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A formula for the number of $(n-2)$-gaps in digital n-objects

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Abstract. We provide a formula that expresses the number of $(n-2)$-gaps of a generic digital n-object. Such a formula has the advantage to involve only a few simple intrinsic parameters of the object and it is obtained by using a combinatorial technique based on incidence structure and on the notion of free cells. This approach seems suitable as a model for an automatic computation, and also allow us to find some expressions for the maximum number of i-cells that bound or are bounded by a fixed j-cell.

1. Introduction

With the word "gap" in Digital Geometry we mean some basic portion of a digital object that a discrete ray can cross without intersecting any voxel of the object itself. Since such a notion is strictly connected with some applications in the field of Computer graphics (e.g. the rendering of a 3D image by the ray-tracing technique), many papers (see for example [1–4]) concerned the study of 0- and 1-gaps of 3-dimensional objects. Recently (see [5]), we have found a formula for expressing the number of 1-gaps of a digital 3-object by means of the number of its free cells of dimension 1 and 2. During the submission process of that paper, the anonymous referee raised to our attention the existence of another recent and more general formula presented in [6] which gives the number of a generical $(n-2)$-gaps of any digital n-object. Unfortunately, such formula involves some parameters (the number of $(n-2)$-blocks and of n-, $(n-1)$- and $(n-2)$- cells) that are non-intrinsic or that can not be easily obtained by the geometrical knowledge of the object. For such a reason, in the present paper, we propose a generalization of the formula obtained in [5] that allow us to express the number of $(n-2)$-gaps using only two basic parameters, that is the number of free $(n-2)$- and $(n-1)$-cells of the object itself. Although we prove the equivalence between these two formulas, the latter approach seems simpler and more suitable as a model for an automatic computation.

In order to obtain our formula, we adopt a combinatorial technic based on the notion of incidence structure, which also allow us to find a couple of interesting expressions for the maximum number of i-cells that bound or are bounded by a fixed j-cell.

In the next section we recall and formalize some basic notions and notations of digital geometry. In Section 3, we introduce the notions of tandem and gap, and we give some elementary facts about them. In Section 4, we prove some propositions concerning, in particular, the number of $(n-1)$-cells of the boundary of a digital object that are bounded by a given $(n-2)$-cell satisfying some particular condition, and we use such results to obtain our main formula for the number of $(n-2)$-gaps. Finally, in Section 5, we summarize the goal of the paper and we give some suggestions for other future researches.

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2. Preliminaries

Throughout this paper we use the *grid cell model* for representing digital objects, and we adopt the terminology from [7, 8].

Let $x = (x_1, \dots x_n)$ be a point of $\mathbb{Z}^n$, $\theta \in \{-1, 0, 1\}^n$ be an n-word over the alphabet $\{-1, 0, 1\}$, and $i \in \{1, \dots n\}$. We define i-cell related to x and θ, and we denote it by $e = (x, \theta)$, the Cartesian product, in a certain fixed order, of $n - i$ singletons $\left\{x_j \pm \frac{1}{2}\right\}$ by i closed sets $\left[x_j - \frac{1}{2}, x_j + \frac{1}{2}\right]$, i.e. we set

$$e = (x, \theta) = \prod_{j=1}^{n} \left[x_j + \frac{1}{2}\theta_j - \frac{1}{2}[\theta_j = 0], x_j + \frac{1}{2}\theta_j + \frac{1}{2}[\theta_j = 0]\right],$$

where $[\bullet]$ denotes the Iverson bracket [9]. The word θ is called the *direction* of the cell (x, θ) related to the point x.

Let us note that an i-cell can be related to different point $x \in \mathbb{Z}^n$, and, once we have fixed it, can be related to different direction. So, when we talk generically about i-cell, we mean one of its possible representation.

The dimension of a cell $e = (x, \theta)$, denoted by $\dim(e) = i$, is the number of non-trivial intervals of its product representation, i.e. the number of null components of its direction θ. Thus, $\dim(e) = \sum_{j=1}^{n}[\theta_j = 0]$ or, equivalently, $\dim(e) = n - \theta \cdot \theta$. So, e is an i-cell if and only if it has dimension i.

We denote by $\mathbb{C}_n^{(i)}$ the set of all i-cells of $\mathbb{R}^n$ and by $\mathbb{C}_n$ the set of all cells defined in $\mathbb{R}^n$, i.e. we set $\mathbb{C}_n = \bigcup_{j=0}^{n} \mathbb{C}_n^{(j)}$. An n-cell of $\mathbb{C}_n$ is also called an n-voxel. So, for convenience, an n-voxel is denoted by v, while we use other lower case letter (usually e) to denote cells of lower dimension. A finite collection D of n-voxels is a digital n-object. For any $i = 0, \dots, n$, we denote by $C_i(D)$ the set of all i-cells of the object D, that is $D \cap \mathbb{C}_n^{(i)}$, and by $c_i(D)$ (or simply by c_i if no confusion arise) its cardinality $|C_i(D)|$.

Definition 2.1. Let $e = (x, \theta)$ be an i-cell. The *center* of e is defined by $cnt(e) = x + \frac{1}{2}\theta$.

Remark 2.2. Let us note that for a cell $e = (x, \theta)$, we have $cnt(e) = x$ if and only if $\dim(e) = n$. Moreover, thanks to Definition 2.1, an i-cell related to x and θ can be shortly represented in the following way:

$$e = \prod_{j=1}^{n} \left[cnt(e)_j - \frac{1}{2}[\theta_j = 0], cnt(e)_j + \frac{1}{2}[\theta_j = 0]\right].$$

Definition 2.3. Let $e = (x, \theta)$ be an i-cell related to the point x and to the direction θ. We define the *dual e' of e*, the cell represented by the following cartesian product:

$$e' = \prod_{j=1}^{n} \left[cnt(e)_j - \frac{1}{2}[\theta_j \neq 0], cnt(e)_j + \frac{1}{2}[\theta_j \neq 0]\right].$$

By the above expression and the definition of dimension of a cell, we have that the dimension of the dual e' of a cell $e = (x, \theta)$ coincides with the number of non-null components of the direction θ, that is $\dim(e') = \sum_{j=1}^{n}[\theta_j \neq 0]$. Consequently, the dual e' of an i-cell e is an $(n - i)$-cell.

Definition 2.4. Let D be a digital object. The *dual D' of D* is the set of all dual cells e', with $e \in D$.

We say that two n-cells v_1, v_2 are i-adjacent ($i = 0, 1, \dots, n - 1$) if $v_1 \neq v_2$ and there exists at least an i-cell $\bar{e}$ such that $\bar{e} \subseteq v_1 \cap v_2$, that is if they are distinct and share at least an i-cell. Two n-cells v_1, v_2 are *strictly* i-adjacent, if they are i-adjacent but not j-adjacent, for any $j > i$, that is if $v_1 \cap v_2 \in \mathbb{C}_n^{(i)}$. The set of all n-cells that are i-adjacent to a given n-voxel v is denoted by $A_i(v)$ and called the *i-adjacent neighborhoods* of v. Two cells $v_1, v_2 \in \mathbb{C}_n$ are *incident* each other, and we write $e_1 I e_2$, if $e_1 \subseteq e_2$ or $e_2 \subseteq e_1$.

Definition 2.5. Let $e_1, e_2 \in \mathbb{C}_n$. We say that e_1 *bounds* e_2 (or that e_2 *is bounded by* e_1), and we write $e_1 < e_2$, if $e_1 I e_2$ and $\dim(e_1) < \dim(e_2)$. The relation $<$ is called the *bounding relation*.

Definition 2.6. Let e be an i-cell of a digital n-object D (with $i = 0, \dots n-1$). We say that e is *simple* if e bounds one and only one n-cell.

Definition 2.7. Let D and G be two finite subsets of $\mathbb{C}_n$. We say that D and G form a *dual pair* iff there exists a bijection $\varphi\colon D \to G$ that inverts the bounded relation, that is for any couple $e, f \in D$, if $e < f$ then $\varphi(f) < \varphi(e)$, and for any $e \in D$, $\dim(\varphi(e)) = n - \dim(e)$.

Proposition 2.8. *Let D be a digital n-object and D' its dual. Then D and D' form a dual pair.*

Proof. Let us consider the mapping $\varphi\colon D \to D'$ that associates to each cell $e = (x, \theta) \in D$ its dual $\varphi(e) = e'$. Since, by Remark 2.2 and Definition 2.3, both e and e' are uniquely determined by the point x and the direction θ, it is clear that φ is a bijection.
By a basic property of the Iverson notation, for every cell $e = (x, \theta)$, we have

$$\dim(\varphi(e)) = \dim(e') = \sum_{j=1}^{n} [\theta_j \neq 0] = \sum_{j=1}^{n} \left(1 - [\theta_j = 0]\right) = n - \sum_{j=1}^{n} [\theta_j = 0] = n - \dim(e).$$

Moreover, φ inverts the bounding relation $<$ over $\mathbb{C}_n$. Indeed, for every pair of cells $e = (x, \theta)$ and $f = (y, \psi)$ in D such that $e < f$, without loss of generality, we have $e \subseteq f$ and $\dim(e) < \dim(f)$. Thus, by Remark 2.2, we get

$$\prod_{j=1}^{n} \left[cnt(e)_j - \frac{1}{2}[\theta_j = 0], cnt(e)_j + \frac{1}{2}[\theta_j = 0]\right] \subseteq \prod_{j=1}^{n} \left[cnt(f)_j - \frac{1}{2}[\psi_j = 0], cnt(f)_j + \frac{1}{2}[\psi_j = 0]\right].$$

Hence, for every $j = 1, \dots, n$, we have

$$cnt(f)_j - \frac{1}{2}[\psi_j = 0] \leq cnt(e)_j - \frac{1}{2}[\theta_j = 0] \leq cnt(e)_j + \frac{1}{2}[\theta_j = 0] \leq cnt(f)_j + \frac{1}{2}[\psi_j = 0].$$

and so, we obtain

$$\begin{aligned} cnt(e)_j - \frac{1}{2}[\theta_j \neq 0] &= cnt(e)_j - \frac{1}{2}\left(1 - [\theta_j = 0]\right) = cnt(e)_j + \frac{1}{2}[\theta_j = 0] - \frac{1}{2} \leq cnt(f)_j + \frac{1}{2}[\psi_j = 0] - \frac{1}{2} \\ &= cnt(f)_j - \frac{1}{2}[\psi_j \neq 0] \leq cnt(f)_j + \frac{1}{2}[\psi_j \neq 0] = cnt(f)_j + \frac{1}{2}\left(1 - [\psi_j = 0]\right) \\ &= cnt(f)_j - \frac{1}{2}[\psi_j = 0] + \frac{1}{2} \leq cnt(e)_j - \frac{1}{2}[\theta_j = 0] + \frac{1}{2} = cnt(e)_j + \frac{1}{2}[\theta_j \neq 0], \end{aligned}$$

which implies

$$\prod_{j=1}^{n} \left[cnt(f)_j - \frac{1}{2}[\psi_j \neq 0], cnt(f)_j + \frac{1}{2}[\psi_j \neq 0]\right] \subseteq \prod_{j=1}^{n} \left[cnt(e)_j - \frac{1}{2}[\theta_j \neq 0], cnt(e)_j + \frac{1}{2}[\theta_j \neq 0]\right].$$

Thus, $f' \subseteq e'$, i.e. $\varphi(f) \subseteq \varphi(e)$. Finally, since $\dim(e) < \dim(f)$, we have $\dim(\varphi(f)) = n - \dim(f) < n - \dim(e) = \dim(\varphi(e))$ and so $\varphi(f) < \varphi(e)$. □

Definition 2.9. An *incidence structure* (see [10]) is a triple $(V, \mathcal{B}, \mathcal{I})$ where V and $\mathcal{B}$ are any two disjoint sets and $\mathcal{I}$ is a binary relation between V and $\mathcal{B}$, that is $\mathcal{I} \subseteq V \times \mathcal{B}$. The elements of V are called *points*, those of $\mathcal{B}$ *blocks*. Instead of $(p, B) \in \mathcal{I}$, we simply write $p\mathcal{I}B$ and say that "the point p lies on the block B" or "p and B are incident".

If p is any point of V, we denote by (p) the set of all blocks incident to p, i.e. $(p) = \{B \in \mathcal{B} : p\mathcal{I}B\}$. Similarly, if B is any block of $\mathcal{B}$, we denote by (B) the set of all points incident to B, i.e. $(B) = \{p \in V : p\mathcal{I}B\}$. For a point p, the number $r_p = |(p)|$ is called the degree of p, and similarly, for a block B, $k_B = |(B)|$ is the degree of B.

We remind the following fundamental proposition of incidence structures.

Proposition 2.10. *Let $(V, \mathcal{B}, \mathcal{I})$ be an incidence structure. We have*

$$\sum_{p \in V} r_p = \sum_{B \in \mathcal{B}} k_B, \tag{1}$$

where r_p and k_B are the degrees of any point $p \in V$ and any block $B \in \mathcal{B}$, respectively.

3. Theoretical backgrounds

In [3] and [5], a constructive definition of gap for a digital object D in spaces of dimensions 2 and 3 was proposed, and a relation between the number of such a gaps and the numbers of free cells was found.

In order to generalize those results for the n-dimensional space, we need to introduce some definitions and to make some considerations.

Definition 3.1. Let e be an i-cell (with $0 \le i \le n-1$) of $\mathbb{C}_n$. Then:

(1) An *i-block centered on e* is the union of all the n-voxels bounded by e, i.e. $B_i(e) = \bigcup\{v \in \mathbb{C}_n^{(n)} : e < v\}$.

(2) An *L-block centered on e* is an $(n-2)$-block centered on e from which we take away one of its four n-cells, that is $L(e) = B_{n-2}(e) \setminus \{v\}$, where $v \in C_n(B_{n-2}(e))$.

Remark 3.2. Let us note that, for any i-cell e, $B_i(e)$ is the union of exactly 2^{n-i} n-voxels, $e \in B_i(e)$, and that an L-block is exactly composed of three n-voxels.

Definition 3.3. Let v_1, v_2 be two n-voxels of a digital object D, and e be an i-cell ($i = 0, \dots, n-1$). We say that $t_i = \{v_1, v_2\}$ forms an *i-tandem of D over e* if $D \cap B_i(e) = \{v_1, v_2\}$, v_1 and v_2 are strictly i-adjacent and $v_1 \cap v_2 = e$.

Definition 3.4. Let D be a digital n-object and e be an i-cell (with $i = 0, \dots, n-2$). We say that D has an *i-gap over e* if there exists an i-block $B_i(e)$ such that $B_i(e) \setminus D$ is an i-tandem over e. The cell e is called *i-hub* of the related i-gap. Moreover, we denote by $g_i(D)$ (or simply by g_i if no confusion arises) the number of i-gap of D.

Examples of gaps for 3D case are given in Figure 1.

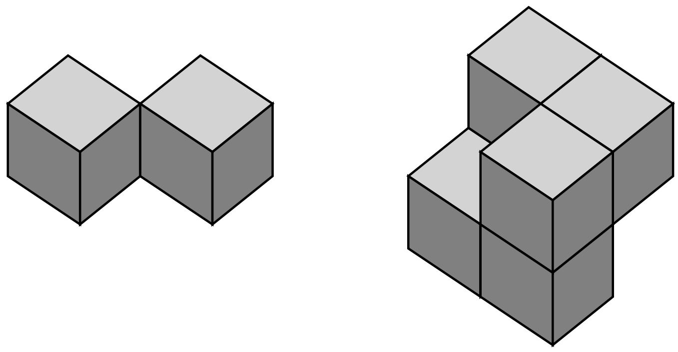

Figure 1: Configurations of 1- and 0-gaps in $\mathbb{C}_3$.

Proposition 3.5. *A digital n-object D has an $(n-2)$-gap over an $(n-2)$-hub e iff there exist two n-voxels v_1 and v_2 such that:*

1) $e < v_1$ and $e < v_2$;

2) $v_1 \in A_{n-2}(v_2) \setminus A_{n-1}(v_2)$*;*

3) $A_{n-1}(v_1) \cap A_{n-1}(v_2) \cap D = \emptyset$*.*

Proof. Let us suppose that D has an $(n-2)$-gap over an $(n-2)$-hub e. Then there exists an $(n-2)$-block $B = B_{n-2}(e)$ such that $B \setminus D$ is an $(n-2)$-tandem over e. Hence $B \setminus D$ is composed of two strictly $(n-2)$-adjacent n-voxel, let us say v_1, v_2, and $v_1 \cap v_2 = e$. This implies that $e \subset v_1$ and $e \subset v_2$, and so $e < v_1$ and $e < v_2$.

Now, let us suppose that $v_1 \notin A_{n-2}(v_2) \setminus A_{n-1}(v_2)$. Then it should be $v_1 \notin A_{n-2}(v_2)$ or $v_1 \in A_{n-1}(v_2)$. Both expressions lead to a contradiction, since v_1 and v_2 are strictly $(n-2)$-adjacent.

Finally, let us suppose that $A_{n-1}(v_1) \cap A_{n-1}(v_2) \cap D \neq \emptyset$. Then it should exists an n-voxel $v_3 \in D$ such that $v_3 \in A_{n-1}(v_1)$ and $v_3 \in A_{n-1}(v_2)$. Hence $\{v_1, v_2, v_3\}$ forms an L-block. A contradiction since v_1 and v_2 are strictly $(n-2)$-adjacent.

Conversely, let us suppose that conditions 1), 2), and 3) hold, and, by contradiction, that for any $(n-2)$-cell $e \in D$, $E = B_{n-2}(e) \setminus D$ is not an $(n-2)$-tandem over e. Then E is either an i-block ($i = n-2, n-1$) or an L-block whose facts contradict our hypothesis. □

Definition 3.6. An i-cell e (with $i = 0, \dots, n-1$) of a digital n-object D is *free* iff $B_i(e) \nsubseteq D$.

For any $i = 0, \dots, n-1$, we denote by $C_i^*(D)$ (respectively by $C_i'(D)$) the set of all free (respectively non-free) i-cells of the object D. Moreover, we denote by $c_i^*(D)$ (or simply by c_i^*) the number of free i-cells of D, and by $c_i'(D)$ (or simply by c_i') the number of non-free cells. It is evident that $\{C_i^*(D), C_i'(D)\}$ forms a partition of $C_i(D)$ and that $c_i = c_i^* + c_i'$.

Definition 3.7. The *i-border* ($i = 1, \dots, n-1$) $bd_i(D)$ of a digital n-object D is the set of all its i-cells such that $B_i(e)$ intersects both D and $\mathbb{C}_n \setminus D$. The union of all i-borders ($0 \leq i \leq n-1$) is called border of D and denoted by $bd(D)$.

An immediate consequence of Definitions 3.6 and 3.7 is given by the following proposition.

Proposition 3.8. *An i-cell e* ($i = 0, \dots, n-1$) *of a digital object D is free iff* $e \in bd(D)$*.*

Remark 3.9. The border $bd(D)$ of a digital n-object is composed of the set of all free cells of D. Moreover, c_i' coincides with the number of all i-blocks $B_i(e)$ such that $B_i(e) \subseteq D$.

4. Main results

Definition 4.1. Let e be an i-cells of $\mathbb{C}_n$. The *j-flower* of e ($i < j \leq n$) is the set of cells $F_j(e)$ constituted by all j-cells that are bounded by e, that is we set $F_j(e) = \{c \in \mathbb{C}_n^{(j)} : e < c\}$. The cell e is called the *center of the flower*, while an element of $F_j(e)$ is called a *j-petal* (or simply petal if confusion does not arise) of the j-flower $F_j(e)$.

Let us note that Definition 4.1 is a generalization of the notion of i-block given in Definition 3.1. Indeed an i-block centered on an i-cell e can be regarded as the n-flower of e.

Notation 4.2. Let i, j be two natural number such that $0 \leq i < j$. We denote by $c_{i \to j}$ the maximum number of i-cells of $\mathbb{C}_n$ that bound a j-cell. Moreover, we denote by $c_{i \leftarrow j}$ the maximum number of j-cell of $\mathbb{C}_n$ that are bounded by an i-cell.

Let us note that, for any $0 \leq i < j$, $c_{i \leftarrow j}$ represents the number of j-petal of the j-flower $F_j(e)$, where e is a cell of dimension i.

Proposition 4.3. *For any* $i, j \in \mathbb{N}$ *such that* $0 \leq i < j$*, holds*

$$c_{i \to j} = 2^{j-i}\binom{j}{i}.$$

Proof. Since a j-cell of $\mathbb{C}_n$ can be regarded as an hypercube of dimension j, the number $c_{i\to j}$ corresponds with the number of i-faces of this hypercube which is $2^{j-i}\binom{j}{i}$ (see, for example, [11]). $\square$

Proposition 4.4. *For any $i, j \in \mathbb{N}$ such that $0 \le i < j$, holds*

$$c_{i\leftarrow j} = 2^{j-i}\binom{n-i}{j-i}.$$

Proof. Let e be an i-cell of $\mathbb{C}_n$, and let $F_j(e)$ be the related j-flower. The dual Φ' of $\Phi = F_j(e) \cup \{e\}$ is an incidence structure $(V, \mathcal{B}, \mathcal{I})$, where $V = \{p' : p \in F_j(e)\}$, $\mathcal{B} = \{e'\}$ and $\mathcal{I}$ is the dual relation of the bounding relation $<$. Moreover, we have $\dim(e') = n - i$ and $\dim(p') = n - j$. Hence, up to a bijection, Φ' is the set composed of the $(n - i)$-cell e' and by all the possible $(n - j)$-cells which bound e' . It follows that the maximum number $c_{i\leftarrow j}$ of j-cells that are bounded by a given i-cell coincides with the maximum number of $(n - j)$-cells that bound an $(n - i)$-cell, that is, by Proposition 4.3,

$$c_{i\leftarrow j} = c_{n-j\to n-i} = 2^{n-i-n+j}\binom{n-i}{n-j} = 2^{j-i}\binom{n-i}{j-i}.$$

$\square$

Lemma 4.5. *Let D be a digital n-object. Then*

$$c_{n-1} = 2nc_n - c'_{n-1}.$$

Proof. Let us consider the set

$$F = \bigcup_{v\in C_n(D)} \{(e, v) : e \in C_{n-1}(D), e < v\}.$$

It is evident that $\left|F\right| = \left|\{(e, v) : e \in C_{n-1}(D), e < v\}\right| \cdot \left|C_n(D)\right| = c_{n-1\to n} \cdot c_n = 2nc_n$. Let us set $F^* = F \cap (C^*_{n-1}(D) \times C_n(D))$ and $F' = F \cap (C'_{n-1}(D) \times C_n(D))$. The map $\phi : F^* \to C^*_{n-1}(D)$, defined by $\phi(e, v) = e$, is a bijection. In fact, besides being evidently surjective, it is also injective, since, if by contradiction there were two distinct pairs (e, v_1) and $(e, v_2) \in F^*$ associated to e, then $B_{n-1}(e) = \{v_1, v_2\}$ should be an $(n - 1)$-block contained in D. This contradicts the fact that the $(n - 1)$-cell e is free. Thus $|F^*| = |C^*_{n-1}(D)| = c^*_{n-1}$.
On the other hand, $\left|F'\right| = \left|\bigcup_{v\in C_n(D)} \{(e, v) : e \in C'_{n-1}(D), e < v\}\right| = \left|\bigcup_{e\in C'_{n-1}(D)} \{(e, v) : v \in C_n(D), e < v\}\right| = \left|\{(e, v) : v \in C_n(D), e < v\}\right| \cdot \left|C'_{n-1}(D)\right| = c_{n-1\leftarrow n} \cdot c'_{n-1} = 2c'_{n-1}$. Since $\{F^*, F'\}$ is a partition of F, we finally have $|F| = |F^*| + |F'|$, that is $2nc_n = c^*_{n-1} + 2c'_{n-1} = c_{n-1} - c'_{n-1} + 2c'_{n-1} = c_{n-1} + c'_{n-1}$, and then the thesis. $\square$

Notation 4.6. Let e be an i-cell of a digital n-object D, and $0 \le i < j$. We denote by $b_j(e, D)$ (or simply by $b_j(e)$ if no confusion arises) the number of j-cells of $bd(D)$ that are bounded by e.

Let us note that if e is a non-free i-cell, then $b_j(e) = 0$.

Definition 4.7. A free i-cell of a digital n-object that is not an i-hub is called *i-nub*.

Notation 4.8. For any $i = 0, \ldots, n - 1$, we denote by $\mathcal{H}_i(D)$ and by $\mathcal{N}_i(D)$ (or simply by $\mathcal{H}_i$ and by $\mathcal{N}_i$ if no confusion arises) the sets of i-hubs and i-nubs of D, respectively. We have $|\mathcal{H}_i| = g_i$ and $|\mathcal{N}_i| = c^*_i - g_i$.

We are interested in classifying all the possible configurations of n-voxels bounded by an $(n - 2)$-cell e.

Lemma 4.9. *Let e be an $(n - 2)$-cell of $\mathbb{C}_n$, and $V = \{v \in \mathbb{C}^{(n)}_n : e < v\}$ be the set of n-voxels bounded by e. Then one and only one of the following five cases occurs (See Figure 2 for an example for 3D case):*

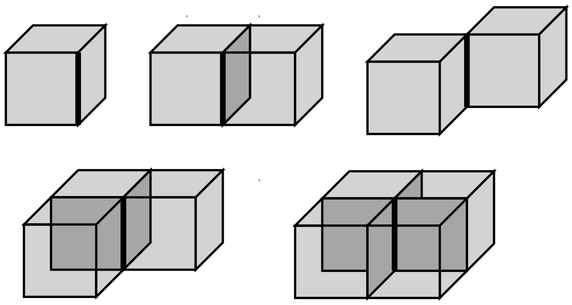

Figure 2: The five possible cases for the set $V = \{v \in \mathbb{C}_n^{(n)} : e < v\}$ in 3D case. The black thick segment represents the edge e.

- *V is a singleton and e is a simple cell;*
- *V is an $(n-1)$-block centered on an $(n-1)$-cell that is bounded by e;*
- *V is $(n-2)$-gap and e is its $(n-2)$-hub;*
- *V is an L-block and e is its center;*
- *V is an $(n-2)$-block and e is its center.*

Proof. By Definition 3.1(1), the largest set of n-voxels bounded by e is the $(n-2)$-block centered on e. Moreover, by Remark 3.2, $c_n(B_{n-2}(e)) = 4$. Hence, the number $c_n(V)$ of n-voxels of V have to be between one and four and, up to symmetries, we can distinguish the following cases.
If $c_n(V) = 1$, V is a single n-voxel. If $c_n(V) = 2$, we have two configurations, depending on the relative position of the two n-voxels v_1 and v_2. More precisely, if v_1 and v_2 are strictly $(n-1)$-adjacent, then they form an $(n-1)$-block centered on an $(n-1)$-cell that is bounded by e; instead, if they are strictly $(n-2)$-adjacent, they form an $(n-2)$-gap having e as $(n-2)$-hub. If $c_n(V) = 3$, by Definition 3.1(2) and Remark 3.2, the unique possible configuration is given by the L-block centered on e. Finally, if $c_n(V) = 4$, V coincides with the $(n-2)$-block centered on e. $\square$

Proposition 4.10. *Let v be an n-voxel and e be one of its i-cells, $i = 0, \ldots, n-1$. Then, for any $i < j \leq n$, it results:*

$$b_j(e) = \frac{c_{i\to j}c_{j\to n}}{c_{i\to n}}.$$

Proof. Let us consider the incidence structure $I = (C_i(v), C_j(v), <)$. By Proposition 2.10, it is $\sum_{a\in C_i(v)} r_a = \sum_{a\in C_j(v)} k_a$. Evidently, $|C_i(v)| = c_i = c_{i\to n}$ and $|C_j(v)| = c_j = c_{j\to n}$, while, for any i-cell a of $C_i(v)$ (respectively j-cell a of $C_j(v)$), $r_a = b_j(e)$ (respectively $k_a = c_{i\to j}$). Hence we have $b_j(e)c_{i\to n} = c_{i\to j}c_{j\to n}$, from which we get the thesis. $\square$

Corollary 4.11. *Let v be an n-voxel and e be one of its i-cell, $i = 0, \ldots, n-1$. Then, for any $i < j \leq n$, we have*

$$b_j(e) = \binom{n-i}{j-i}.$$

Proof. By Proposition 4.10, it is

$$b_j(e) = \frac{c_{i\to j}c_{j\to n}}{c_{i\to n}} = \frac{2^{j-i}\binom{j}{i}2^{n-j}\binom{n}{j}}{2^{n-i}\binom{n}{i}} = \frac{j!}{(j-i)!i!}\cdot\frac{n!}{(n-j)!j!}\cdot\frac{(n-i)!i!}{n!} = \frac{(n-i)!}{(n-j)!(j-i)!} = \binom{n-i}{j-i}.$$

$\square$

Lemma 4.12. *Let e be an $(n-1)$-cell of $\mathbb{C}_n$. Then the number of i-cells of the $(n-1)$-block centered on e is*

$$c_i(B_{n-1}(e)) = \frac{3n+i}{2n} c_{i\to n}.$$

Proof. By Remark 3.2, $B_{n-1}(e)$ is composed of two $(n-1)$-adjacent n-voxels. Each of such voxels has exactly $c_{i\to n}$ i-cells, but some of these cells are in common. The number of these common i-cells coincides with the number of i-cells of the center e of the given block. So, we have $c_i(B_{n-1}(e)) = 2c_{i\to n} - c_{i\to n-1} = 2\cdot 2^{n-i}\binom{n}{i} - 2^{n-1-i}\binom{n-1}{i} = 2\cdot 2^{n-i}\binom{n}{i} - 2^{n-i-1}\binom{n}{i}\frac{n-1}{n} = 2^{n-i}\binom{n}{i}\left(2-\frac{n-i}{2n}\right) = \frac{3n+i}{2n}c_{i\to n}$. □

Lemma 4.13. *Let e be an $(n-1)$-cell of $\mathbb{C}_n$. Then the number of free $(n-1)$-cells of the $(n-1)$-block centered on e is:*

$$c^*_{n-1}(B_{n-1}(e)) = 2(2n-1).$$

Proof. By applying Lemma 4.5 to the digital object $B_{n-1}(e)$, we have $c'_{n-1} + c^*_{n-1} = 2nc_n - c'_{n-1}$. But for an $(n-1)$-block it is $c_n = 2$ and $c'_{n-1} = 1$. Then $c^*_{n-1} = 2(2n-1)$. □

Proposition 4.14. *Let e be a free $(n-2)$-cells that belongs to the center of an $(n-1)$-block $B_{n-1}(f)$, then $b_{n-1}(e) = 2$.*

Proof. Let us consider the incidence structure $(C_{n-2}(B_{n-1}(f)), C^*_{n-1}(B_{n-1}(f)), <)$. By Lemma 4.12, it is $|C_{n-2}(B_{n-1}(f))| = c_{n-2} = 2(n-1)(2n-1)$, and by Lemma 4.13, we have $|C^*_{n-1}(B_{n-1}(f))| = c^*_{n-1} = 4n-2$.
Moreover, by Proposition 2.10, it is

$$\sum_{a\in C_{n-2}(B_{n-1}(f))} r_a = \sum_{a\in C^*_{n-1}(B_{n-1}(f))} k_a.$$

Since for any $a \in C^*_{n-1}(B_{n-1}(f))$ it is $k_a = c_{n-2\to n-1}$, we have

$$\sum_{a\in C^*_{n-1}(B_{n-1}(f))} k_a = c^*_{n-1}\cdot c_{n-2\to n-1} = (4n-2)\cdot 2\cdot (n-1) = 4(2n-1)(n-1).$$

Let us consider the sets

$$F = \{a \in C_{n-2}(B_{n-1}(f)) \colon a < f\}$$

and

$$G = \{a \in C_{n-2}(B_{n-1}(f)) \colon a \nless f\}.$$

Since $\{F, G\}$ forms a partition of $C_{n-2}(B_{n-1}(f))$, we can write

$$\sum_{a\in C_{n-2}(B_{n-1}(f))} r_a = \sum_{a\in F} r_a + \sum_{a\in G} r_a.$$

For any $a \in F$, $r_a = b_{n-1}(e)$, and so

$$\sum_{a\in F} r_a = |F| b_{n-1}(e) = c_{n-2\to n-1} b_{n-1}(e) = 2(n-1)b_{n-1}(e).$$

Instead, thanks to Proposition 4.10, for any $a \in G$, we have

$$r_a = b_{n-1}(e) = \frac{c_{n-2\to n-1}\cdot c_{n-1\to n}}{c_{n-2\to n}} = 2.$$

Hence, we get that

$$\sum_{a\in G} r_a = 2(c_{n-2} - c_{n-2\to n-1}) = 2(2(n-1)(2n-1) - 2(n-1)) = 4(n-1)(2n-1) - 4(n-1).$$

To sum up, we can write $4(n-1)(2n-1) - 4(n-1) + 2(n-1)b_{n-1}(e) = 4(2n-1)(n-1)$, from which we get the thesis. □

Lemma 4.15. *Let e be an $(n-2)$-cell of $\mathbb{C}_n$. Then the number of i-cells of the L-block centered on e is:*

$$c_i(L(e)) = \left(\frac{2n+i}{n}\right) c_{i\to n}.$$

Proof. By Remark 3.2, $L(e)$ is composed of three n-voxels, which are pairwise $(n-1)$-adjacent in exactly two non-free $(n-1)$-cells. Each of these three voxels has exactly $c_{i\to n}$ i-cells, but some of these cells are in common. The number of such common i-cells coincides with the number of i-cells of the two non-free $(n-1)$-cells. So, we have $c_i(L(e)) = 3c_{i\to n} - 2c_{i\to n-1} = 3\cdot 2^{n-i}\binom{n}{i} - 2\cdot 2^{n-i-1}\binom{n-1}{i} = 3\cdot 2^{n-i}\binom{n}{i} - 2^{n-i}\binom{n}{i}\frac{n-i}{n} = 2^{n-i}\binom{n}{i}\left(3-\frac{n-i}{n}\right) = \left(\frac{2n+i}{2n}\right)c_{i\to n}$. $\square$

Lemma 4.16. *Let e be an $(n-1)$-cell of $\mathbb{C}_n$. Then the number of free $(n-1)$-cells of the L-block centered on e is:*

$$c^*_{n-1}(L(e)) = 2(3n-2).$$

Proof. By applying Lemma 4.5 to the digital object $L(e)$, we have $c'_{n-1} + c^*_{n-1} = 2nc_n - c'_{n-1}$. But for an L-block it is $c_n = 3$ and $c'_{n-1} = 2$. Then $c^*_{n-1} = 2(3n-2)$. $\square$

Proposition 4.17. *Let e be a free $(n-2)$-cells which is the center of an L-block $L(e)$. Then $b_{n-1}(e) = 2$.*

Proof. Let us consider the incidence structure $(C_{n-2}(L(e)), C^*_{n-1}(L(e)), <)$. By Lemma 4.15, we have $|C_{n-2}(L(e))| = c_{n-2} = 2(n-1)(3n-2)$, and by Lemma 4.16, it is $|C^*_{n-1}(L(e))| = c^*_{n-1} = 2(3n-2)$.

By Proposition 2.10, it is

$$\sum_{a\in C_{n-2}(L(e))} r_a = \sum_{a\in C^*_{n-1}(L(e)} k_a. \tag{2}$$

Since for any $a \in C^*_{n-1}(L(e))$ it is $k_a = c_{n-1\to n-2}$, we have

$$\sum_{a\in C^*_{n-1}(L(e))} k_a = c^*_{n-1}\cdot c_{n-1\to n-2} = 2(3n-2)\cdot 2\cdot(n-1) = 4(3n-2)(n-1).$$

Let us set $F = \mathbb{C}'_{n-1}(L(e))$, and let us consider the sets:

$A = \{e\}$,

$B = \{c \in \mathbb{C}_{n-2}(L(e))\colon c \nless f, \text{ for some } f \in F\}$.

$C = \{c \in \mathbb{C}_{n-2}(L(e))\colon c < f, \text{ for some } f \in F\}$.

Let us observe that $|F| = 2$ because the number of $(n-1)$-block of $L(e)$ is 2. Since $\{A, B, C\}$ forms a partition of $C_{n-2}(L(e))$, it results

$$\sum_{a\in C_{n-2}(L(e))} r_a = r_e + \sum_{a\in B} r_a + \sum_{a\in C} r_a, \tag{3}$$

where, evidently, $r_e = b_{n-1}(e)$.
Moreover, by Proposition 4.14, it is $\sum_{a\in B} r_a = (2c_{n-2\to n-1} - 2)\cdot 2 = (2\cdot 2(n-1)-2)\cdot 2 = 8(n-1)-4$. Finally, by Proposition 4.10, we have $\sum_{a\in C} r_a = 2(c_{n-2} - 2c_{n-2\to n-1} + 1) = 2(2(3n-2)(n-1) - 2\cdot 2(n-1)+1) = 4(3n-2)(n-1) - 8(n-1) + 2$.

Thus, replacing these results into Formulas (3) and (2), we obtain $4(3n-2)(n-1) = b_{n-1}(e) + 8(n-1) - 4 + 4(3n-2)(n-1) - 8(n-1) + 2$, from which we get the thesis. $\square$

Proposition 4.18. *Let D be a digital object of $\mathbb{C}_n$ and $e \in \mathcal{H}_{n-2}$. Then $b_{n-1}(e) = 4$.*

Proof. Let v_1 and v_2 be the two n-voxels of the $(n-2)$-gap through e. Then the number $b_{n-1}(e)$ of free $(n-1)$-cells of D bounded by e coincides with the maximum number of $(n-1)$-cells bounded by an $(n-2)$-cell, that is, by Proposition 4.4:

$$b_{n-1}(e) = c_{n-2 \leftarrow n-1} = 2^{(n-1)-(n-2)} \binom{n-(n-2)}{(n-1)-(n-2)} = 4.$$

□

Proposition 4.19. *Let D be a digital object of $\mathbb{C}_n$ and $e \in \mathcal{N}_{n-2}$. Then $b_{n-1}(e) = 2$.*

Proof. Every free $(n-2)$-cell that is not an $(n-2)$-hub is either a simple cell, or bounds the center of an $(n-1)$-block, or is the center of an L-block. Hence, by Corollary 4.11 and Propositions 4.14 and 4.17, we get the thesis. □

Proposition 4.20. *Let D be a digital n-object, and $i < j \le n-1$. Then*

$$\sum_{e \in bd_i(D)} b_j(e) = c_{i \to j} c_j^*.$$

Proof. The i-border $bd_i(D)$ of D can be considered as an incidence structure $(V, \mathcal{B}, \mathcal{I})$, where $V = bd_i(D)$, $\mathcal{B} = bd_j(D)$, and the incidence relation $\mathcal{I}$ is the bounding relation $<$.
In such a structure, the point degree of every vertex $e \in V$ coincides with the number $b_j(e)$ of j-cells of $bd(D)$ bounded by e. Moreover, the block degree k_β of every block $\mathcal{B}$ coincides with the maximum number $c_{i \to j}$ of i-cells that bound a j-cell. Hence, by Proposition 2.10, $\sum_{e \in bd_i(D)} b_j(e) = \sum_{\beta \in bd_j(D)} c_{i \to j} = c_{i \to j} |bd_j(D)| = c_{i \to j} c_j^*$. □

Theorem 4.21. *The number of $(n-2)$-gaps of a digital object D of $\mathbb{C}_n$ is given by the formula:*

$$g_{n-2} = (n-1)c_{n-1}^* - c_{n-2}^*. \tag{4}$$

Proof. Let us consider the sets $\mathcal{H}_{n-2}$ and $\mathcal{N}_{n-2}$ of all $(n-2)$-hubs and $(n-2)$-nubs of D, respectively. Evidently $\{\mathcal{H}_{n-2}, \mathcal{N}_{n-2}\}$ is a partition of $bd_{n-2}(D)$. Moreover, for $i = n-1$ and $j = n-2$, Proposition 4.20 give us

$$\sum_{e \in bd_{n-2}(D)} b_{n-1}(e) = c_{n-2 \to n-1} c_{n-1}^* = 2(n-1)c_{n-1}^*.$$

Since

$$\sum_{e \in bd_{n-2}} b_{n-1}(e) = \sum_{e \in \mathcal{H}_{n-2}} b_{n-1}(e) + \sum_{e \in \mathcal{N}_{n-2}} b_{n-1}(e),$$

by Lemmas 4.18 and 4.19, we obtain

$$\sum_{e \in bd_{n-2}} b_{n-1}(e) = 4|\mathcal{H}_{n-2}| + 2|\mathcal{N}_{n-2}| = 4g_{n-2} + 2(c_{n-2}^* - g_{n-2})$$

and hence the thesis. □

In [6], it was proved that the number of $(n-2)$-gap of a digital n-object D can be expressed by

$$g_{n-2} = -2n(n-1)c_n + 2(n-1)c_{n-1} - c_{n-2} + \beta_{n-2}, \tag{5}$$

where β_{n-2} is the number of all $(n-2)$-blocks contained in D.

Such a formula is equivalent to the expression (4) obtained in Theorem 4.21. Indeed, we have the following theorem.

Theorem 4.22. *The formulas*

$$g_{n-2} = (n-1)c^*_{n-1} - c^*_{n-2} \tag{6}$$

and

$$g_{n-2} = -2n(n-1)c_n + 2(n-1)c_{n-1} - c_{n-2} + \beta_{n-2} \tag{7}$$

are equivalent.

Proof. By Lemma 4.5, we have

$$c^*_{n-1} = c_{n-1} - c'_{n-1} = c_{n-1} + c_{n-1} - 2nc_n = 2c_{n-1} - 2nc_n.$$

Hence, replacing the latter expression in (6) , we obtain

$$g_{n-2} = (n-1)c^*_{n-1} - c^*_{n-2} = 2(n-1)c_{n-1} - 2(n-1)c_n - c_{n-2} + c'_{n-2}.$$

Finally, since c'_{n-2} is the number β_{n-2} of $(n-2)$-blocks contained in D, we get Formula (7).

Conversely, by Lemma 4.5, we have $c_n = \frac{c_{n-1}+c'_{n-1}}{2n}$. Thus Formula (7) becomes

$$g_{n-2} = -2n(n-1)\frac{c_{n-1} + c'_{n-1}}{2n} + 2(n-1)c_{n-1} + c^*_{n-2} = -(n-1)c'_{n-1} + (n-1)c_{n-1} + c^*_{n-2} = (n-1)c^*_{n-1} + c^*_{n-2},$$

that is Formula (6). This completes our proof. □

5. Conclusion and perspective

In this paper we have found a new formula for expressing the number of $(n-2)$-gaps of a digital n-object by means of its free cells. Unlike the equivalent formula (5) given in [6], our expression has the advantage to involve only few intrinsic parameters. We conjecture that such information could be obtained from some appropriate data structure related to the digital n-object. This will be the focus of a forthcoming research.

Another field of investigation could consist in finding a formula, analogous to (4), which express the number of any k-gaps with $0 \leq k \leq n-3$, by means of same basic parameters of the digital n-object.

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