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Some New Results for Jacobi Matrix Polynomials

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Abstract. The main aim of this paper is to obtain some recurrence relations and generating matrix function for Jacobi matrix polynomials (JMP). Also, various integral representations satisfied by JMP are derived.

1. Introduction

Special matrix functions seen on statistics, Lie group theory and number theory are well known in [6, 16]. In the recent papers, matrix polynomials have significant emergent in [7–9, 11–14] and some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials in [1–4, 7, 8, 15]. In [13], these polynomials are orthogonal as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Jacobi matrix polynomials have been introduced and studied in [8] for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. Our main aim in this paper is to prove new properties for the Jacobi matrix polynomials. The structure of this paper is the following:

In section 2, recurrence relations for Jacobi matrix polynomials (JMP) are given. A generating matrix function for JMP is also obtained in section 3. Furthermore, we show the integral representations for JMP.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A and B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, then from the properties of the matrix functional calculus in [10], it follows that

$$f(A)g(B) = g(B)f(A)$$

where $AB = BA$.

The Jacobi matrix polynomials have been given in [8], $P_n^{(A,B)}(x)$ for parameter matrices A and B whose eigenvalues, z, all satisfy $Re(z) > -1$. For $n \in \mathbb{N}$, the n-th Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by

$$P_n^{(A,B)}(x) = \frac{1}{n!}\, F\left(A + B + (n+1)\, I, -nI; A + I; \frac{1-x}{2}\right)(A + I)_n \tag{1}$$

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or

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{n!} F\left(A + B + (n+1) I, -nI; B + I; \frac{1+x}{2}\right)(B+I)_n \tag{2}$$

where hypergeometric matrix function $F(A', B'; C'; z)$ has been given in the form [11]

$$F(A', B'; C'; z) = \sum_{k=0}^{\infty} \frac{(A')_k (B')_k}{k!} \left[(C')_k\right]^{-1} z^k \tag{3}$$

for matrices A', B' and C' in $\mathbb{C}^{N\times N}$ such that $C' + kI$ is invertible for all integer $k \geq 0$ and for $|z| < 1$. Here

$$(A')_k = A'(A' + I)(A' + 2I)...(A' + (k-1)I);\ k \geq 1;\ (A')_0 = I. \tag{4}$$

These polynomials have the following Rodrigues formula:

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n\, n!}(1-x)^{-A}(1+x)^{-B}\frac{d^n}{dx^n}\left[(1-x)^{A+nI}(1+x)^{B+nI}\right]. \tag{5}$$

Let P and Q be positive stable matrices in $\mathbb{C}^{N\times N}$, then Beta matrix function in [12] is defined by

$$\mathcal{B}(P, Q) = \int_0^1 t^{P-I}\ (1-t)^{Q-I}\ dt.$$

If P, Q and $P + Q$ are positive stable matrices in $\mathbb{C}^{N\times N}$ and $PQ = QP$, then

$$\mathcal{B}(P, Q) = \Gamma(P)\,\Gamma(Q)\,\Gamma^{-1}(P+Q)$$

[12]. Furthermore, in [8], the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{N\times N}$, the Riesz-Dunford functional calculus [10] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence: if $C \in \mathbb{C}^{N\times N}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C + nI)\Gamma^{-1}(C). \tag{6}$$

Lemma 1.1. *Assume that $\Phi(y)$ is analytic in a neighborhood of $y = x$,*

$$r = \frac{y-x}{\Phi(y)} = \sum_{n=1}^{\infty} a_n (y-x)^n\ ,\ a_1 \neq 0 \tag{7}$$

and f is analytic in a neighborhood of $y = x$. Then $f(y)$ can be expanded in powers of r :

$$f(y) = f(x) + \sum_{n=1}^{\infty} \frac{r^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left(f'(x)(\Phi(x))^n\right) \tag{8}$$

in [5].

2. Recurrence Relations for Jacobi Matrix Polynomials

In this section, some recurrence relations satisfied by Jacobi matrix polynomials (JMP) are given.

Theorem 2.1. *Let A and B be matrices in* $\mathbb{C}^{N\times N}$ *whose eigenvalues, z, all satisfy* $Re(z) > -1$. *JMP satisfy*

(i) $$\frac{d}{dx}P_n^{(A,B)}(x) = \frac{(n+1)\,I + A + B}{2}P_{n-1}^{(A+I,B+I)}(x),$$

(ii) $$\frac{d^k}{dx^k}P_n^{(A,B)}(x) = \frac{((n+1)\,I + A + B)_k}{2^k}P_{n-k}^{(A+kI,B+kI)}(x)$$

$$\text{for } 0 \le k \le n,$$

(iii) $$P_n^{(A,B)}(-x) = (-1)^n P_n^{(B,A)}(x).$$

Proof. **(i)** By using (1), it can be proved.
(ii) It is enough to use (**i**).
(iii) Taking $(-x)$ instead of x in (2), we have desired relation. □

For $n \in \mathbb{N}$ and $AB = BA$, the n-th Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by

$$P_n^{(A,B)}(x) = \frac{1}{n!}\left(\frac{x+1}{2}\right)^n F\left(-(B+nI)\,,-nI; A+I; \frac{x-1}{x+1}\right)(A+I)_n \tag{9}$$

or

$$P_n^{(A,B)}(x) = \frac{1}{n!}\left(\frac{x-1}{2}\right)^n F\left(-(A+nI)\,,-nI; B+I; \frac{x+1}{x-1}\right)(B+I)_n \tag{10}$$

[3]. With the help of these equalities, we can give the following theorem:

Theorem 2.2. *Let A and B be matrices in* $\mathbb{C}^{N\times N}$ *whose eigenvalues, z, all satisfy* $Re(z) > -1$. *For the Jacobi matrix polynomials (JMP), the following recurrence relations*

$$(x+1)\,\frac{d}{dx}P_n^{(A,B)}(x) = nP_n^{(A,B)}(x) + (B+nI)\,P_{n-1}^{(A+I,B)}(x) \tag{11}$$

and

$$(x-1)\,\frac{d}{dx}P_n^{(A,B)}(x) = nP_n^{(A,B)}(x) - (A+nI)\,P_{n-1}^{(A,B+I)}(x) \tag{12}$$

hold.

Proof. Differentiating (9) with respect to x, we can write that

$$\frac{d}{dx}P_n^{(A,B)}(x) = n\,(x+1)^{-1}\,P_n^{(A,B)}(x) + (B+nI)\,\frac{(x+1)^{-1}}{(n-1)!}\left(\frac{x+1}{2}\right)^{n-1}$$

$$\times \sum_{k=0}^{\infty} \frac{(-(n-1)\,I)_k\,(-(B+(n-1)\,I))_k\,\left[(A+2I)_k\right]^{-1}}{k!}\left(\frac{x-1}{x+1}\right)^k (A+2I)_{n-1}$$

$$= n\,(x+1)^{-1}\,P_n^{(A,B)}(x) + (B+nI)\,(x+1)^{-1}\,P_{n-1}^{(A+I,B)}(x).$$

Therefore, we obtain

$$(x+1)\,\frac{d}{dx}P_n^{(A,B)}(x) = nP_n^{(A,B)}(x) + (B+nI)\,P_{n-1}^{(A+I,B)}(x).$$

Similarly, if we differentiate (10) with respect to x and use (10) again, we find the second relation. □

Corollary 2.3. *As a consequence of Theorem 2.1(**i**),(11) and (12), we have the following recurrence relations:*

$$2\frac{d}{dx}P_n^{(A,B)}(x) = (B + nI)\,P_{n-1}^{(A+I,B)}(x) + (A + nI)\,P_{n-1}^{(A,B+I)}(x)$$

and

$$(A + B + (n + 2)\,I)\,P_n^{(A+I,B+I)}(x) = (B + (n + 1)\,I)\,P_n^{(A+I,B)}(x) + (A + (n + 1)\,I)\,P_n^{(A,B+I)}(x).$$

Lemma 2.4. *Let A', B' and C' be matrices in $\mathbb{C}^{N\times N}$ and A' and B' be commutative. For the hypergeometric matrix function $F(A', B'; C'; z)$, the equality*

$$F(A', B'; C'; z) = F(A' - I, B' + I; C'; z) + (B' + I - A')zF(A', B' + I; C' + I; z)(C')^{-1}$$

holds where $A' - I$, $B' + kI$ and $C' + kI$ are invertible for all integer $k \geq 0$.

Proof. If we rearrange equation in (3), we can write that

$$\begin{aligned}
&F(A', B'; C'; z)\\
&= \sum_{k=0}^{\infty} \frac{(A')_k\,(B')_k}{k!}\,[(C')_k]^{-1}\; z^k\\
&= \sum_{k=0}^{\infty} (A' + (k-1)\,I)(A' - I)^{-1}B'(B' + kI)^{-1}(A' - I)_k\,(B' + I)_k\,[(C')_k]^{-1}\,\frac{z^k}{k!}\\
&= \sum_{k=0}^{\infty} \left[I - k(A' - I)^{-1}(B' + kI)^{-1}\,(A' - B' - I)\right](A' - I)_k\,(B' + I)_k\,[(C')_k]^{-1}\,\frac{z^k}{k!}\\
&= \sum_{k=0}^{\infty} (A' - I)_k\,(B' + I)_k\,[(C')_k]^{-1}\,\frac{z^k}{k!} + (B' + I - A')z\sum_{k=0}^{\infty} (A')_k\,(B' + I)_k\,[(C' + I)_k]^{-1}\,(C')^{-1}\,\frac{z^k}{k!}\\
&= F(A' - I, B' + I; C'; z) + (B' + I - A')\,zF(A', B' + I; C' + I; z)\,(C')^{-1}
\end{aligned}$$

which completes the proof. □

Theorem 2.5. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. JMP satisfy as follows:*

$$P_n^{(A,B)}(x)(A + (n + 1)I) = (n + 1)\,P_{n+1}^{(A,B)}(x) + (A + B + 2(n + 1)I)\,\frac{(1 - x)}{2}P_n^{(A+I,B)}(x)$$

where $A + B + kI$ is invertible for all integer $k \geq 0$.

Proof. In Lemma 2.4, taking

$$A' = -nI\,,\; B' = A + B + (n + 1)I\,,\; C' = A + I\,,\; z = \frac{1}{2}\,(1 - x)\,,$$

we have

$$\begin{aligned}
&F\left(-nI, A + B + (n + 1)I; A + I; \frac{1 - x}{2}\right)\\
&= F\left(-(n + 1)\,I, A + B + (n + 2)I; A + I; \frac{1 - x}{2}\right)\\
&+\tfrac{1}{2}\,(1 - x)\,(A + B + 2(n + 1)\,I)F\left(-nI, A + B + (n + 2)I; A + 2I; \frac{1 - x}{2}\right)(A + I)^{-1}.
\end{aligned}$$

With the help of (1), we obtain

$$P_n^{(A,B)}(x)(A + (n + 1)I) = (n + 1)\,P_{n+1}^{(A,B)}(x) + (A + B + 2(n + 1)I)\,\frac{(1 - x)}{2}P_n^{(A+I,B)}(x).$$

□

If the hypergeometric matrix function $F(A', B'; C'; z)$ given by (3) is rearranged, we can give the following lemma.

Lemma 2.6. *Let A', B' and C' be matrices in $\mathbb{C}^{N\times N}$ and A' and B' be commutative. For the hypergeometric matrix function $F(A', B'; C'; z)$, the equality*

$$(A' - B') F(A', B'; C'; z) = A'F(A' + I, B'; C'; z) - B'F(A', B' + I; C'; z)$$

holds.

Theorem 2.7. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. We have for JMP*

$$P_{n-1}^{(A,B+I)}(x)(A + nI) + (A + B + (n + 1)I) P_n^{(A,B+I)}(x) = (A + B + (2n + 1)I) P_n^{(A,B)}(x).$$

Proof. In Lemma 2.6, getting

$$A' = -nI \, , \; B' = A + B + (n + 1)I \, , \; C' = A + I \, , \; z = \frac{1}{2}(1 - x) \, ,$$

and using (1), we have desired recurrence relation. □

Corollary 2.8. *As a result of Corollary 2.3 and Theorem 2.7, we can give as follows:*

$$P_{n-1}^{(A+I,B+I)}(x)(A + (n + 1) I) = (A + (n + 1) I)P_n^{(A+I,B)}(x) - (A + (n + 1) I)P_n^{(A,B+I)}(x)$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$.

3. Generating Matrix Function for Jacobi Matrix Polynomials

In this section, a generating matrix function satisfied by JMP is given.

Theorem 3.1. *Assume that A and B are commutative matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. A generating matrix function for JMP is*

$$\sum_{n=0}^{\infty} P_n^{(A,B)}(x)r^n = 2^{A+B}R^{-1}(1 - r + R)^{-A}(1 + r + R)^{-B}$$

where $R = (1 - 2xr + r^2)^{1/2}$ and $|r| < 1$.

Proof. Taking $\Phi(y) = \frac{y^2-1}{2}$ in Lemma 1.1, we have $y = \frac{1}{r} - \frac{R}{r}$. Taking $(1 - x)^A(1 + x)^B$ instead of $f'(x)$ in (8) and differentiating (8) with respect to x, we get

$$(1 - y)^A(1 + y)^B \frac{1}{R} = (1 - x)^A(1 + x)^B + \sum_{n=1}^{\infty} \frac{r^n}{n!} \frac{d^n}{dx^n} \left((1 - x)^A(1 + x)^B \left(\tfrac{x^2-1}{2}\right)^n\right).$$

Using (5) in this equation and multiplying $(1 - x)^{-A}(1 + x)^{-B}$, theorem can be proved. □

4. Integral Representations for Jacobi Matrix Polynomials

In this section, integral representations are given for JMP.

Theorem 4.1. *Let A, B, C and M be matrices in $\mathbb{C}^{N\times N}$ satisfying following conditions*
$Re(\mu) > 0$ for all eigenvalue $\mu \in \sigma(C)$,
$Re(\mu) > 0$ for all eigenvalue $\mu \in \sigma(M)$,
$C + M + kI$ is invertible for all natural number k,
and these matrices are commutative. Then

$$x^{C+M-I}\ F(A, B; C + M; x) = \Gamma(C + M)\Gamma^{-1}(C)\Gamma^{-1}(M) \int_0^x (x - t)^{M-I} t^{C-I}\ F(A, B; C; t)dt \tag{13}$$

Proof. *Starting right-side of the equation in (13) and using Beta matrix function in [12], theorem can be proved.* □

Theorem 4.2. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. Also, let M be matrix in $\mathbb{C}^{N\times N}$ whose eigenvalues, z, all satisfy $Re(z) > 0$ and these matrices be commutative. JMP satisfy following equalities:*

(i) $(1 - x)^{A+M} P_n^{(A+M,B-M)}(x) \left[P_n^{(A+M,B-M)}(1)\right]^{-1} =$

$$\Gamma(A + M + I)\Gamma^{-1}(A + I)\Gamma^{-1}(M) \int_x^1 (1 - y)^A P_n^{(A,B)}(y) \left[P_n^{(A,B)}(1)\right]^{-1} (y - x)^{M-I} dy$$

where $Re(\lambda) > -1$ and $Re(\mu) > -1$ for $\forall\lambda \in \sigma(A + M)$ and $\forall\mu \in \sigma(B - M)$.

(ii) $(1 + x)^{B+M} P_n^{(A-M,B+M)}(x) \left[P_n^{(B+M,A-M)}(1)\right]^{-1} =$

$$\Gamma(B + M + I)\Gamma^{-1}(B + I)\Gamma^{-1}(M) \int_{-1}^x (1 + y)^B P_n^{(A,B)}(y) \left[P_n^{(B,A)}(1)\right]^{-1} (x - y)^{M-I} dy$$

where $Re(\lambda) > -1$ and $Re(\mu) > -1$ for $\forall\lambda \in \sigma(A - M)$ and $\forall\mu \in \sigma(B + M)$.

(iii) $(1 - x)^{A+M}\ (1 + x)^{-A-nI-I} P_n^{(A+M,B)}(x) \left[P_n^{(A+M,B)}(1)\right]^{-1} =$

$$2^M \Gamma(A + M + I)\Gamma^{-1}(A + I)\Gamma^{-1}(M) \int_x^1 (1 - y)^A (1 + y)^{-A-M-nI-I} P_n^{(A,B)}(y) \left[P_n^{(A,B)}(1)\right]^{-1} (y - x)^{M-I} dy$$

where $Re(\lambda) > -1$ for $\forall\lambda \in \sigma(A + M)$.

(iv) $(1 + x)^{B+M}(1 - x)^{-B-nI-I} P_n^{(A,B+M)}(x) \left[P_n^{(B+M,A)}(1)\right]^{-1} =$

$$2^M \Gamma(B + M + I)\Gamma^{-1}(B + I)\Gamma^{-1}(M) \int_{-1}^x (1 + y)^B (1 - y)^{-B-M-nI-I} P_n^{(A,B)}(y) \left[P_n^{(B,A)}(1)\right]^{-1} (x - y)^{M-I} dy$$

where $Re(\lambda) > -1$ for $\forall\lambda \in \sigma(B + M)$.

Proof. ***(i)*** *To prove (i), taking $A \to -nI$, $B \to A + B + (n + 1) I$, $C \to A + I$, $x \to \frac{1-x}{2}$ and $t \to \frac{1-y}{2}$ in Theorem 4.1.*

(ii) *Taking $A \to B$ and $B \to A$ and $(-x)$ instead of x and $(-y)$ instead of y in equation **(i)** and using Theorem 2.1(**iii**), which completes of proof (**ii**).*

(iii) *Taking $x \to \frac{x}{x-1}$, $t \to \frac{t}{t-1}$ and $B \to C - B$ in Theorem 4.1, then taking $A \to -nI$, $B \to A + B + (n + 1) I$, $C \to A + I$, $x \to \frac{1-x}{2}$ and $t \to \frac{1-y}{2}$, theorem can be proved.*

(iv) *Taking $A \to B$ and $B \to A$ and $(-x)$ instead of x and $(-y)$ instead of y in equation **(iii)** , using Theorem 2.1(**iii**), we complete the proof.* □

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