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T_0 and T_1 semiuniform convergence spaces

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Abstract. In previous papers, various notions of T_0 and T_1 objects in a topological category were introduced and compared. In this paper, we characterize each of these classes of objects in categories of various types of uniform convergence spaces and compare them with the usual ones as well as examine how these generalizations are related.

1. Introduction

Semiuniform convergence spaces which form a strong topological universe, i.e., a cartesian closed and hereditary topological category such that products of quotients are quotients, are introduced in [28–30], and [15]. It is well known, [28, 30], that the construct ***Conv*** of Kent convergence spaces can be bicoreflectively embedded in ***SUConv*** of semiuniform convergence spaces, and consequently, each semiuniform convergence spaces has an underlying Kent convergence space, namely its bicoreflective ***Conv***-modification. The strong topological universe ***SUConv*** contains all (symmetric) limit spaces as well as uniform convergence spaces [17] as a generalization of Weil's uniform spaces [31] and thus all (symmetric) topological spaces and all uniform spaces. Since topological and uniform concepts are available in ***SUConv***, it is shown, in [29], that semiuniform convergence spaces are the suitable framework for studying continuity, Cauchy continuity, uniform continuity, completeness, total boundedness, compactness, and connectedness as well as convergence structures in function spaces such as simple convergence, continuous convergence, and uniform convergence. There are other known attempts to embed topological and uniform spaces into a common topological supercategory (e.g. quasiuniform spaces by L. Nach [19], syntopogeneous spaces by A. Császár [18], generalized topological spaces (= super topological spaces) by D.B. Doitchinov [21], merotopopic spaces (= seminearness spaces) by M. Katétov [23], and nearness spaces by H. Herrlich [22]) that have not even led to cartesian closed topological categories.

Various generalizations of the usual separation properties at a point p are given in [2] and [3]. One of the uses of separation properties at a point p is to define the notions of (strong) closedness in set-based topological categories which are introduced in [2, 3]. These notions are used in [2, 8, 12, 13] to generalize each of the notions of compactness, connectedness, Hausdorffness, and perfectness to arbitrary set-based topological categories. Moreover, it is shown, in [11, 12, 14] that they form appropriate closure operators in the sense of Dikranjan and Giuli [20] in some well-known topological categories.

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There are several ways to generalize the usual T_0- axiom of topology to topological categories ([2, 5, 25, 32]) and the relationships among various forms of generalized T_0-axiom in topological categories have been investigated in ([5, 32]). One of the uses of T_0 objects is to define various forms of Hausdorff objects ([2, 6]) in arbitrary topological categories.

Also, there is a generalization of the usual T_1- axiom of topology to topological categories [2] and it is used to define each of T_3, T_4, regular, completely regular, and normal objects of an arbitrary topological category [9, 10].

The main goal of this paper is

1. to give the definition of each of the of T_i, $i = 0, 1$ semiuniform convergence spaces and to examine how these generalizations are related with the usual T_i, $i = 0, 1$ semiuniform convergence spaces,
2. to give the definition of each of the of T_i, $i = 0, 1$ semiuniform convergence spaces at a point p and to examine how these are related with the T_i, $i = 0, 1$ semiuniform convergence spaces.

2. Preliminaries

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$ if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small (i.e., sets) fibers, and for which every $\mathcal{U}-$source has an initial lift or, equivalently, for which each $\mathcal{U}-$sink has a final lift [1] or [27]. Note that a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding).

A filter α on a set B is said to be proper (improper) if and only if α does not contain (resp., α contains) the empty set, ϕ. Let $F(B)$ denote the set of filters on B. Let $M \subset B$ and $[M] = \{A \subset B : M \subset A\}$ and $[x] = [\{x\}]$. Note that $\alpha \cup \beta$ is the filter $[\{\mathrm{U} \cap \mathrm{V} \mid \mathrm{U} \in \alpha,\ \mathrm{V} \in \beta\}]$, $\alpha \cap \beta = [\{\mathrm{U} \cup \mathrm{V} \mid \mathrm{U} \in \alpha,\ \mathrm{V} \in \beta\}]$, and $\alpha \times \beta = [\{U \times V : U \in \alpha, V \in \beta\}]$. If $\alpha, \beta \in F(B \times B)$, then $\alpha^{-1} = \{U^{-1} : U \in \alpha\}$, where $U^{-1} = \{(x, y) : (y, x) \in U\}$. If $U \circ V = \{(x, y)$: there exists $z \in B$ with $(x, z) \in V$ and $(z, y) \in U\} \neq \phi$ for every $U \in \alpha$ and every $V \in \beta$, then $\alpha \circ \beta$ is the filter generated by $\{U \circ V : U \in \alpha, V \in \beta\}$.

Lemma 2.1. *For a set B, let σ and δ be filters on $B \times B$ and let $f : B \to C$ be a function. Then*
(i) $(f \times f)(\sigma \cap \delta) = (f \times f)(\sigma) \cap (f \times f)(\delta)$,
(ii) If $\sigma \subset \delta$, then $(f \times f)(\sigma) \subset (f \times f)(\delta)$.

Definition 2.2. (cf. [28, 30])

1. A semiuniform convergence space is a pair $(B, \mathfrak{I})$, where B is a set $\mathfrak{I}$ is a set of filters on $B \times B$ such that the following conditions are satisfied:
   (UC_1) $[x] \times [x]$ belongs to $\mathfrak{I}$ for each $x \in B$.
   (UC_2) $\beta \in \mathfrak{I}$ whenever $\alpha \in \mathfrak{I}$ and $\alpha \subset \beta$.
   (UC_3) $\alpha \in \mathfrak{I}$ implies $\alpha^{-1} \in \mathfrak{I}$.
   If $(B, \mathfrak{I})$ is a semiuniform convergence space, then the elements of $\mathfrak{I}$ are called uniform filters.
2. A semiuniform convergence space $(B, \mathfrak{I})$ is called a semiuniform limit space provided that the following is satisfied:
   (UC_4) $\alpha \in \mathfrak{I}$ and $\beta \in \mathfrak{I}$ imply $\alpha \cap \beta \in \mathfrak{I}$.
3. A semiuniform limit space $(B, \mathfrak{I})$ is called uniform limit space provided that the following is satisfied:
   (UC_5) $\alpha \in \mathfrak{I}$ and $\beta \in \mathfrak{I}$ imply $\alpha \circ \beta \in \mathfrak{I}$ (whenever $\alpha \circ \beta$ exists).
4. A map $f : (B, \mathfrak{I}) \to (B', \mathfrak{I}')$ between semiuniform convergence spaces is called uniformly continuous provided that $(f \times f)(\alpha) \in \mathfrak{I}'$ for each $\alpha \in \mathfrak{I}$.
5. The consctruct of semiuniform convergence spaces (and uniformly continuous maps) is denoted by ***SUConv***, whereas its full subconsctructs of semiuniform limit spaces and uniform limit spaces are denoted by ***SULim*** and ***ULim***, respectively.

2.2 A source $\{f_i : (B, \mathfrak{I}) \to (B_i, \mathfrak{I}_i), i \in I\}$ in **SUConv** is an initial lift if and only if $\alpha \in \mathfrak{I}$ precisely when $(f_i \times f_i)(\alpha) \in \mathfrak{I}_i$ for all $i \in I$ (cf. [28], [30] p. 33 or [16] p. 67).

2.3 An epi sink $\{f_i : (B_i, \mathfrak{I}_i) \to (B, \mathfrak{I})\}$ in **SUConv** is a final lift iff $\alpha \in \mathfrak{I}$ implies that there exist $i \in I$ and $\beta_i \in \mathfrak{I}_i$ such that $(f_i \times f_i)(\beta_i) \subset \alpha$ (cf. [28], [16] p. 67 or [30] p. 263).

2.4 The discrete semiuniform convergent structure $\mathfrak{I}_d$ on B is given by $\mathfrak{I}_d = \{[\phi], [x] \times [x] : x \in B\}$.

2.5 The indiscrete semiuniform convergent structure on B is given by $\mathfrak{I} = F(B \times B)$.

3. T_0 and T_1 semiuniform convergence spaces at a point

In this section, we give the definition of each of the of T_i, $i = 0, 1$ semiuniform convergence spaces at a point p.

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p [2], i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed p-axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$. The fold map at p, $\triangledown_p : B \vee_p B \to B$ is given by $\triangledown_p(x_i) = x$ for $i = 1, 2$ [2, 3].

Note that the maps A_p, S_p and $\triangledown_p$ are the unique maps arising from the above pushout diagram for which $A_p i_1 = (id, f)$, $S_p i_1 = (id, id) : B \to B^2$, $A_p i_2 = S_p i_2 = (f, id) : B \to B^2$, and $\triangledown_p i_j = id$, $j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $f : B \to B$ is the constant map at p [12].

Let $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$ and p is a point in B.

Definition 3.1. (cf. [2, 3])
(1) X is $\overline{T_0}$ at p if and only if the initial lift of the $\mathcal{U}$-source $\{A_p : B \bigvee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
(2) X is T_0' at p if and only if the initial lift of the $\mathcal{U}$-source $\{id : B \bigvee_p B \to \mathcal{U}(B \bigvee_p B)' = B \bigvee_p B$ and $\nabla_p : B \bigvee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $(B \bigvee_p B)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \bigvee_p B\}$, i_1 and i_2 are the canonical injections.
(3) X is T_1 at p if and only if the initial lift of the $\mathcal{U}$-source $\{S_p : B \bigvee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \bigvee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.

Remark 3.2. (1) Note that for the category $\textbf{\textit{Top}}$ of topological spaces, $\overline{T_0}$ at p, T_0' at p, or T_1 at p reduce to the usual T_0 at p or T_1 at p, respectively, where a topological space X is called T_0 at p (resp. T_1 at p) if for each $x \neq p$, there exists a neighborhood of x not containing p or (resp. and) there exists a neighborhood of p not containing x [7].
(2) A topological space X is T_i $i = 0, 1$ if and only if X is T_i, $i = 0, 1$, at p for all points p in X ([7], Theorem 1.5(5)).
(3) Let $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ be a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ be a retract of X, i.e., the initial lift $h : \bar{1} \to X$ of the $\mathcal{U}$-source $p : 1 \to \mathcal{U}(X)$ is a retract, where 1 is the terminal object in $\textbf{\textit{Set}}$. Then if X is $\overline{T_0}$ at p (resp. T_1 at p), then X is T_0' at p but the reverse implication is not true, in general ([4], Theorem 2.10).
(4) If $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ is a normalized topological functor, then each of $\overline{T_0}$ at p and T_1 at p implies T_0' at p ([4], Corollary 2.11).
(5) One of the uses of $\overline{T_0}$ at p and T_1 at p is to define the notions of (strong) closedness in set-based topological categories which are introduced in [2, 3]. These notions are used in [2, 8, 12, 13] to generalize each of the notions of compactness, connectedness, Hausdorffness, and perfectness to arbitrary set-based topological categories. Moreover, it is shown, in [11, 12, 14] that they form appropriate closure operators in the sense of Dikranjan and Giuli [20] in some well-known topological categories.

Theorem 3.3. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and let $p \in B$. $(B, \mathfrak{I})$ is T_1 at p if and only if for each $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$.*

Proof. Suppose that $(B, \mathfrak{I})$ is T_1 at p. If $[x] \times [p] \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = [x_1] \times [x_2]$. Clearly, $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [x] \times [p] \in \mathfrak{I}$, $(\pi_2 S_p \times \pi_2 S_p)(\alpha) = [x] \times [x] \in \mathfrak{I}$, where $\pi_i : B^2 \to B$, $i = 1, 2$, are the projection maps, and $(\triangledown_p \times \triangledown_p)([x_1] \times [x_2]) = [x] \times [x] \in \mathfrak{I}_d$, the discrete semiuniform convergence structure on B. Since $(B, \mathfrak{I})$ is T_1 at p, we get a contradiction. Hence, $[x] \times [p] \notin \mathfrak{I}$ for all $x \neq p$.

If $([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$ for some $x \neq p$, then let $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$. By Lemma 2.1 (*i*), $\left(\pi_1 S_p \times \pi_1 S_p\right)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$, $\left(\pi_2 S_p \times \pi_2 S_p\right)(\alpha) = [x] \times [x] \in \mathfrak{I}$, $(\triangledown_p \times \triangledown_p)(\alpha) = [x] \times [x] \in \mathfrak{I}_d$, a contradiction since $(B, \mathfrak{I})$ is T_1 at p. Hence, we must have $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$ for all $x \neq p$.

Conversely, suppose that for each $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$ and $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$. We need to show that $(B, \mathfrak{I})$ is T_1 at p ,i.e., by 2.2, 2.4, and Definition 3.1, we must show that the semiuniform convergence structure $\mathfrak{I}_W$ on $B \vee_p B$ induced by $S_p : B \vee_p B \to U((B^2, \mathfrak{I}^2)) = B^2$ and $\triangledown_p : B \vee_p B \longrightarrow U((B, \mathfrak{I}_d)) = B$ is discrete, where $\mathfrak{I}^2$ and $\mathfrak{I}_d$ are the product semiuniform convergence structure on B^2 and the discrete semiuniform convergence structure on B, respectively. Let α be any filter in $\mathfrak{I}_W$, i.e., $(\pi_i S_p \times \pi_i S_p)(\alpha) \in \mathfrak{I}$ $i = 1, 2$ and $\left(\triangledown_p \times \triangledown_p\right)(\alpha) \in \mathfrak{I}_d$. We need to show that $\alpha = [x_i] \times [x_i]$ $(i = 1, 2)$ or $\alpha = [p] \times [p]$ or $\alpha = [\phi]$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [p] \times [p]$, then $\alpha = [p_i] \times [p_i]$ $(i = 1, 2)$ since $\left(\triangledown_p\right)^{-1}\{p\} = \{p_i = (p, p)\}$ $(i = 1, 2)$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [\phi]$, then $\alpha = [\phi]$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [x] \times [x]$ for some $x \in B$, then it follows easily that $\alpha = [x_i] \times [x_j]$ $(i, j = 1, 2)$ or $\alpha \supset [\{x_1, x_2\}] \times [\{x_1, x_2\}]$ or $\alpha \supset [x_i] \times [\{x_i, x_j\}]$ or $\alpha \supset [\{x_i, x_j\}] \times [x_i]$ $(i, j = 1, 2$ and $i \neq j)$.

If $\alpha = [x_i] \times [x_j]$, $i \neq j$, then, in particular, $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [x] \times [p]$ or $[p] \times [x] \in \mathfrak{I}$ $(i = 1, j = 2$ or $i = 2, j = 1$, respectively), a contradiction. Hence, $\alpha \neq [x_i] \times [x_j]$, $i \neq j$.

If $\alpha = [x_i] \times [\{x_i, x_j\}]$, $i \neq j$, then (for $i = 1$) $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [x] \times [\{x, p\}] \subset [x] \times [p]$ and consequently $[x] \times [p] \in \mathfrak{I}$, a contradiction, (for $i = 2$) $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [p] \times [\{p, x\}] \subset [p] \times [x]$ and consequently, $[p] \times [x] \in \mathfrak{I}$, a contradiction. Hence, $\alpha \neq [x_i] \times [\{x_i, x_j\}]$, $i \neq j$.

We next show that if $[\phi] \neq \alpha \neq [x_i] \times [\{x_i, x_j\}]$ $(i \neq j)$, then $\alpha \supset [x_i] \times [\{x_i, x_j\}]$ if and only if $\alpha = [x_i] \times [x_j]$ or $[x_i] \times [x_i]$. If $\alpha = [x_i] \times [x_j]$ or $[x_i] \times [x_i]$, then clearly $\alpha \supset [x_i] \times [\{x_i, x_j\}]$. If $\alpha \supset [x_i] \times [\{x_i, x_j\}]$ and $[\phi] \neq \alpha \neq [x_i] \times [\{x_i, x_j\}]$, then there exist $U \in \alpha$ such that $U \neq \phi$, $U \neq \{x_i\} \times \{x_i, x_j\}$. Since $U \in \alpha$, $\{x_i\} \times \{x_i, x_j\} \in \alpha$, and α is a filter, then $U \cap (\{x_i\} \times \{x_i, x_j\}) = \{x_i\} \times \{x_i\}$ or $\{x_i\} \times \{x_j\}$ is in α i.e., $\alpha = [x_i] \times [x_i]$ or $[x_i] \times [x_j]$. We have already shown that $\alpha \neq [x_i] \times [x_j]$, $i \neq j$. Hence, $\alpha = [x_i] \times [x_i]$, $i = 1, 2$.

The case $\alpha \supset [\{x_i, x_j\}] \times [x_i]$ can be handled similarly.

If $\alpha = [\{x_1, x_2\}] \times [\{x_1, x_2\}]$, then, in particular, $(\pi_1 S_p \times \pi_1 S_p)(\alpha) = [\{x, p\}] \times [\{x, p\}] \subset [x] \times [p]$ and consequently, $[x] \times [p] \in \mathfrak{I}$, a contradiction. Hence, $\alpha \neq [\{x_1, x_2\}] \times [\{x_1, x_2\}]$.

If $\alpha \supset [\{x_1, x_2\}] \times [\{x_1, x_2\}]$ and $[\phi] \neq \alpha \neq [\{x_1, x_2\}] \times [\{x_1, x_2\}]$, then there exist $U \in \alpha$ such that $U \neq \phi$ and $U \neq \{x_1, x_2\} \times \{x_1, x_2\}$. Since $U \in \alpha$, $\{x_1, x_2\} \times \{x_1, x_2\} \in \alpha$, and α is a filter, then $U \cap (\{x_1, x_2\} \times \{x_1, x_2\}) \in \alpha$. Note that $U \cap (\{x_1, x_2\} \times \{x_1, x_2\}) = \{(x_i, x_j)\}$ or $\{x_i\} \times \{(x_i, x_j)\}$ or $\{(x_i, x_j)\} \times \{x_i\}$ or $\{(x_1, x_2), (x_2, x_1)\}$ or $\{(x_1, x_1), (x_2, x_2)\}$ or $(\{x_1\} \times \{x_1, x_2\}) \cup (\{(x_2, x_1)\})$ or $(\{x_1\} \times \{x_1, x_2\}) \cup (\{(x_2, x_2)\})$ or $(\{x_1, x_2\} \times \{x_1\}) \cup (\{x_2, x_2)\}$ or $(\{x_1, x_2\} \times \{x_2\}) \cup (\{(x_2, x_1)\})$.

If $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$, then $\left(\pi_1 S_p \times \pi_1 S_p\right)(\alpha) = ([x] \times [x]) \cap ([p] \times [p]) \in \mathfrak{I}$, a contradiction.

If $\alpha = ([x_1] \times [x_2]) \cap ([x_2] \times [x_1])$, then $\left(\pi_1 S_p \times \pi_1 S_p\right)(\alpha) = [\{x, p\}] \cap [\{x, p\}] \subset [x] \times [p]$ and consequently, $[x] \times [p] \in \mathfrak{I}$, a contradiction. If $\alpha = ([\{x_1\} \times \{x_1, x_2\}]) \cap ([x_2] \times [x_1])$, then $\left(\pi_1 S_p \times \pi_1 S_p\right)(\alpha) = ([x] \times [\{x, p\}]) \cap ([p] \times [x]) \subset [x] \times [p]$ and consequently, $[x] \times [p] \in \mathfrak{I}$, a contradiction.

By using the similar argument as above, for the remaining of α's, we must have $\alpha = [x_i] \times [x_i]$ $i = 1, 2$ and consequently, by Definition 3.1, 2.2, and 2.4, $(B, \mathfrak{I})$ is T_1 at p. □

Theorem 3.4. *Let $(B, \mathfrak{I})$ be a semiuniform convergence space and $p \in B$. $(B, \mathfrak{I})$ is $\overline{T}_0$ at p iff for each $x \neq p$ in B, the following conditions hold.*

(i) $[x] \times [p] \notin \mathfrak{I}$

(ii) $([x] \times [x]) \cap ([p] \times [p]) \notin \mathfrak{I}$ *or* $([p] \times [p]) \cap ([x] \times [x]) \notin \mathfrak{I}$.

Proof. Note that if $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$, then by Lemma 2.1(i), $\left(\pi_1 A_p \times \pi_1 A_p\right)(\alpha) = \left(\pi_1 A_p \times \pi_1 A_p\right)(([x_1] \times [x_1]) \cap ([x_2] \times [x_2])) = ([x] \times [x]) \cap ([p] \times [p])$ and $\left(\pi_2 A_p \times \pi_2 A_p\right)(\alpha) = ([p] \times [p]) \cap ([x] \times [x])$. By using the same argument in the proof of Theorem 3.3 and replacing S_p by A_p, we obtain the proof. □

Theorem 3.5. *All semiuniform convergence spaces are $T_0^{'}$ at p.*

Proof. Suppose that $(B, \mathfrak{I})$ is a semiuniform convergence space and and $p \in B$. By Definition 3.1, 2.2, 2.4, we will show that for any filter α on $(B \vee_p B)^2$, if $\alpha \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$, $k = 1$ or 2 and $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [\phi]$ or $[p] \times [p]$ or $[x] \times [x]$ for some $x \in B$, then $\alpha = [\phi]$ or $[p] \times [p]$ or $[x_m] \times [x_m]$, $m = 1, 2$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [p] \times [p]$, then $\alpha = [p] \times [p]$, $(m = 1, 2)$ since $\left(\triangledown_p\right)^{-1}\{p\} = \{p_m = p\}$ $(m = 1, 2)$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [\phi]$, then $\alpha = [\phi]$.

If $\left(\triangledown_p \times \triangledown_p\right)(\alpha) = [x] \times [x]$ for some $x \neq p$ in B, then it follows easily that $\alpha = [x_m] \times [x_n]$ $(m, n = 1, 2)$ or $\alpha \supset [x_m] \times [\{x_m, x_n\}]$ $(m, n = 1, 2)$ or $\alpha \supset [\{x_m, x_n\}] \times [x_m]$ or $\alpha \supset [\{x_1, x_2\}] \times [\{x_1, x_2\}]$.

If $\alpha = [x_1] \times [x_2] \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), then $\{x_1\} \times \{x_2\} \in (i_1 \times i_1)(V)$ for all $V \in \beta$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction since $x \neq p$.

If $\alpha = [x_2] \times [x_1] \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), then $\{x_2\} \times \{x_1\} \in (i_1 \times i_1)(V)$ for all $V \in \beta$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction since $x \neq p$.

If $\alpha = [x_m] \times [\{x_m, x_n\}] \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), $(m \neq n,\ m, n = 1, 2)$. Then it follows easily that $\{x_1\} \times \{x_1, x_2\} \subset (i_1 \times i_1)(V)$ for all $V \in \beta$ and consequently x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction.

If $[\phi] \neq \alpha \neq [x_m] \times [\{x_m, x_n\}]$ and $\alpha \supset [x_m] \times [\{x_m, x_n\}]$ $(m \neq n,\ m, n = 1, 2)$, then $\alpha = [x_m] \times [x_m]$ or $\alpha = [x_m] \times [x_n]$ (see the proof of Theorem 3.1). By the same argument used above, $\alpha = [x_m] \times [x_n]$, $m \neq n$, $m, n = 1, 2$ can not occur. Similarly if $[\phi] \neq \alpha \neq [\{x_m, x_n\}] \times [x_m]$ and $\alpha \supset [\{x_m, x_n\}] \times [x_m]$ $(m \neq n,\ m, n = 1, 2)$, then $\alpha = [x_m] \times [x_m]$.

If $\alpha = [\{x_1, x_2\}] \times [\{x_1, x_2\}] \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), then $\{x_1, x_2\} \times \{x_1, x_2\} \in (i_1 \times i_1)(V)$ for all $V \in \beta$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction since $x \neq p$.

If $[\phi] \neq \alpha \neq [\{x_1, x_2\}] \times [\{x_1, x_2\}]$ and $\alpha \supset [\{x_1, x_2\}] \times [\{x_1, x_2\}]$, then (see the proof of Theorem 3.1) $\alpha = [x_m] \times [x_m]$ or $\alpha = ([x_1] \times [x_2\}]) \cap ([x_2] \times [x_1\}]$ or $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$.

If $\alpha = ([x_1] \times [x_2]) \cap ([x_2] \times [x_1\}] \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), then $(\{x_1\} \times \{x_2\}) \cup (\{x_2\} \times \{x_1\}) \in (i_1 \times i_1)(V)$ for all $V \in \beta$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction since $x \neq p$.

If $\alpha = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2]) \supset (i_k \times i_k)(\beta)$ for some $\beta \in \mathfrak{I}$ and $k = 1$ (resp. $k = 2$), then $(\{x_1\} \times \{x_1\}) \cup (\{x_2 \times x_2\}) \in (i_1 \times i_1)(V)$ for all $V \in \beta$ which shows that x_2 (resp. x_1) must be in the first (resp. second) component of $B \vee_p B$, a contradiction since $x \neq p$. Hence, we must have $\alpha = [x_m] \times [x_m]$, $m = 1, 2$. By 2.2, 2.4 and Definition 3.1, $(B, \mathfrak{I})$ is $T_0^{'}$ at p. □

Remark 3.6. Let $(B, \mathfrak{I})$ be a semiuniform limit space (uniform limit space) and $p \in B$. It follows from Theorem 3.3, Theorem 3.4 and Definition 2.2 that $(B, \mathfrak{I})$ is $\overline{T}_0$ at p if and only if $(B, \mathfrak{I})$ is T_1 at p if and only if for each $x \neq p$, $[x] \times [p] \notin \mathfrak{I}$.

4. T_0 and T_1 semiuniform convergence spaces

There are several ways to generalize the usual T_0- axiom of topology to topological categories ([2, 5, 25, 32]) and the relationships among various forms of generalized T_0-axiom in topological categories have been investigated in [5, 32]. One of the uses of T_0 objects is to define various forms of Hausdorff objects ([2, 6]) in arbitrary topological categories.

Also, there is a generalization of the usual T_1- axiom of topology to topological categories [2] and it is used to define each of T_3, T_4, regular, completely regular, and normal objects of an arbitrary topological category [9] and [10].

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal, i.e., the result of pushing out Δ along itself. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$ ($(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \vee_\Delta B^2$. Clearly $(x, y)_1 = (x, y)_2$ if and only if $x = y$ [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [2].

Definition 4.1. Let $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.

1. X is $\overline{T_0}$ if and only if the initial lift of the $\mathcal{U}$ -source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$ [2].
2. X is T_0' if and only if the initial lift of the $\mathcal{U}$ -source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$, i_1 and i_2 are the canonical injections, and $\mathcal{D}(B^2)$ is the discrete structure on B^2 [2, 5].
3. X is T_0 if and only if X does not contain an indiscrete subspace with (at least) two points [25, 32].
4. X is T_1 if and only if the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete [2].

Remark 4.2. (1) Note that for the category ***Top*** of topological spaces, $\overline{T_0}$, T_0', T_0 or T_1 reduce to the usual T_0, or T_1 separation axioms, respectively [2, 25, 32].

(2) For an arbitrary topological category, we have $\overline{T}_0$ implies T_0' ([5], Theorem 3.2) but the reverse implication is generally not true (see [5] or Theorem 4.2 and Theorem 4.3, below). Moreover, there are no implications between T_0 and each of $\overline{T}_0$ and T_0' (see [5], Remark 3.6).

(3) Let $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ be a topological functor, X an object in $\mathcal{E}$ and $p \in \mathcal{U}(X)$ be a retract of X, i.e., the initial lift $h : \bar{1} \to X$ of the $\mathcal{U}$-source $p : 1 \to \mathcal{U}(X)$ is a retract, where 1 is the terminal object in ***Set***. Then if X is $\overline{T}_0$ (resp. T_1), then X is $\bar{T}_0$ at p (resp. T_1 at p) but the reverse implication is not true, in general ([4], Theorem 2.6).

(4) If $\mathcal{U} : \mathcal{E} \to \textbf{\textit{Set}}$ is a normalized topological functor, then $\overline{T}_0$ (resp. T_1) implies $\overline{T_0}$ at p (resp. T_1 at p) ([4], Corollary 2.9).

Theorem 4.3. *A semiuniform convergence space $(B, \mathfrak{I})$ is T_0 if and only if for each distinct pair x and y in B, $[\{x, y\}] \times [\{x, y\}] \notin \mathfrak{I}$.*

Proof. Suppose that $(B, \mathfrak{I})$ is T_0 and $[\{x, y\}]\times[\{x, y\}] \in \mathfrak{I}$ for some distinct pair x and y in B. Let $A = \{x, y\}$. Note that $(A, \mathfrak{I}_A)$ is subspace of $(B, \mathfrak{I})$, where $\mathfrak{I}_A$ is the subsemiuniform convergence structure on A induced by the inclusion map $i : A \to B$. Since $(i\times i)([\{x, y\}]\times[\{x, y\}] = [A]\times[A] \in \mathfrak{I}$, it follows from 2.2 that $[A]\times[A] \in \mathfrak{I}_A$ and consequently, $\mathfrak{I}_A = F(A \times A)$, the indiscrete semiuniform structure on $A \times A$, a conctradiction. Hence, we must have for each distinct pair x and y in B, $[\{x, y\}] \times [\{x, y\}] \notin \mathfrak{I}$.

Conversely, suppose that $[\{x, y\}] \times [\{x, y\}] \notin \mathfrak{I}$ for each distinct pair x and y in B. Let $A = \{x, y\} \subset B$ with $x \neq y$. Note that $(A, \mathfrak{I}_A)$ is not an indiscrete semiuniform convergence subspace of $(B, \mathfrak{I})$. Hence, by Definition 4.1, $(B, \mathfrak{I})$ is T_0. □

Theorem 4.4. *A semiuniform convergence space $(B, \mathfrak{I})$ is $\overline{T}_0$ if and only if for every distinct pair x and y in B, the conditions (i) and (ii) hold.*

(i) $[x] \times [y] \notin \mathfrak{I}$

(ii) $([x] \times [x]) \cap ([y] \times [y]) \notin \mathfrak{I}$

Proof. Suppose that $(B, \Im)$ is $\overline{T}_0$ and either (*i*) or (*ii*) does not hold. If the condition (*i*) does not hold, then let $\alpha = [(x, y)_1] \times [(x, y)_2]$. Note that, $(\pi_1 A \times \pi_1 A)(\alpha) = [x] \times [x] \in \Im$, $(\pi_2 A \times \pi_2 A)(\alpha) = [y] \times [x] \in \Im$, $(\pi_3 A \times \pi_3 A)(\alpha) = [x] \times [y] \in \Im$, where $\pi_i : B^3 \to B$, $i = 1, 2, 3$ are the projection maps, and $(\nabla \times \nabla)(\alpha) = [(x, y)] \times [(x, y)] \in \Im^2_d$, the discrete semiuniform convergence structure on B^2, a contradiction since $(B, \Im)$ is $\overline{T}_0$. Hence, $[x] \times [y] \notin \Im$ or $[y] \times [x] \notin \Im$ for each $x \neq y$.

If the condition (*ii*) does not hold, then let $\alpha = ([(x, y)_1] \times [(x, y)_1]) \cap ([(x, y)_2] \times [(x, y)_2])$. Clearly, by Lemma 2.1 (*i*), $(\pi_1 A \times \pi_1 A)(\alpha) = [x] \times [x \in \Im$, $(\pi_i A \times \pi_i A)(\alpha) = ([y] \times [[y]) \cap ([x] \times [[x]) \in \Im$, $i = 2, 3$, and $(\nabla \times \nabla)(\alpha) = [(x, y)] \times [(x, y)] \in \Im^2_d$, a contradiction since $(B, \Im)$ is $\overline{T}_0$. Hence, $([x] \times [x]) \cap ([y] \times [y]) \notin \Im$ for all $x \neq y$.

Conversely, suppose that the conditions hold. We need to show that $(B, \Im)$ is $\overline{T}_0$, i.e., by Definition 4.1, 2.2, and 2.4, we must show that the semiuniform convergence structure $\Im^A_W$ on $B^2 \vee_\Delta B^2$ induced by $A : B^2 \vee_\Delta B^2 \to \mathcal{U}\left((B^3, \Im^3)\right) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \longrightarrow \mathcal{U}\left((B^2, \Im^2_d)\right) = B^2$ is discrete, where $\Im^3$ and $\Im^2_d$ are the product semiuniform convergence structure on B^3 and the discrete semiuniform convergence structure on B^2, respectively. Let α be any filter in $\Im^A_W$, i.e., $(\pi_i A \times \pi_i A)(\alpha) \in \Im$ $i = 1, 2, 3$ and $(\nabla \times \nabla)(\alpha) \in \Im^2_d$. We need to show that $\alpha = [(x, y)_i] \times [(x, y)_i]$ $(i = 1, 2)$ or $\alpha = [\phi]$.

If $(\nabla \times \nabla)(\alpha) = [\phi]$, then $\alpha = [\phi]$.

If $(\nabla \times \nabla)(\alpha) = [(x, x)] \times [(x, x)]$ for some $x \in B$, then $\alpha = [(x, x)_i] \times [(x, x)_i]$ since $\nabla^{-1}(\{(x, x)\}) = \{(x, x)_i = (x, x)\}$, $i = 1, 2$.

If $(\nabla \times \nabla)(\alpha) = [(x, y)] \times [(x, y)]$ for some $(x, y) \in B^2$ with $x \neq y$, then it follows that $\alpha = [(x, y)_i] \times [(x, y)_j]$ $(i, j = 1, 2)$ or $\alpha \supset [\{(x, y)_i, (x, y)_j\}] \times [\{(x, y)_i, (x, y)_j\}]$ or

$\alpha \supset [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$

or $\alpha \supset [\{(x, y)_i, (x, y)_j\}] \times [(x, y)_i]$ $(i, j = 1, 2$ and $i \neq j)$.

If $\alpha = [(x, y)_i] \times [(x, y)_j]$, $i \neq j$, then, in particular, $(\pi_2 A \times \pi_2 A)(\alpha) = [x] \times [y]$ or $[y] \times [x] \in \Im$ $(i = 2, j = 1$ or $i = 1, j = 2$, respectively), a contradiction. Hence, $\alpha \neq [(x, y)_i] \times [(x, y)_j]$, $i \neq j$.

If $\alpha = [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$, $i \neq j$, then (for $i = 1$) $(\pi_2 A \times \pi_2 A)(\alpha) = [y] \times [\{y, x\}] \subset [y] \times [x]$ and (for $i = 2$) $(\pi_2 A \times \pi_2 A)(\alpha) = [x] \times [\{x, y\}] \subset [x] \times [y]$ and consequently $[y] \times [x] \in \Im$, $[x] \times [y] \in \Im$, a contradiction. Hence, $\alpha \neq [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$, $i \neq j$.

We next show that if $[\phi] \neq \alpha \neq [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$ $(i \neq j)$, then $\alpha \supset [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$ if and only if $\alpha = [(x, y)_i] \times [(x, y)_j]$ or $[(x, y)_i] \times [(x, y)_i]$. If $\alpha = [(x, y)_i] \times [(x, y)_j]$ or $[(x, y)_i] \times [(x, y)_i]$, then clearly, $\alpha \supset [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$. If $\alpha \supset [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$ and $[\phi] \neq \alpha \neq [(x, y)_i] \times [\{(x, y)_i, (x, y)_j\}]$, then there exists $U \in \alpha$ such that $U \neq \phi$, $U \neq \{(x, y)_i\} \times \{(x, y)_i, (x, y)_j\}$. Since $U \in \alpha$, $\{(x, y)_i\} \times \{(x, y)_i, (x, y)_j\} \in \alpha$, and α is a filter, then $U \cap (\{(x, y)_i\} \times \{(x, y)_i, (x, y)_j\}) = \{(x, y)_i\} \times \{(x, y)_i\}$ or $\{(x, y)_i\} \times \{(x, y)_j\}$ is in α i.e., $\alpha = [(x, y)_i] \times [(x, y)_i]$ or $[(x, y)_i] \times [(x, y)_j]$. We have already shown that $\alpha \neq [(x, y)_i] \times [(x, y)_j]$, $i \neq j$. Hence, $\alpha = [(x, y)_i] \times [(x, y)_i]$, $i = 1, 2$.

Similarly, the case $\alpha \supset [\{(x, y)_i, (x, y)_j\}] \times [(x, y)_i]$ can be handled.

If $\alpha = ([(x, y)_1] \times [(x, y)_1]) \cap ([(x, y)_2] \times [(x, y)_2])$, then $(\pi_2 A \times \pi_2 A)(\alpha) = ([x] \times [x]) \cap ([y] \times [y]) \in \Im$, a contradiction. If $[\phi] \neq \alpha \neq ([(x, y)_1] \times [(x, y)_1]) \cap ([(x, y)_2] \times [(x, y)_2])$ and $\alpha \supset ([(x, y)_1] \times [(x, y)_1]) \cap ([(x, y)_2] \times [(x, y)_2])$, then $\alpha = [(x, y)_i] \times [(x, y)_i]$ $i = 1, 2$ (see the proof of Theorem 3.3).

If $\alpha = [\{(x, y)_1, (x, y)_2\}] \times [\{(x, y)_1, (x, y)_2\}]$, then $(\pi_2 A \times \pi_2 A)(\alpha) = [\{x, y\}] \times [\{x, y\}] \subset [x] \times [y]$ and $[\{x, y\}] \times [\{x, y\}] \subset [y] \times [x]$, and consequently, $[x] \times [y] \in \Im$ and $[y] \times [x] \in \Im$, a contradiction. Hence, $\alpha \neq [\{(x, y)_1, (x, y)_2\}] \times [\{(x, y)_1, (x, y)_2\}]$.

If $[\phi] \neq \alpha \neq [\{(x, y)_1, (x, y)_2\}] \times [\{(x, y)_1, (x, y)_2\}]$ and $\alpha \supset [\{(x, y)_1, (x, y)_2\}] \times [\{(x, y)_1, (x, y)_2\}]$, then by using a similar argument as above and in the proof of Theorem 3.3, we must have $\alpha = [(x, y)_i] \times [(x, y)_i]$ $i = 1, 2$.

Hence, by Definition 4.1, 2.2, and 2.4, $(B, \Im)$ is $\overline{T}_0$. □

Theorem 4.5. *All semiuniform convergence spaces are* T'_0.

Proof. Let $(B, \Im)$ be any semiuniform convergence space. By Definition 4.1, 2.2, and 2.4, we will show that for any filter α on $(B^2 \vee_\Delta B^2)^2$, if $\alpha \supset (i_k \times i_k)(\beta)$ for some $\beta \in \Im^2$, $k = 1$ or 2 and $(\nabla \times \nabla)(\alpha) = [\phi]$ or $[(x, y)] \times [(x, y)] \in \Im^2_d$ for some $(x, y) \in B^2$, where $\Im^2$ and $\Im^2_d$ are the product semiuniform convergence structure and the discrete semiuniform convergence structure on B^2, respectively, then $\alpha = [\phi]$ or $[(x, y)_n] \times [(x, y)_n]$, $n = 1, 2$.

If $(\nabla\times\nabla)(\alpha)=[\phi]$, then $\alpha=[\phi]$. If $(\nabla\times\nabla)(\alpha)=[(x,x)]\times[(x,x)]$ for some $x\in B$, then $\alpha=[(x,x)_i]\times[(x,x)_i]$ since $\nabla^{-1}(\{(x,x)\})=\{(x,x)_i=(x,x)\}$, $i=1,2$.

If $(\nabla\times\nabla)(\alpha)=[(x,y)]\times[(x,y)]$ for some $(x,y)\in B^2$ with $x\neq y$, then it follows that $\alpha=[(x,y)_m]\times[(x,y)_n]$ $(m,n=1,2)$ or $\alpha\supset[(x,y)_m]\times[\{(x,y)_m,(x,y)_n\}]$ or $\alpha\supset[\{(x,y)_m,(x,y)_n\}]\times[(x,y)_m]$, $m\neq n$, $m,n=1,2$ or $\alpha\supset[\{(x,y)_1,(x,y)_2\}]\times[\{(x,y)_1,(x,y)_2\}]$.

If $\alpha=[(x,y)_1]\times[(x,y)_2]\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), then $\{(x,y)_1\}\times\{(x,y)_2\}\in(i_k\times i_k)(V)$ for all $V\in\beta$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$.

If $\alpha=[(x,y)_2]\times[(x,y)_1]\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), then $\{(x,y)_2\}\times\{(x,y)_1\}\in(i_k\times i_k)(V)$ for all $V\in\beta$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$.

If $\alpha=[(x,y)_m]\times[\{(x,y)_m,(x,y)_n\}]\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), $(m\neq n, m,n=1,2)$, then it follows that $\{(x,y)_1\}\times\{(x,y)_1,(x,y)_2\}\subset(i_k\times i_k)(V)$ for all $V\in\beta$ and consequently, $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$.

If $[\phi]\neq\alpha\neq[(x,y)_m]\times[\{(x,y)_m,(x,y)_n\}]$ and $\alpha\supset[(x,y)_m]\times[\{(x,y)_m,(x,y)_n\}]$ $(m\neq n,\ m,n=1,2)$, then $\alpha=[(x,y)_m]\times[(x,y)_m]$ or $\alpha=[(x,y)_m]\times[(x,y)_n]$ (see the proof of Theorem 4.4). By the same argument used above, $\alpha=[(x,y)_m]\times[(x,y)_n]$, $m\neq n$, $m,n=1,2$ can not occur.

If $\alpha=[\{(x,y)_1,(x,y)_2\}]\times[\{(x,y)_1,(x,y)_2\}]\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), then $\{(x,y)_1,(x,y)_2\}\times\{(x,y)_1,(x,y)_2\}\in(i_k\times i_k)(V)$ for all $V\in\beta$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$.

If $[\phi]\neq\alpha\neq[\{(x,y)_1,(x,y)_2\}]\times[\{(x,y)_1,(x,y)_2\}]$ and $\alpha\supset[\{(x,y)_1,(x,y)_2\}]\times[\{(x,y)_1,(x,y)_2\}]$, then $\alpha=[(x,y)_m]\times[(x,y)_m]$ or $\alpha=([(x,y)_1]\times[(x,y)_2\}])\cap([(x,y)_1]\times[(x,y)_2\}]$ or $\alpha=([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])$ (see the proof of Theorem 3.3 and Theorem 4.4).

If $\alpha=([(x,y)_1]\times[(x,y)_2\}])\cap([(x,y)_1]\times[(x,y)_2\}]\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), then $\{(x,y)_1\times(x,y)_2\}\cup\{(x,y)_2\times(x,y)_1\}\in(i_k\times i_k)(V)$ for all $V\in\beta$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$.

If $\alpha=([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])\supset(i_k\times i_k)(\beta)$ for some $\beta\in\mathfrak{I}^2$ and $k=1$ (resp. $k=2$), then $\{(x,y)_1\times(x,y)_1\}\cup\{(x,y)_2\times(x,y)_2\}\in(i_1\times i_1)(V)$ for all $V\in\beta$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $B^2\vee_\Delta B^2$, a contradiction since $x\neq y$. Hence, if $x\neq y$, then we must have $\alpha=[(x,y)_m]\times[(x,y)_m]$, $m=1,2$. By 2.2, 2.4 and Definition 4.1, $(B,\mathfrak{I})$ is T_0'. □

Theorem 4.6. *A semiuniform convergence space $(B,\mathfrak{I})$ is T_1 if and only if for each distinct pair x and y in B, $[x]\times[y]\notin\mathfrak{I}$ and $([x]\times[x])\cap([y]\times[y])\notin\mathfrak{I}$.*

Proof. Suppose that $(B,\mathfrak{I})$ is T_1. If $[x]\times[y]\in\mathfrak{I}$ for some $x\neq y$ in B, then let $\alpha=[(x,y)_2]\times[(x,y)_1]$. Note that $(\pi_1S\times\pi_1S)(\alpha)=[x]\times[x]\in\mathfrak{I}$, $(\pi_2S\times\pi_2S)(\alpha)=[x]\times[y]\in\mathfrak{I}$, $(\pi_3S\times\pi_3S)(\alpha)=[y]\times[y]\in\mathfrak{I}$ and $(\nabla\times\nabla)(\alpha)=[(x,y)]\times[(x,y)]\in\mathfrak{I}^2_d$, where $\mathfrak{I}^2_d$ is the discrete semiuniform convergence structure on B^2, a contradiction since $(B,\mathfrak{I})$ is T_1. Hence, we must have $[x]\times[y]\notin\mathfrak{I}$ for each $x\neq y$.

If $([x]\times[x])\cap([y]\times[y])\in\mathfrak{I}$, then let $\alpha=([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])$. By Lemma 2.1 (*i*), $(\pi_1S\times\pi_1S)(\alpha)=[x]\times[x]\in\mathfrak{I}$, $(\pi_2S\times\pi_2S)(\alpha)=([y]\times[y])\cap([x]\times[x])\in\mathfrak{I}$, $(\pi_3S\times\pi_3S)(\alpha)=([y]\times[y])\in\mathfrak{I}$, and $(\nabla\times\nabla)(\alpha)=[(x,y)]\times[(x,y)]\in\mathfrak{I}^2_d$, a contradiction since $(B,\mathfrak{I})$ is T_1. Hence, $([x]\times[x]\cap([y]\times[y])\notin\mathfrak{I}$ for all $x\neq y$.

Conversely, suppose that for each $x\neq y$, $[x]\times[y]\notin\mathfrak{I}$ and $([x]\times[x])\cap([y]\times[y])\notin\mathfrak{I}$. We need to show that $(B,\mathfrak{I})$ is T_1, i.e., by Definition 4.1, 2.2, and 2.4, we must show that the semiuniform convergence structure $\mathfrak{I}^S_W$ on $B^2\vee_\Delta B^2$ induced by $S:B^2\vee_\Delta B^2\to U\big((B^3,\mathfrak{I}^3)\big)=B^3$ and $\nabla:B^2\vee_\Delta B^2\longrightarrow U\big((B^2,\mathfrak{I}^2_d)\big)=B^2$ is discrete, where $\mathfrak{I}^3$ and $\mathfrak{I}^2_d$ is the product semiuniform convergence structure on B^3 and the discrete semiuniform convergence structure on B^2, respectively. Let α be any filter in $\mathfrak{I}^S_W$, i.e., $(\pi_iS\times\pi_iS)(\alpha)\in\mathfrak{I}$, $i=1,2,3$ and $(\nabla\times\nabla)(\alpha)\in\mathfrak{I}^2_d$. We must show that $\alpha=[(x,y)_i]\times[(x,y)_i]$ $(i=1,2)$ or $\alpha=[\phi]$.

If $(\nabla\times\nabla)(\alpha)=[\phi]$, then $\alpha=[\phi]$.

If $(\nabla\times\nabla)(\alpha)=[(x,x)]\times[(x,x)]$ for some $x\in B$, then $\alpha=[(x,x)_i]\times[(x,x)_i]$ since $\nabla^{-1}(\{(x,x)\})=\{(x,x)_i=(x,x)\}$, $i=1,2$.

If $(\nabla\times\nabla)(\alpha) = [(x,y)]\times[(x,y)]$ for some $(x,y)\in B^2$ with $x\neq y$, then it follows that $\alpha = [(x,y)_i]\times[(x,y)_j]$ $(i,j=1,2)$ or $\alpha\supset[\{(x,y)_i,(x,y)_j\}]\times[\{(x,y)_i,(x,y)_j\}]$ or $\alpha\supset[(x,y)_i]\times[\{(x,y)_i,(x,y)_j\}]$ or $\alpha\supset[\{(x,y)_i,(x,y)_j\}]\times[(x,y)_i]$ $(i,j=1,2$ and $i\neq j)$.

If $\alpha=[(x,y)_2]\times[(x,y)_1]$, then $(\pi_2 S\times\pi_2 S)(\alpha) = ([x]\times[y]\in\mathfrak{I}$, a contradiction. If $\alpha=[(x,y)_1]\times[(x,y)_2]$, then $(\pi_2 S\times\pi_2 S)(\alpha)=([y]\times[x]\in\mathfrak{I}$, a contradiction. Hence, $\alpha\neq[(x,y)_i]\times[(x,y)_j]$, $(i,j=1,2$ and $i\neq j)$.

If $\alpha=[(x,y)_i]\times[\{(x,y)_i,(x,y)_j\}]$, $i\neq j$, then (for $i=1$) $(\pi_2 S\times\pi_2 S)(\alpha)=[y]\times[\{y,x\}]\subset[y]\times[x]$ and, (for $i=2$) $(\pi_2 S\times\pi_2 S)(\alpha)=[x]\times[\{x,y\}]\subset[x]\times[y]$ and consequently, $[y]\times[x]\in\mathfrak{I}$, $[x]\times[y]\in\mathfrak{I}$, a contradiction. Hence, $\alpha\neq[(x,y)_i]\times[\{(x,y)_i,(x,y)_j\}]$, $i\neq j$.

If $\alpha=([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])$, then $(\pi_2 S\times\pi_2 S)(\alpha)=([y]\times[y])\cap([x]\times[x])\in\mathfrak{I}$, a contradiction. If $[\phi]\neq\alpha\neq([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])$ and $\alpha\supset([(x,y)_1]\times[(x,y)_1])\cap([(x,y)_2]\times[(x,y)_2])$, then $\alpha=[(x,y)_i]\times[(x,y)_i]$ $i=1,2$ (see the proof of Theorem 3.3).

For the remaining cases, by using the same argument used in the proof of Theorem 4.4 and by the assumptions, we must have $\alpha=[(x,y)_i]\times[(x,y)_i]$, $i=1,2$. Hence, by Definition 4.1, 2.2, and 2.4, $(B,\mathfrak{I})$ is T_1. □

Recall that a Kent convergence space [24] (in [26] p. 1374, it is called a local filter convergence space) is a pair (B,q), where B is a set and $q\subset F(B)\times B$ such that the following are satisfied:

C_1) $([x],x)\in q$ for each $x\in B$.

C_2) $(\beta,x)\in q$ whenever $(\alpha,x)\in q$ and $\beta\supset\alpha$.

C_3) $(\alpha\cap[x],x)\in q$ whenever $(\alpha,x)\in q$.

C_4) A map $f:(B,q)\to(B',q')$ between Kent convergence spaces is called continuous provided that $((f(\alpha),f(x))\in q'$ for each $(\alpha,x)\in q$.

C_5) The category of Kent (local filter) convergence spaces and continuous maps is denoted by ***Conv*** in [24] (resp., ***LFCO*** in [26]).

Note that every semiuniform convergence spaces $(B,\mathfrak{I})$ has an underlying Kent convergence space $(B,q_{\gamma_\mathfrak{I}})$ defined as follows: $q_{\gamma_\mathfrak{I}}=\{(\alpha,x):\alpha\cap[x]\in\gamma_\mathfrak{I}\}$, where $\gamma_\mathfrak{I}=\{\beta\in F(B):\beta\times\beta\in\mathfrak{I}\}$ [28, 30].

We recall from [30] (p. 148),

1. A Kent convergence space (B,q) is called

(a) a T_0-space if and only if for each pair $(x,y)\in B\times B$, $([x],y)\in q$ and $([y],x)\in q$ imply $x=y$.

(b) a T_1-space if and only if for each pair $(x,y)\in B\times B$, $([x],y)\in q$ implies $x=y$.

2. A semiuniform convergence space $(B,\mathfrak{I})$ is called a T_0-space (resp.T_1-space) (we will refer to them as the usual ones) if and only if $(B,q_{\gamma_\mathfrak{I}})$ is a T_0-space (resp.T_1-space).

Remark 4.7. (1) Let $(B,\mathfrak{I})$ be a semiuniform convergence space.

(i) By 2.2, Definition 2.2, Theorem 4.4, and Theorem 4.6, $(B,\mathfrak{I})$ is T_1 if and only if it is $\overline{T}_0$ if and only if for any distinct pair of points x and y in B, $[x]\times[y]\notin\mathfrak{I}$ and $([x]\times[x])\cap([y]\times[y])\notin\mathfrak{I}$.

(ii) By Theorem 4.3, Theorem 4.4, and Theorem 4.5, $\overline{T}_0\Rightarrow T_0\Rightarrow T_0'$ but the reverse of each implication is not true, in general.

(*iii*) If $(B,\mathfrak{I})$ is T_1 (in our sense), then it is T_1 (in the usual sense given above). If $(B,\mathfrak{I})$ is $\overline{T}_0$, then it is T_0 (in the usual sense given above) which is equivalent to our T_0 (Theorem 4.1).

(iv) By Theorem 3.3, Theorem 3.4, Theorem 4.4, and Theorem 4.6, $(B,\mathfrak{I})$ is T_1 if and only if it is $\overline{T}_0$ if and only if $(B,\mathfrak{I})$ is T_1 at p for all points p in B if and only if $(B,\mathfrak{I})$ is $\overline{T}_0$ at p for all points p in B.

(v) By Theorem 3.5 and Theorem 4.5, $(B,\mathfrak{I})$ is T_0' if and only if it is T_0' at p for all points p in B.

(2) Let $(B,\mathfrak{I})$ be a semiuniform limit space (resp. uniform limit space).

(a) By 2.2, Definition 2.2, Theorem 4.3, Theorem 4.4, and Theorem 4.6, then the followings are equivalent:

(i) $(B,\mathfrak{I})$ is T_1.

(ii) $(B,\mathfrak{I})$ is $\overline{T}_0$.

(iii) $(B,\mathfrak{I})$ is T_0.

(iv) $(B,\mathfrak{I})$ is T_1 (in the usual sense).

(v) $(B,\mathfrak{I})$ is T_0 (in the usual sense).

(vi) For any distinct pair of points x and y in B, $[x]\times[y]\notin\mathfrak{I}$.

(vi) $(B, \mathfrak{I})$ is T_1 at p for all points p in B.
(vii) $(B, \mathfrak{I})$ is $\overline{T}_0$ at p for all points p in B.
(b) By Definition 2.2, Theorem 3.5, and Theorem 4.5, $(B, \mathfrak{I})$ is $T_0^{'}$ if and only if it is $T_0^{'}$ at p for all points p in B.
(c) By Theorem 4.4, and Theorem 4.5, $\overline{T}_0 \Rightarrow T_0^{'}$ but the reverse implication is not true, in general.

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