha = \frac{\bar{R} - R}{2(n-1)}. \tag{3.12}$$

Substituting (3.12) into (3.11), then we get

$$f\pi_{ji} = \frac{\bar{R}_{ji} - R_{ji}}{n-2} - \frac{\bar{R} - R}{2(n-1)(n-2)} g_{ji}. \tag{3.13}$$

Moreover we have

$$\begin{aligned}
\bar{R}^h_{kji} - R^h_{kji} &= \delta^h_k \{\frac{\bar{R}_{ji} - R_{ji}}{n-2} - \frac{\bar{R} - R}{2(n-1)(n-2)} g_{ji}\} \\
&\quad -\delta^h_j \{\frac{\bar{R}_{ki} - R_{ki}}{n-2} - \frac{\bar{R} - R}{2(n-1)(n-2)} g_{ki}\} \\
&\quad +g_{ij} \{\frac{\bar{R}^h_k - R^h_k}{n-2} - \frac{\bar{R} - R}{2(n-1)(n-2)} \delta^h_k\} \\
&\quad -g_{ki} \{\frac{\bar{R}^h_j - R^h_j}{n-2} - \frac{\bar{R} - R}{2(n-1)(n-2)} \delta^h_j\}.
\end{aligned}$$

and

$$\begin{aligned}
&\bar{R}^h_{kji} - \frac{1}{n-2}(\delta^h_k \bar{R}_{ji} - \delta^h_j \bar{R}_{ki} + \bar{R}^h_k g_{ji} - \bar{R}^h_k g_{ji}) + \frac{\bar{R}}{(n-1)(n-2)}(\delta^h_k g_{ji} - \delta^h_j g_{ki}) \\
&= R^h_{kji} - \frac{1}{n-2}(\delta^h_k R_{ji} - \delta^h_j R_{ki} + R^h_k g_{ji} - R^h_k g_{ji}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ji} - \delta^h_j g_{ki}).
\end{aligned}$$

Therefore, the tensor with the form below

$$\begin{aligned}
S^h_{kji} &= R^h_{kji} - \frac{1}{n-2}(\delta^h_k R_{ji} - \delta^h_j R_{ki} + R^h_k g_{ji} - R^h_j g_{ki}) \\
&\quad + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ji} - \delta^h_j g_{ki}),
\end{aligned} \tag{3.14}$$

is an invariant. This completes the proof of Theorem 3.4 □

Theorem 3.5. *Assume that φ_{ij} and $\bar{\varphi}_{ij}$ are skew-symmetric, if $\varphi^h_j - \bar{\varphi}^h_j = f\delta^h_j, \pi_j = \bar{\pi}_j$, then the tensor*

$$X^h_{kji} = R^h_{kji} + \delta^h_i R_{kj}, \tag{3.15}$$

is an invariant under the connection transformation from D to $\bar{D}$.

Proof. Since φ_{ij} and $\bar{\varphi}_{ij}$ are skew-symmetric, $U_{ij} = 0$ and $V^j_i = \varphi^j_i$, so the curvature tensors of $D, \bar{D}$ are given as follows

$$R^h_{kji} = K^h_{kji} - \varphi^h_i(\nabla_k \pi_j - \nabla_j \pi_k) - (\pi_j \nabla_k \varphi^h_i - \pi_k \nabla_j \varphi^h_i).$$

$$\bar{R}^h_{kji} = K^h_{kji} - \bar{\varphi}^h_i(\nabla_k \bar{\pi}_j - \nabla_j \bar{\pi}_k) - (\bar{\pi}_j \nabla_k \bar{\varphi}^h_i - \bar{\pi}_k \nabla_j \bar{\varphi}^h_i).$$

Using the hypotheses, we get

$$\varphi^h_j - \bar{\varphi}^h_j = f\delta^h_j, \pi_j = \bar{\pi}_j, \tag{3.16}$$

Then we arrive at

$$\begin{aligned}\bar{R}^h_{kji} - R^h_{kji} &= (\varphi^h_i - \bar{\varphi}^h_i)(\nabla_k\pi_j - \nabla_j\pi_k) + \pi_j\nabla_k(\varphi^h_i - \bar{\varphi}^h_i) - \pi_k\nabla_j(\varphi^h_i - \bar{\varphi}^h_i)\\ &= f\delta^h_i(\nabla_k\pi_j - \nabla_j\pi_k) + \pi_j\nabla_k(f\delta^h_i) - \pi_k\nabla_j(f\delta^h_i)\\ &= f\delta^h_i(\nabla_k\pi_j - \nabla_j\pi_k) + \delta^h_i\pi_j\nabla_k f - \delta^h_i\pi_k\nabla_j f\\ &= \delta^h_i\{f(\nabla_k\pi_j - \nabla_j\pi_k) + \pi_j\nabla_k f - \pi_k\nabla_j f\}.\end{aligned}$$

Let $k = h = \alpha$, we have

$$\bar{R}_{ji} - R_{ji} = f(\nabla_i\pi_j - \nabla_j\pi_i) + \pi_j\nabla_i f - \pi_i\nabla_j f.$$

therefore we get

$$\bar{R}^h_{kji} - R^h_{kji} = -\delta^h_i(\bar{R}_{kj} - R_{kj}),$$

namely,

$$\bar{R}^h_{kji} + \delta^h_i\bar{R}_{kj} = R^h_{kji} + \delta^h_i R_{kj}, \tag{3.17}$$

which means that the tensor

$$X^h_{kji} = R^h_{kji} + \delta^h_i R_{kj}, \tag{3.18}$$

is an invariant under the connection transformation from D to $\bar{D}$. This completes the proof of Theorem 3.5 □

In the next subsection, we define the mutual connection $\tilde{D}$ of the quarter-symmetric metric connection D by

$$\tilde{\Gamma}^k_{ji} = \Gamma^k_{ji} - T^k_{ji}, \tag{3.19}$$

where $\tilde{\Gamma}^k_{ji}$ is the coefficient of $\tilde{D}$ and T^k_{ji} is the torsion tensor of D defined by (2.4).

When φ_{ji} is of skew-symmetric, according to (3.6), we have

$$\tilde{\Gamma}^k_{ji} = \left\{{k \atop ji}\right\} + 2\pi_j\varphi^k_i - \pi_i\varphi^k_j - \varphi_{ji}\pi^k. \tag{3.20}$$

By a direct computation, we get the curvature tensor of $\tilde{D}$ as follows

$$\begin{aligned}\tilde{R}^h_{kji} = {} & K^h_{kji} - \nabla_j(\varphi_{ki}\pi^h + \varphi^h_k\pi_i - 2\pi_k\varphi^h_i) + \nabla_k(\varphi_{ji}\pi^h + \varphi^h_j\pi_i - 2\pi_j\varphi^h_i)\\ & -2\pi_i\pi_j\varphi^\alpha_k\varphi^h_\alpha + 2\pi_i\pi_k\varphi^\alpha_j\varphi^h_\alpha - 2\pi^\alpha\pi_j\varphi_{ki}\varphi^h_\alpha + 2\pi^\alpha\pi_k\varphi_{ji}\varphi^h_\alpha\\ & +2\pi_\alpha\pi_j\varphi^\alpha_i\varphi^h_k - 2\pi_\alpha\pi_k\varphi^\alpha_i\varphi^h_j + 2\pi^h\pi_j\varphi^\alpha_i\varphi_{k\alpha} - 2\pi^h\pi_k\varphi^\alpha_i\varphi_{j\alpha}\\ & +\pi_\alpha\pi_i\varphi^\alpha_k\varphi^h_j - \pi_\alpha\pi_i\varphi^\alpha_j\varphi^h_k + \pi^h\pi_i\varphi^\alpha_k\varphi_{j\alpha} - \pi^h\pi_i\varphi^\alpha_j\varphi_{k\alpha}\\ & +\pi^\alpha\pi^h\varphi_{ki}\varphi_{j\alpha} - \pi^\alpha\pi^h\varphi_{ji}\varphi_{k\alpha} + \pi^\alpha\pi_\alpha\varphi_{ji}\varphi^h_k - \pi^\alpha\pi_\alpha\varphi_{ki}\varphi^h_j.\end{aligned}$$

Let

$$\begin{aligned}A^h_{kji} = {} & \nabla_k(\varphi_{ji}\pi^h + \varphi^h_j\pi_i - 2\pi_j\varphi^h_i) + 2\pi_i\pi_k\varphi^\alpha_j\varphi^h_\alpha + 2\pi^\alpha\pi_k\varphi_{ji}\varphi^h_\alpha\\ & -2\pi^h\pi_k\varphi^\alpha_i\varphi_{j\alpha} - 2\pi_\alpha\pi_k\varphi^\alpha_i\varphi^h_j - \pi^\alpha\pi^h\varphi_{ji}\varphi_{k\alpha} + \pi_\alpha\pi_i\varphi^\alpha_j\varphi^h_j\\ & -\pi^h\pi_i\varphi^\alpha_j\varphi_{k\alpha} - \pi^\alpha\pi_\alpha\varphi_{ki}\varphi^h_j.\end{aligned}$$

Then, we get

$$\tilde{R}^h_{kji} = K^h_{kji} + A^h_{kji} - A^h_{jki}.$$

So there exists the following

Theorem 3.6. *When $A^h_{kji} = A^h_{jki}$, then the curvature tensor will keep unchanged under the connection transformation $\nabla \to \tilde{D}$.*

When φ_{ji} is of skew-symmetric, the formula (3.19) becomes

$$\tilde{\Gamma}^k_{ji} = \{^k_{ji}\} - 2\pi_i\varphi^k_j + \pi_j\varphi^k_i. \tag{3.21}$$

By a similar computation just as above, we can give out the curvature tensor of $\tilde{D}$ as below

$$\begin{aligned}\tilde{R}^h_{kji} \ = \ & K^h_{kji} + \nabla_j(\varphi_{ki}\pi^h + \varphi^h_k\pi_i - 2\pi_k\varphi^h_i) - \nabla_k(\varphi_{ji}\pi^h + \varphi^h_j\pi_i - 2\pi_j\varphi^h_i) \\ & -2\pi_i\pi_j\varphi^\alpha_k\varphi^h_\alpha + 2\pi_\alpha\pi_j\varphi^\alpha_i\varphi^h_k - \pi_i\pi_\alpha\varphi^\alpha_j\varphi^h_k \\ & +2\pi_i\pi_k\varphi^\alpha_j\varphi^h_\alpha - 2\pi_\alpha\pi_k\varphi^\alpha_i\varphi^h_j + \pi_i\pi_\alpha\varphi^\alpha_k\varphi^h_j.\end{aligned}$$

Set

$$B^h_{kji} = -\nabla_k(\varphi_{ji}\pi^h + \varphi^h_j\pi_i - 2\pi_j\varphi^h_i) + 2\pi_i\pi_k\varphi^\alpha_j\varphi^h_\alpha - 2\pi_\alpha\pi_k\varphi^\alpha_i\varphi^h_j + \pi_i\pi_\alpha\varphi^\alpha_k\varphi^h_j.$$

Then, we know

$$\tilde{R}^h_{kji} = K^h_{kji} + B^h_{kji} - B^h_{jki}.$$

Therefore, there is the following

Theorem 3.7. *When $B^h_{jik} = B^h_{ijk}$, then the curvature tensor will keep unchanged under the connection transformation $\nabla \to \tilde{D}$.*

Remark 3.8. *According to the conclusions given above, it is not hard to see that the connection transformation $\nabla \to \tilde{D}$ will change a constant mean curvature space into a constant mean curvature space.*

4. Examples

By Remark 2.2, if $V_{ij} = 0$, $U_{ij} = g_{ij}$, then we get the semi-symmetric metric connection $\bar{D}$ with coefficients

$$\Gamma^h_{ji} = \{^h_{ji}\} + \pi_j\delta^h_i - \pi^h g_{ij},$$

and it is easy to see that there is the following

$$R^h_{kji} = K^h_{kji} + \delta^h_j\pi_{ki} - \delta^h_k\pi_{ji} + \pi^h_j g_{ki} - \pi^h_k g_{ji}. \tag{4.1}$$

By a contractive operation, we can write down the Ricci tensor and the scalar curvature of semi-symmetric metric connections as

$$R_{ji} = K_{ji} - (n-2)\pi_{ji} - ag_{ji}, \tag{4.2}$$

$$\bar{R} = r - 2(n-1)a, \tag{4.3}$$

where K^h_{kji}, K_{ji}, r are the curvature tensor, the Ricci tensor and the scalar curvature, respectively, with respect to the Levi-Civita connection ∇, and a is the trace of α. Then we have the following properties about semi-symmetric metric connections according to [21].

Proposition 4.1. *For a Riemannian manifold (M, g) admitting a semi-symmetric metric connection, one knows that the Ricci tensor is symmetric if and only if the 1-form π is closed.*

Proposition 4.2. *A Riemannian manifold (M, g) admits a semi-symmetric metric connection, and the 1-form π is closed, then there are the following*

$$R^h_{kji} + R^h_{jik} + R^h_{ikj} = 0,$$
$$R_{kjih} = R_{ihkj}.$$

Let M be a Riemannian manifold admitting the semi-symmetric metric connection $\bar{D}$, we call M an Einstein manifold with respect to this semi-symmetric metric connection, if

$$R_{ji} = \mu g_{ji},$$

where μ is a scalar function on M. By (4.2), we know R_{ji} is symmetric, then the 1-form π is closed, in particular, if $\pi = 0$, then we would have $R_{ji} = K_{ji}$, where K is the Ricci tensor of ∇. Thus, we have $K_{ji} = \mu g_{ji}$, and M reduces to an Einstein manifold in the usual sense. Hence by a direct calculation we have

$$R_{kjih} = \frac{\mu}{n-1}(g_{ji}g_{kh} - g_{ki}g_{jh}), \tag{4.4}$$

therefore we arrive at

Example 4.3. *If M is an Einstein manifold with respect to the semi-symmetric metric connection, then M is conformally flat if and only if the curvature tensor of $\bar{D}$ satisfies* (4.4).

An odd dimension differentiable manifold $M^m(m = 2n + 1)$ is called a contact manifold if it carries a global differentiable 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^m, the 1-form η is called a contact form of M^{2n+1}. A Riemannian metric g is said to be associated with a contact manifold if there exists a $(1, 1)$ tensor field ϕ and a contra-variant global vector field ξ, called the characteristic vector field of the manifold, such that

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, g(X, \xi) = \eta(\xi),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields X, Y on M. Then the structure (ϕ, ξ, η, g) is said to be a contact metric structure and the manifold M^m equipped with such a structure is said to be a contact metric manifold. For a contact metric manifold, the following relations hold

$$\phi\xi = 0, \eta \circ \phi = 0, d\eta(\xi, X) = 0, g(X, \phi Y) + g(\phi X, Y) = 0.$$

A contact metric manifold is said to be an $\eta-$ Einstein manifold if its Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta,$$

where a, b are smooth functions on the manifold. In a contact metric manifold, we define a $(1, 1)$ tensor field h by

$$h = \frac{1}{2} L_\xi \phi,$$

where L denotes the Lie differentiation. Then h is self-adjoint and satisfies

$$h\xi = 0, h\phi = -\phi h, Tr(h) = Tr(\phi h) = 0.$$

A contact metric manifold is said to be a (k, μ)-contact metric manifold if it satisfies the relation

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

for all vector fields X, Y on M, where k, μ are real constants and R is the Riemannian curvature tensor of the manifold of type $(1, 3)$.

The class of (k, μ)-contact metric manifolds contains both the class of Sasakian $(k = 1, h = 0)$ and non-Sasakian $(k \neq 1, h \neq 0)$ manifolds. For example, the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure is a non-Sasakian (k, μ)-contact metric manifold.

It is not hard to see that the following theorem about the quarter-symmetric metric connection, according to [25], holds.

Theorem 4.4. *There exists a unique quarter-symmetric metric connection on a Riemannian manifold.*

Example 4.5. *In a non-Sasakian $(k, \mu)-$ contact metric manifold (M, g), a linear connection $\tilde{\nabla}$ is a quarter-symmetric metric connection if and only if*

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)hX - g(hX, Y)\xi, \forall X, Y \in \chi(M), \tag{4.5}$$

Example 4.6. *The curvature tensor $\tilde{R}$ of a non-Sasakian $(k, \mu)-$ contact metric manifold with respect to the quarter-symmetric metric connection satisfies*

$$\begin{aligned}
&(a) \quad \tilde{R}(X, Y)Z = -\tilde{R}(Y, X)Z;\\
&(b) \quad g(\tilde{R}(X, Y)Z, W) = -g(\tilde{R}(X, Y)W, Z);\\
&(c) \quad (g(\tilde{R}(X, Y)Z, W) = g(\tilde{R}(Z, W)X, Y);\\
&(d) \quad \tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 2\{d\eta(X, Z)hY - d\eta(Y, Z)hX\\
&\quad - d\eta(X, Y)hZ + (1 - k)[d\eta(Y, Z)\eta(X)\xi + d\eta(X, Y)\eta(Z)\xi + d\eta(X, Z)\eta(Y)\xi]\},
\end{aligned}$$

for all vector fields $X, Y, Z \in \chi(M)$.

By the formula (d) above, we know that

$$\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0, \tag{4.6}$$

if and only if

$$d\eta = 0, \tag{4.7}$$

this means that the curvature tensor of a non-Sasakian (k, μ)-contact metric manifold with respect to the quarter-symmetric metric connection satisfies the Bianchi identity. This implies that the Ricci tensor is symmetric if and only if the contact form η is closed.

Theorem 4.7. *If the Ricci tensor of an non-Sasakian (k, μ)-contact metric manifold with respect to the quarter-symmetric metric connection vanishes, then the manifold is locally isometric to either an η-Einstein or a 3-dimensional non-Sasakian (k, μ)-contact metric manifold.*

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