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Approximation Theorems for q-Bernstein-Kantorovich Operators

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Abstract. In the present paper we introduce a q-analogue of the Bernstein-Kantorovich operators and investigate their approximation properties. We study local and global approximation properties and Voronovskaja type theorem for the q-Bernstein-Kantorovich operators in case $0 < q < 1$.

1. Introduction

In the last two decades interesting generalizations of Bernstein polynomials were proposed by Lupaş [15] and by Phillips [20]. Generalizations of the Bernstein polynomials based on the q-integers attracted a lot of interest and was studied widely by a number of authors. A survey of the obtained results and references on the subject can be found in [19]. Recently some new generalizations of well known positive linear operators, based on q-integers were introduced and studied by several authors, see [23], [5], [6], [8], [21], [22], [16].

The classical Kantorovich operator B_n^*, $n = 1, 2, ...$ is defined by (cf. [14])

$$B_n^*(f;x) := (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{k/n+1}^{k+1/n+1} f(t)\,dt$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_0^1 f\left(\frac{k+t}{n+1}\right)dt, \quad f:[0,1]\to\mathbb{R}. \tag{1}$$

These operators have been extensively considered in the mathematical literature. Also, a number of generalizations have been introduced by different authors (see, for instance [24], [25], [26]).

In this paper, inspired by (1), we introduce a q-type generalization of Bernstein-Kantorovich polynomial operators as follows.

$$B_{n,q}^*(f,x) := \sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 f\left(\frac{[k]+q^k t}{[n+1]}\right)d_q t,$$

where $f \in C[0,1]$, $0 < q < 1$.

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The paper is organized as follows. In Section 2, we give standard notations that will be used throughout the paper, introduce q-Bernstein-Kantorovich operators and evaluate the moments of $B_{n,q}^{*}$. In Section 3 we study local and global convergence properties of the q-Bernstein-Kantorovich operators and prove Voronovskaja-type asymptotic formula. In the final section we give statistical approximation result for the q-Bernstein-Kantorovich operators.

2. q-Bernstein-Kantorovich operators

Let $q > 0$. For any $n \in N \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + ... + q^{n-1}, \quad [0] := 0;$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! := [1]\,[2] \ldots [n], \quad [0]! := 1.$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right] := \frac{[n]!}{[k]!\,[n-k]!}.$$

The q-analogue of integration in the interval $[0, A]$ (see [13]) is defined by

$$\int_0^A f(t)\, d_q t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n)\, q^n, \quad 0 < q < 1.$$

Let $0 < q < 1$. Based on the q-integration we propose the Kantorovich type q-Bernstein polynomial as follows.

$$B_{n,q}^{*}(f, x) = \sum_{k=0}^{n} p_{n,k}(q; x) \int_0^1 f\left(\frac{[k] + q^k t}{[n+1]}\right) d_q t, \quad 0 \leq x \leq 1, n \in \mathbb{N}$$

where

$$p_{n,k}(q; x) := \left[\begin{array}{c} n \\ k \end{array} \right] x^k (1-x)_q^{n-k}, \quad (1-x)_q^n := \prod_{s=0}^{n-1} (1 - q^s x).$$

It can be seen that for $q \to 1^-$ the q-Bernstein-Kantorovich operator becomes the classical Bernstein-Kantorovich operator.

Lemma 2.1. *For all $n \in \mathbb{N}$, $x \in [0, 1]$ and $0 < q \leq 1$ we have*

$$B_{n,q}^{*}(t^m, x) = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) \frac{[n]^j}{[n+1]^m\,[m-j+1]} \sum_{i=0}^{m-j} \left(\begin{array}{c} m-j \\ i \end{array} \right) (q^n - 1)^i\, B_{n,q}\left(t^{j+i}, x\right). \quad (2)$$

Proof. The recurrence formula can be derived by direct computation.

$$\begin{aligned}
B_{n,q}^{*}\left(t^{m},x\right) &= \sum_{k=0}^{n} p_{n,k}\left(q;x\right) \sum_{j=0}^{m} \int_{0}^{1} \binom{m}{j} \frac{[k]^{j}\, q^{k(m-j)} t^{m-j}}{[n+1]^{m}} d_{q}t \\
&= \sum_{k=0}^{n} p_{n,k}\left(q;x\right) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{k(m-j)}\, [k]^{j}}{[n+1]^{m}\, [m-j+1]} \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j}}{[n+1]^{m}\, [m-j+1]} \sum_{k=0}^{n} \left(q^{k}-1+1\right)^{m-j} \frac{[k]^{j}}{[n]^{j}} p_{n,k}\left(q;x\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j}}{[n+1]^{m}\, [m-j+1]} \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} \left(q^{k}-1\right)^{i} \frac{[k]^{j}}{[n]^{j}} p_{n,k}\left(q;x\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j}}{[n+1]^{m}\, [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} \left(q^{n}-1\right)^{i} \sum_{k=0}^{n} \frac{[k]^{j+i}}{[n]^{j+i}} p_{n,k}\left(q;x\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]^{j}}{[n+1]^{m}\, [m-j+1]} \sum_{i=0}^{m-j} \binom{m-j}{i} \left(q^{n}-1\right)^{i} B_{n,q}\left(t^{j+i},x\right). \square
\end{aligned}$$

Lemma 2.2. *For all* $n \in \mathbb{N}$, $x \in [0,1]$ *and* $0 < q \leq 1$ *we have*

$$\begin{aligned}
&B_{n,q}^{*}\left(1,x\right) = 1, \quad B_{n,q}^{*}\left(t,x\right) = \frac{2q}{[2]} \frac{[n]}{[n+1]} x + \frac{1}{[2]} \frac{1}{[n+1]}, \\
&B_{n,q}^{*}\left(t^{2},x\right) = \frac{q\left(q+2\right)}{[3]} \frac{q\,[n]\,[n-1]}{[n+1]^{2}} x^{2} + \frac{4q+7q^{2}+q^{3}}{[2]\,[3]} \frac{[n]}{[n+1]^{2}} x + \frac{1}{[3]} \frac{1}{[n+1]^{2}}.
\end{aligned}$$

Proof. Taking into account (2), by direct computation, we obtain explicit formulas for $B_{n,q}^{*}\left(t,x\right)$ and $B_{n,q}^{*}\left(t^{2},x\right)$ as follows.

$$\begin{aligned}
B_{n,q}^{*}\left(t,x\right) &= \frac{1}{[n+1]\,[2]} \left(B_{n,q}\left(1,x\right) + \left(q^{n}-1\right) B_{n,q}\left(t,x\right)\right) + \frac{[n]}{[n+1]} B_{n,q}\left(t,x\right) \\
&= \left(\frac{q^{n}-1}{[2]\,[n+1]} + \frac{[n]}{[n+1]}\right) x + \frac{1}{[2]\,[n+1]} = \frac{2q}{[2]} \frac{[n]}{[n+1]} x + \frac{1}{[2]\,[n+1]}
\end{aligned}$$

and

$$\begin{aligned}
B_{n,q}^{*}\left(t^{2},x\right) &= \frac{1}{[3]\,[n+1]^{2}} \left(B_{n,q}\left(1,x\right) + 2\left(q^{n}-1\right) B_{n,q}\left(t,x\right) + \left(q^{n}-1\right)^{2} B_{n,q}\left(t^{2},x\right)\right) \\
&+ \frac{2\,[n]}{[2]\,[n+1]^{2}} \left(B_{n,q}\left(t,x\right) + \left(q^{n}-1\right) B_{n,q}\left(t^{2},x\right)\right) + \frac{[n]^{2}}{[n+1]^{2}} B_{n,q}\left(t^{2},x\right) \\
&= \frac{1}{[3]\,[n+1]^{2}} + \left(\frac{[n]^{2}}{[n+1]^{2}} + \frac{2\,[n]\left(q^{n}-1\right)}{[2]\,[n+1]^{2}} + \frac{\left(q^{n}-1\right)^{2}}{[3]\,[n+1]^{2}}\right)\left(1 - \frac{1}{[n]}\right) x^{2} \\
&+ \left(\frac{[n]^{2}}{[n]\,[n+1]^{2}} + \frac{2\,[n]\left(q^{n}-1\right)}{[2]\,[n]\,[n+1]^{2}} + \frac{\left(q^{n}-1\right)^{2}}{[3]\,[n]\,[n+1]^{2}} + \frac{2\,[n]}{[2]\,[n+1]^{2}} + \frac{2\left(q^{n}-1\right)}{[3]\,[n+1]^{2}}\right) x \\
&= \frac{2q+3q^{2}+q^{3}}{[2]\,[3]} \frac{q\,[n]\,[n-1]}{[n+1]^{2}} x^{2} + \frac{4q+7q^{2}+q^{3}}{[2]\,[3]} \frac{[n]}{[n+1]^{2}} x + \frac{1}{[3]\,[n+1]^{2}}. \square
\end{aligned}$$

Remark 2.3. *It is observed from the above lemma that for $q = 1$, we get the moments of the Bernstein-Kantorovich operators.*

Lemma 2.4. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q \leq 1$ we have*

$$B^*_{n,q}\left((t-x)^2, x\right) \leq \frac{4}{[n]}\left(x(1-x) + \frac{1}{[n]}\right), \quad B^*_{n,q}\left((t-x)^4, x\right) \leq \frac{C}{[n]^2}\left(x(1-x) + \frac{1}{[n]^2}\right),$$

where C is a positive absolute constant.

Proof. Note that estimation of the moments for the q-Bernstein operators is given in [17]. The proof is based on the estimations of the second and fourth order central moments of the q-Bersntein polynomials.

$$B_{n,q}\left((t-x)^2, x\right) = \frac{1}{[n]}x(1-x), \quad B_{n,q}\left((t-x)^4, x\right) \leq \frac{C}{[n]^2}x(1-x).$$

Indeed

$$\begin{aligned} &B^*_{n,q}\left((t-x)^2, x\right) \\ &= \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]+q^k t}{[n+1]} - x\right)^2 d_q t = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]} + \frac{[k]}{[n]} - x\right)^2 d_q t \\ &\leq 2\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]}\right)^2 d_q t + 2\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]}{[n]} - x\right)^2 d_q t \\ &\leq \frac{4}{[3][n+1]^2} + \frac{4}{[n+1]^2} + \frac{2}{[n]}x(1-x) \leq \frac{4}{[n]}\left(x(1-x) + \frac{1}{[n]}\right). \end{aligned}$$

A similar calculus reveals:

$$\begin{aligned} &B^*_{n,q}\left((t-x)^4, x\right) \\ &= \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]+q^k t}{[n+1]} - x\right)^4 d_q t = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]} + \frac{[k]}{[n]} - x\right)^4 d_q t \\ &\leq 4\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{q^k t}{[n+1]} - \frac{q^n [k]}{[n][n+1]}\right)^4 d_q t + 4\sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]}{[n]} - x\right)^4 d_q t \\ &\leq \frac{32}{[5][n+1]^4} + \frac{32}{[n+1]^4} + \frac{4}{[n]^2}Cx(1-x) \leq \frac{C}{[n]^2}\left(x(1-x) + \frac{1}{[n]^2}\right). \square \end{aligned}$$

Lemma 2.5. *Assume that $0 < q_n < 1$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}(t-x;x) = -\frac{1+a}{2}x + \frac{1}{2},$$

$$\lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}\left((t-x)^2;x\right) = -\frac{1}{3}x^2 - \frac{2}{3}ax^2 + x.$$

Proof. To prove the lemma we use formulas for $B^*_{n,q_n}(t;x)$ and $B^*_{n,q_n}\left(t^2;x\right)$ given in Lemma 2.2.

$$\begin{aligned} \lim_{n\to\infty} [n]_{q_n} B^*_{n,q_n}(t-x;x) &= \lim_{n\to\infty} \left\{ [n]_{q_n} \left(\frac{2q_n}{[2]_{q_n}} \frac{[n]_{q_n}}{[n+1]_{q_n}} - 1 \right) x + \frac{1}{[2]_{q_n}} \frac{[n]_{q_n}}{[n+1]_{q_n}} \right\} \\ &= \lim_{n\to\infty} \left\{ -\frac{[n]_{q_n}}{[n+1]_{q_n}} \frac{1+q_n^{n+1}}{[2]_{q_n}} x + \frac{1}{[2]_{q_n}} \frac{[n]_{q_n}}{[n+1]_{q_n}} \right\} = -\frac{1+a}{2}x + \frac{1}{2}. \end{aligned}$$

$$\lim_{n\to\infty}[n]_{q_n}B^*_{n,q}\left((t-x)^2,x\right)$$
$$=\lim_{n\to\infty}[n]_{q_n}\left(B^*_{n,q}\left(t^2,x\right)-x^2-2xB^*_{n,q}(t-x,x)\right)$$
$$=\lim_{n\to\infty}[n]_{q_n}\left(\frac{q_n(q_n+2)}{[3]_{q_n}}\frac{[n]^2_{q_n}-[n]_{q_n}}{[n+1]^2_{q_n}}-1\right)x^2+\lim_{n\to\infty}[n]_{q_n}\frac{4q_n+7q_n^2+q_n^3}{[2]_{q_n}[3]_{q_n}}\frac{[n]_{q_n}}{[n+1]^2_{q_n}}x$$
$$-\lim_{n\to\infty}[n]_{q_n}2xB^*_{n,q_n}(t-x,x)$$
$$=\lim_{n\to\infty}q_n(1-q_n^n)\left(2q_n+q_n^2+2\right)x^2-\lim_{n\to\infty}\left(4q_n+3q_n^2+2q_n^3\right)x^2+\lim_{n\to\infty}\frac{4q_n+7q_n^2+q_n^3}{[2]_{q_n}[3]_{q_n}}x$$
$$-\lim_{n\to\infty}[n]_{q_n}2xB^*_{n,q_n}(t-x,x)$$
$$=\frac{5}{3}(1-a)x^2-3x^2+2x+(1+a)x^2-x$$
$$=-\frac{1}{3}x^2-\frac{2}{3}ax^2+x.\square$$

3. Local and global approximation

We begin by considering the following K-functional:

$$K_2\left(f,\delta^2\right):=\inf\left\{\left\|f-g\right\|+\delta^2\left\|g''\right\|:g\in C^2[0,1]\right\},\quad \delta\geq 0,$$

where

$$C^2[0,1]:=\{g:g,g',g''\in C[0,1]\}.$$

Then, in view of a known result [7], there exists an absolute constant $C_0>0$ such that

$$K_2\left(f,\delta^2\right)\leq C_0\omega_2(f,\delta) \tag{3}$$

where

$$\omega_2(f,\delta):=\sup_{0<h\leq\delta}\sup_{x\pm h\in[0,1]}\left|f(x-h)-2f(x)+f(x+h)\right|$$

is the second modulus of smoothness of $f\in C[0,1]$.

Our first main result is stated below.

Theorem 3.1. *There exists an absolute constant $C>0$ such that*

$$\left|B^*_{n,q}(f;x)-f(x)\right|\leq C\omega_2\left(f,\sqrt{\frac{\delta_n(x)}{[n]}}\right)+\omega\left(f,\left|\frac{\left(1+q^{n+1}\right)x-1}{[2][n+1]}\right|\right),$$

where $f\in C[0,1]$, $\delta_n(x)=\varphi^2(x)+\frac{1}{[n]}$, $0\leq x\leq 1$ and $0<q<1$.

Proof. Let

$$\widetilde{B}^*_{n,q}(f;x)=B^*_{n,q}(f;x)+f(x)-f(a_nx+b_n),$$

where $f\in C[0,1]$, $a_n=\frac{2q}{1+q}\frac{[n]}{[n+1]}$ and $b_n=\frac{1}{1+q}\frac{1}{[n+1]}$. Using the Taylor formula

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-s)g''(s)\,ds,\quad g\in C^2[0,1],$$

we have

$$\widetilde{B}_{n,q}^{*}(g;x)=g(x)+B_{n,q}^{*}\left(\int_x^t (t-s)\,g''(s)\,ds;x\right)-\int_x^{a_nx+b_n}(a_nx+b_n-s)\,g''(s)\,ds,\quad g\in C^2[0,1].$$

Hence

$$\begin{aligned}\left|\widetilde{B}_{n,q}^{*}(g;x)-g(x)\right| &\le B_{n,q}^{*}\left(\left|\int_x^t |t-s|\left|g''(s)\right|ds\right|;x\right)+\left|\int_x^{a_nx+b_n}|a_nx+b_n-s|\left|g''(s)\right|ds\right|\\ &\le \left\|g''\right\|B_{n,q}^{*}\left((t-x)^2;x\right)+\left\|g''\right\|(a_nx+b_n-x)^2\\ &\le \left\|g''\right\|\left\{\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)+\frac{4}{[n]^2}x^2+\frac{2}{[n]^2}\right\}\\ &=\frac{10}{[n]}\delta_n(x)\left\|g''\right\|. \qquad (4)\end{aligned}$$

Using (4) and the uniform boundedness of $\widetilde{B}_{n,q}^{*}$ we get

$$\begin{aligned}\left|B_{n,q}^{*}(f;x)-f(x)\right| &\le \left|\widetilde{B}_{n,q}^{*}(f-g;x)\right|+\left|\widetilde{B}_{n,q}^{*}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|+\left|f(a_nx+b_n)-f(x)\right|\\ &\le 4\left\|f-g\right\|+\frac{10}{[n]}\delta_n(x)\left\|g''\right\|+\omega(f,|(a_n-1)x+b_n|).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in C^2[0,1]$, we obtain

$$\left|B_{n,q}^{*}(f;x)-f(x)\right|\le 10K_2\left(f;\frac{\delta_n(x)}{[n]}\right)+\omega(f,|(a_n-1)x+b_n|),$$

which together with (3) gives the proof of the theorem.

Corollary 3.2. *Assume that* $q_n\in(0,1)$, $q_n\to1$ *as* $n\to\infty$. *For any* $f\in C^2[0,1]$ *we have*

$$\lim_{n\to\infty}\left\|B_{n,q_n}^{*}(f)-f\right\|=0.$$

We next present the direct global approximation theorem for the operators $B_{n,q}^{*}$. In order to state the theorem we need the weighted K-functional of second order for $f\in C[0,1]$ defined by

$$K_{2,\phi}\left(f,\delta^2\right):=\inf\left\{\left\|f-g\right\|+\delta^2\left\|\phi^2g''\right\|:g\in W^2(\varphi)\right\},\quad \delta\ge0,\ \ \varphi^2(x)=x(1-x)$$

where

$$W^2(\varphi):=\left\{g\in C[0,1]:g'\in AC[0,1],\ \ \varphi^2g''\in C[0,1]\right\},$$

and $g'\in AC[0,1]$ means that g is differentiable and g' is absolutely continuous in $[0,1]$. Moreover, the Ditzian-Totik modulus of second order is given by

$$\omega_2^{\varphi}(f,\delta):=\sup_{0<h\le\delta}\ \sup_{x\pm h\varphi(x)\in[0,1]}\left|f(x-\varphi(x)h)-2f(x)+f(x+\varphi(x)h)\right|.$$

It is well known that the K-functional $K_{2,\varphi}\left(f,\delta^2\right)$ and the Ditzian-Totik modulus $\omega_2^{\varphi}(f,\delta)$ are equivalent (see [7]).

Now we state our next main result.

Theorem 3.3. *There exists an absolute constant $C > 0$ such that*

$$\left\|B_{n,q}^{*}(f)-f\right\| \leq C\omega_{2}^{\varphi}\left(f,\frac{1}{\sqrt{[n]}}\right)+\overrightarrow{\omega}_{\psi}\left(f,\frac{1}{[n]}\right),$$

where $f \in C[0,1]$, $0 < q < 1$, $\varphi^2(x) = x(1-x)$, $\psi(x) = 2x+1$.

Proof. Let

$$\widetilde{B}_{n,q}^{*}(f;x) = B_{n,q}^{*}(f;x) + f(x) - f(a_n x + b_n),$$

where $f \in C[0,1]$, $a_n = \frac{2q}{1+q}\frac{[n]}{[n+1]}$ and $b_n = \frac{1}{1+q}\frac{1}{[n+1]}$. Using the Taylor formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)\,g''(s)\,ds, \quad g \in W^2(\varphi),$$

we have

$$\widetilde{B}_{n,q}^{*}(g;x) = g(x) + B_{n,q}^{*}\left(\int_x^t (t-s)\,g''(s)\,ds;x\right) - \int_x^{a_n x+b_n} (a_n x + b_n - s)\,g''(s)\,ds, \quad g \in W^2(\varphi).$$

Hence

$$\left|\widetilde{B}_{n,q}^{*}(g;x) - g(x)\right| \leq B_{n,q}^{*}\left(\left|\int_x^t |t-s|\left|g''(s)\right|ds\right|;x\right) + \left|\int_x^{a_n x+b_n} |a_n x + b_n - s|\left|g''(s)\right|ds\right|. \tag{5}$$

Because the function δ_n^2 is concave on $[0,1]$, we have for $u = t + \tau(x-t)$, $\tau \in [0,1]$, the estimate

$$\frac{|t-s|}{\delta_n^2(s)} = \frac{\tau|x-t|}{\delta_n^2(s)} \leq \frac{\tau|x-t|}{\delta_n^2(t) + \tau\left(\delta_n^2(x) - \delta_n^2(t)\right)} \leq \frac{|x-t|}{\delta_n^2(x)}.$$

Hence, by (5), we find

$$\begin{aligned}
\left|\widetilde{B}_{n,q}^{*}(g;x) - g(x)\right| &\leq \left\|\delta_n^2 g''\right\| B_{n,q}^{*}\left(\left|\int_x^t \frac{|t-s|}{\delta_n^2(s)}ds\right|;x\right) + \left\|\delta_n^2 g''\right\|\left|\int_x^{a_n x+b_n} |a_n x + b_n - s|/\delta_n^2(s)\,ds\right| \\
&\leq \frac{\left\|\delta_n^2 g''\right\|}{\delta_n^2(x)}\left(B_{n,q}^{*}\left((t-x)^2;x\right) + (a_n x + b_n - x)^2\right) \\
&\leq \frac{\left\|\delta_n^2 g''\right\|}{\delta_n^2(x)}\left\{\frac{4}{[n]}\left(x(1-x) + \frac{1}{[n]}\right) + \frac{4}{[n]^2}x^2 + \frac{2}{[n]^2}\right\} \\
&\leq \frac{\left\|\delta_n^2 g''\right\|}{\delta_n^2(x)}\left\{\frac{10}{[n]}\left(x(1-x) + \frac{1}{[n]}\right)\right\} = \frac{10}{[n]}\left\|\delta_n^2 g''\right\|.
\end{aligned}$$

Since

$$\left\|\delta_n^2 g''\right\| \leq \left\|\varphi^2 g''\right\| + \frac{1}{[n+1]}\left\|g''\right\|$$

we have

$$\left|\widetilde{B}_{n,q}^{*}(g;x) - g(x)\right| \leq \frac{10}{[n]}\left(\left\|\varphi^2 g''\right\| + \frac{1}{[n]}\left\|g''\right\|\right). \tag{6}$$

Using (6) and the uniform boundedness of $\widetilde{B}^*_{n,q}$ we get

$$\begin{aligned}\left|B^*_{n,q}(f;x)-f(x)\right| &\le \left|\widetilde{B}^*_{n,q}(f-g;x)\right|+\left|\widetilde{B}^*_{n,q}(g;x)-g(x)\right|+\left|f(x)-g(x)\right|+\left|f(a_nx+b_n)-f(x)\right|\\ &\le 4\left\|f-g\right\|+\frac{10}{[n]}\left(\left\|\varphi^2g''\right\|+\frac{1}{[n]}\left\|g''\right\|\right)+\left|f(a_nx+b_n)-f(x)\right|.\end{aligned}$$

Taking the infimum on the right hand side over all $g\in W^2(\varphi)$, we obtain

$$\left|B^*_{n,q}(f;x)-f(x)\right|\le 10K_{2,\varphi}\left(f;\frac{1}{[n]}\right)+\left|f(a_nx+b_n)-f(x)\right|. \tag{7}$$

On the other hand

$$\begin{aligned}\left|f(a_nx+b_n)-f(x)\right| &= \left|f(x+\psi(x)((a_n-1)x+b_n))-f(x)\right|\\ &\le \sup\left|f\left(x+\psi(t)\left(-\frac{1+q^{n+1}}{\psi(x)[2][n+1]}x+\frac{1}{[2][n+1]\psi(x)}\right)\right)-f(x)\right|\\ &\le \overrightarrow{\omega}_\psi\left(f;\left|-\frac{1+q^{n+1}}{\psi(x)[2][n+1]}x+\frac{1}{[2][n+1]\psi(x)}\right|\right)\\ &\le \overrightarrow{\omega}_\psi\left(f;\frac{\left|B^*_{n,q}(t;x)-x\right|}{\psi(x)}\right)\le \overrightarrow{\omega}_\psi\left(f;\frac{2x+1}{[2][n]\psi(x)}\right).\end{aligned} \tag{8}$$

Hence, by (7) and (8), using the equivalence of $K_{2,\varphi}\left(f,\frac{1}{[n]}\right)$ and the Ditzian-Totik modulus $\omega_2^\varphi\left(f,\sqrt{\frac{1}{[n]}}\right)$ we get the desired estimate.

Next we prove Voronovskaja type result for q-Bernstein-Kantorovich operators.

Theorem 3.4. *Assume that $q_n\in(0,1)$, $q_n\to1$ and $q_n^n\to a$ as $n\to\infty$. For any $f\in C^2[0,1]$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(B^*_{n,q_n}(f;x)-f(x)\right)=f'(x)\left(-\frac{1+a}{2}x+\frac{1}{2}\right)+\frac{1}{2}f''(x)\left(-\frac{1}{3}x^2-\frac{2}{3}ax^2+x\right)$$

uniformly on $[0,1]$.

Proof. Let $f\in C^2[0,1]$ and $x\in[0,1]$ be fixed. By the Taylor formula we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{9}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x)\in C[0,1]$ and $\lim_{t\to x}r(t;x)=0$. Applying B^*_{n,q_n} to (9) we obtain

$$\begin{aligned}[n]_{q_n}\left(B^*_{n,q_n}(f;x)-f(x)\right)&=f'(x)[n]_{q_n}B^*_{n,q_n}(t-x;x)\\ &\quad+\frac{1}{2}f''(x)[n]_{q_n}B^*_{n,q_n}\left((t-x)^2;x\right)+[n]_{q_n}B^*_{n,q_n}\left(r(t;x)(t-x)^2;x\right).\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$B^*_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\le\sqrt{B^*_{n,q_n}(r^2(t;x);x)}\sqrt{B^*_{n,q_n}\left((t-x)^4;x\right)}. \tag{10}$$

Observe that $r^2(x;x)=0$ and $r^2(\cdot;x)\in C[0,1]$. Then it follows from Corollary 3.2 that

$$\lim_{n\to\infty}B^*_{n,q_n}\left(r^2(t;x);x\right)=r^2(x;x)=0 \tag{11}$$

uniformly with respect to $x\in[0,1]$. Now from (10), (11) and Lemma 2.5 we get immediately

$$\lim_{n\to\infty}[n]_{q_n}B^*_{n,q_n}\left(r(t;x)(t-x)^2;x\right)=0.$$

The proof is completed.

4. Statistical approximation

At this moment, we recall the concept of statistical convergence. The density of a subset K of $\mathbb{N}$ is given by $\delta(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k)$, whenever the limit exists, where χ_K is the characteristic function of K. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent to L if for any $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$ and it is denoted by $st - \lim x = L$ (see[10]).

Assume that $\{q_n\}_{n\in\mathbb{N}}$ be sequence from $(0,1]$ such that

$$st - \lim_n q_n = 1. \tag{12}$$

Observe that for any sequence $\{q_n\}_{n\in\mathbb{N}} \subset (0,1]$, satisfying (12) and for fixed $x \in [0,1]$, we have

$$st - \lim_n \frac{\delta_n(x)}{[n]_{q_n}} = st_A - \lim_n \left| \frac{\left(1 + q_n^{n+1}\right)x - 1}{[2]_{q_n}[n+1]_{q_n}} \right| = 0, \tag{13}$$

which yields

$$st - \lim_n \omega_2\left(f, \sqrt{\frac{\delta_n(x)}{[n]_{q_n}}}\right) = 0, \tag{14}$$

and

$$st - \lim_n \omega\left(f, \left| \frac{\left(1 + q_n^{n+1}\right)x - 1}{[2]_{q_n}[n+1]_{q_n}} \right|\right) = 0 \tag{15}$$

respectively. So, Theorem 3.1 gives the following statistical approximation theorem.

Theorem 4.1. *Assume that, $\{q_n\}_{n\in\mathbb{N}}$ is a sequence satisfying (12). Then, for all $f \in C[0,1]$ and fixed $x \in [0,1]$, we have*

$$st - \lim_n \left| B_{n,q}^*(f;x) - f(x) \right| = 0.$$

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